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On Vandiver's arithmetical function – I

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Abstract: We study certain properties of Vandiver's arithmetic function $V(n) = \prod_{d|n}(d+1)$.
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1 Introduction

H. S. Vandiver (1882–1973) was a famous mathematician, best known for his work on Fermat's last theorem (he won the AMS Cole Prize in 1931 essentially for his activity in this field), but he wrote many papers also on cyclotomic and Abelian fields, Bernoulli numbers, the reciprocity laws, finite fields, factorization techniques, semigroups, semirings, and algebras. See D. H. Lehmer [2] for Vandiver's interesting life and activity.

In 1904, Vandiver published a problem (see [6]), which states that for any odd positive integers $n > 1$, one has

$$\prod_{d|n}(d+1) < 2^n. \tag{1}$$

His proof was based on some properties of Euler's totient function. In what follows, we shall call

$$V(n) = \prod_{d|n}(d+1) \tag{2}$$

as Vandiver's arithmetic function. We shall obtain some properties of function $V(n)$. Among others, it is shown that (1) holds also for even integers n, with a few exceptions $(n \neq 2, 4, 6, 8, 12)$.

Arithmetic functions of type

$$F(n) = \prod_{d|n} f(d) \tag{3}$$

for $f(n) = n$ or $f(n) = \varphi(n)$, $f(n) = \sigma(n)$ have been considered by the author, e.g. in papers [3, 5].

The main difference between these functions an Vandiver's one is that in (3) all functions (as $n, \sigma(n), \varphi(n)$, etc.) were *multiplicative* (i.e., satisfying $f(mn) = f(m)f(n)$ for $(m, n) = 1$), while $V(n)$ is clearly not. Therefore, one needs different techniques to handle this function.

Vandiver used the idea that, by Euler's divisibility theorem, one has $2^{\varphi(d)} - 1$ divisible by d (if d is odd), and particularly

$$2^{\varphi(d)} \geq d + 1. \tag{4}$$

The inequality (4) may be combined with Gauss' identity

$$\sum_{d|n} \varphi(d) = n.$$

It is easy to show that at least one inequality in (4) is strict, so in (1) there is a strict inequality.

We will use another argument to obtain stronger relations.

2 Main results

Theorem 1. *For all $n \geq 1$ we have the double inequality*

$$(\sqrt{n} + 1)^{d(n)} \leq V(n) \leq \left(\frac{\sigma(n)}{d(n)} + 1 \right)^{d(n)}, \tag{5}$$

where $d(n)$ and $\sigma(n)$ denote the number, respectively, the sum of divisors of n.

Proof. Let $d_1, d_2, \ldots, d_r$ denote the distinct divisors of n $(r = d(n))$. By the arithmetic mean—geometric mean inequality

$$(x_1 x_2 \ldots x_r)^{1/r} \leq \frac{x_1 + x_2 + \cdots + x_r}{r}, \tag{6}$$

applied to $x_i = d_i + 1 \ \ (i = 1, 2, \ldots, r)$, we get

$$(V(n))^{1/d(n)} \leq \frac{d_1 + \cdots + d_r + r}{r} = \frac{\sigma(n)}{d(n)} + 1,$$

so the right-hand side of (5) follows. Since all divisors d_i are distinct, there is equality only for $r = 1$, i.e., $n = 1$.

For the left-hand side of (5) we shall use the idea that $\prod_{d|n}(d+1) = \prod_{d|n}(\frac{n}{d}+1) = V(n)$, implying

$$(V(n))^2 = \prod_{d|n}(d+1)(\frac{n}{d}+1). \tag{7}$$

On the other hand, as $(d+1)(\frac{n}{d}+1) = n+1+(d+\frac{n}{d}) \geq n+1+2\sqrt{n} = (\sqrt{n}+1)^2$ by inequality (6) for $r=2, x_1=d, x_2=\frac{n}{d}$. Since $\prod_{d|n}((\sqrt{n}+1)^2) = (\sqrt{n}+1)^{2d(n)}$, by relation (7) we get the desired result. □

Remark 1. *Another method of proving the left-hand side of (5) is based on the following Minkowski inequality (see, e.g. [1]):*

$$\sqrt[r]{(a_1+b_1)\cdots(a_r+b_r)} \geq \sqrt[r]{a_1\cdots a_r} + \sqrt[r]{b_1\cdots b_r}, \tag{8}$$

where $a_i, b_i \ (i=1,\ldots,r)$ are positive real numbers. Apply (8) for $b_i=1, a_i=d_i$, and the result follows by the well-known formula $d_1\cdots d_r = n^{r/2}$.

Theorem 2. *For all $n>1$ odd, and for all $n \geq 14$ even, inequality (1) is true.*

Proof. We shall use the right-hand side of (5), and the known inequality (see, e.g., [4])

$$\frac{\sigma(n)}{d(n)} \leq \frac{n+1}{2}. \tag{9}$$

Now, by (5) and (9) we have to study the inequality

$$\left(\frac{n+3}{2}\right)^{d(n)} < 2^n. \tag{10}$$

Since $d(n) < 2\sqrt{n}$ (see, e.g. [4]) for all n, by taking logarithms in (10), it will be sufficient to consider

$$\sqrt{n}\log 2 > 2\log\left(\frac{n+3}{2}\right). \tag{11}$$

Since the function $f(x) = \sqrt{x}\log 2 - 2\log\left(\frac{x+3}{2}\right)$ has a derivative $f'(x) = \frac{(x+3)\log 2 - 4\sqrt{x}}{2(x+3)\sqrt{x}}$, the function f will be strictly increasing if $g(x) = (x+3)\log 2 - 4\sqrt{x} > 0$.

As $g'(x) = \log 2 - \frac{2}{\sqrt{x}} > 0$ <u>iff</u> $x > (\frac{2}{\log 2})^2 = 8.32\ldots,$ g is strictly increasing for such values of x. As $g(36) = 39\log 2 - 24 > 0$, g is strictly increasing for $x \geq 36$.

However, the values of f are negative for such low values of x.

But $f(169) = 13\log 2 - 2\log 86 = 0.102\ldots > 0$, so inequality (11) is valid for any $n \geq 169$. Now, a computer search shows that for $2 \leq n \leq 168$ only $n = 2, 4, 6, 8$ and 12 are exceptions, and this proves the theorem. □

Remark 2. *The above computations can be made somewhat easier by remarking that the upper bound in (5) may be replaced by:*

$$V(n) \leq \left(\sqrt{2(n+1)}\right)^{d(n)}. \tag{12}$$

For the proof of (12) we will use relation (7) and the fact that

$$d + \frac{n}{d} \leq n+1. \tag{13}$$

Inequality (13) holds true, as may be written equivalently as $(d-1)(d-n) \leq 0$, which is valid, as $d \geq 1$ and $d \leq n$.

Therefore, by (7) we get

$$(V(n))^2 \leq \prod_{d|n} 2(n+1)$$

and this implies (12).

Now, inequality

$$\left(\sqrt{2(n+1)}\right)^{d(n)} < 2^n \tag{14}$$

by $d(n) \leq 2\sqrt{n}$, is implied by the weaker inequality

$$(2(n+1))^{2\sqrt{n}} < 2^{2n},$$

or

$$2^{\sqrt{n}-1} > n+1. \tag{15}$$

Now, it is easy to see that the real-variable inequality

$$2^{x-1} > x^2+1 \tag{16}$$

holds true for any $x \geq 7$ (by the graphs of functions $f_1(x) = 2^{x-1}$ and $f_2(x) = x^2+1$); so (15) will be valid for any $n \geq 49$.

Therefore, it is sufficient to verify (1) for $2 \leq n \leq 48$, and this requires fewer computations than the above case.

A number n has been called in [5] as *multiplicatively f-perfect* if it satisfies

$$\prod_{d|n} f(d) = (f(n))^2. \tag{17}$$

for $f(n) = n$, along with generalizations, see [3]; while for example $f(n) = \sigma(n)$, see [5]. In these cases there are infinitely many solutions.

A similar equation to (17) is

$$\prod_{d|n} f(d) = 2f(n). \tag{18}$$

Theorem 3. *When $f(n) = n+1$, equation (17) has no solutions. All solutions of (18) are the prime numbers.*

Proof. Equation (17) in the case $f(n) = n+1$ may be written as

$$V(n) = (n+1)^2. \tag{19}$$

Now, by the left-hand side inequality of Theorem 1, by (19) we can write

$$(\sqrt{n}+1)^{d(n)} \leq (n+1)^2. \tag{20}$$

For $d(n) \geq 4$ we get $(\sqrt{n}+1)^4 \leq (n+1)^2$, or $(\sqrt{n}+1)^2 \leq n+1$, which is a contradiction. Therefore, one must have $2 \leq d(n) \leq 3$.

a) $d(n) = 2$. Then n is a prime, $n = p$, so $V(n) = 2(p+1) = (p+1)^2$ iff $p^2 = 1$, contradiction.

b) $d(n) = 3$. Then $n = p^2$ (p prime); in which case $V(n) = 2(p+1)(p^2+1) = (p^2+1)^2$ iff $2(p+1) = p^2+1$, i.e., $p^2 - 2p + 1 = 2$ or $(p-1)^2 = 2$; impossible as $\sqrt{2}$ is irrational.

Thus, the equation is not solvable for $n > 1$. Since $V(1) = 2 \neq 2^2, n = 1$ is not a solution either.

Related to equation (18), remark that as 1 and n are distinct divisors of $n > 1$,

$$V(n) \geq 2(n+1) \text{ for any } n > 1. \tag{21}$$

There is equality in (21) only when n has only these two divisors; i.e., if n is prime. Thus, equation (18) has only these solutions. □

A generalization of (21) is the following inequality

$$V(kn) \geq (kn+1).V(k) \text{ for } k \geq 1, n \geq 1. \tag{21'}$$

For $k = 1$, (21′) reduces to (21).

Let $k \geq 2$. Then remark that any divisor d of k is also a divisor of kn, so $V(kn)$ is a multiple of $V(k)$. There is at least one divisor of kn, which is not a divisor of k (for $k \geq 2$), namely kn; so by the definition of the function V, (21′) follows.

There is equality when kn has all divisors as the divisors of k, and nk, which is possible only if $k = 1$ and n prime.

Remark 3. *The equation*

$$V(n) = (n+1)^{d(n)} \tag{22}$$

has a single solution, namely $n = 1$.

Indeed, by inequality (12) we get $n + 1 \leq \sqrt{2(n+1)}$, *implying* $n \leq 1$, *i.e.,* $n = 1$. *Since* $V(1) = 2 = 2^{d(1)}$, *this is indeed a solution.*

Now we prove an asymptotic result:

Theorem 4. *One has*

$$\frac{\log V(n)}{d(n)} \sim \frac{1}{2} \log n \text{ as } n \to \infty. \tag{23}$$

Proof. By the left-hand side of (5) and relation (12) we get

$$d(n) \log(\sqrt{n} + 1) \leq \log V(n) \leq d(n) \log \sqrt{2(n+1)}. \tag{24}$$

It is easy to see that $\log(\sqrt{n} + 1) \sim \frac{1}{2} \log n$, and $\log \sqrt{2(n+1)} \sim \frac{1}{2} \log n$, so by (24) the result follows. □

Corollary 1.

$$\lim_{n\to\infty} \sup \log \log V(n). \frac{\log \log n}{\log n} = \log 2. \tag{24'}$$

Proof. This follows by (23) and the well-known result of S. Wiegert (see, e.g. [4])

$$\lim_{n\to\infty} \sup \frac{\log d(n). \log \log n}{\log n} = \log 2. \tag{25}$$

This completes the proof. □

Remark 4. *As, e.g.,* $V(p) = 2(p+1)$ *for primes p, one has clearly*

$$\lim_{n\to\infty} \inf \log\log V(n).\frac{\log\log n}{\log n} = 0. \tag{26}$$

This implies also, by inequality (21), that:

$$\lim_{n\to\infty} \inf \frac{V(n)}{n} = 2. \tag{26'}$$

An arithmetical property of $V(n)$ is contained in:

Theorem 5. $V(n)$ *is a power of* 2 *only if* $n = 1$ *or* n *is a Mersenne prime.*

Proof. One has $V(1) = 2$, while for $n > 1$ clearly $V(n)$ is divisible by $n+1$, so if n is an even number, then $V(n)$ can have odd divisors, as well. When $V(n) = 2^m$, by (2) all divisors $d > 1$ must be of the form $d = 2^a - 1$ for some integer $a > 1$. Remark that a number of the form $4m+1$ cannot be written as $2^a - 1$. Indeed, this would imply $4m = 2^a - 2 = 2(2^{a-1} - 1)$, which is not divisible by 4. Therefore, in the prime factorization of n, we cannot have primes $\equiv 1 \pmod 4$. If there are at least two prime divisors $p, q \equiv -1 \pmod 4$, then $p.q \equiv 1 \pmod 4$ is also a divisor of n, so we get a contradiction, as above. Thus, $n = p^\alpha$ with $p \equiv -1 \pmod 4$. For $\alpha \geq 2$, $p^2 \equiv 1 \pmod 4$ is a divisor of n, so this is impossible. Therefore $\alpha = 1$, when $n = p$ and $V(n) = 2(p+1) = 2^m$ iff $p = 2^{m-1} - 1$ is prime, i.e., a Mersenne prime. □

An improvement of inequality (21) is contained in the following theorem.

Theorem 6. *For any* $n > 1$ *one has*

$$V(n) \geq 2\sigma(n) \geq 2(n+1). \tag{27}$$

Proof. When $n > 1$ is prime, clearly $V(n) = 2\sigma(n) = 2(n+1)$. Suppose that n is composite. Then, by inequality (9) one has $2\sigma(n) \leq (n+1)d(n)$ (in fact, the inequality is strict, as there is equality in (9) only for n prime). Therefore, it is sufficient to prove

$$(n+1)d(n) < (\sqrt{n}+1)^{d(n)}, \tag{28}$$

in view of the left-hand side of (5). Put $d(n) = m \geq 3$ (as n is composite). Then we will prove by induction upon m the inequality

$$(n+1)m < (\sqrt{n}+1)^m. \tag{29}$$

For $m = 3$ one has $(\sqrt{n}+1)^3 = n\sqrt{n} + 3n + 3\sqrt{n} + 1 > 3n + 3$ is trivial.

Now, assuming (29) for m, one has

$$(\sqrt{n}+1)^{m+1} = (\sqrt{n}+1)^m.(\sqrt{n}+1) > (n+1)m(\sqrt{n}+1) > (n+1)(m+1)$$

$$\underline{\text{iff}}\;\; m(\sqrt{n}+1) > m+1,$$

which holds true as $m\sqrt{n} > m$. Therefore, (28) is true for any composite n, and this finishes the proof. □

Another proof can be given using the definitions of $V(n)$ and $\sigma(n)$:

$$\prod_{d|n}(d+1) \geq 2\sum_{d|n} d. \tag{30}$$

Let $d_1, d_2, \ldots, d_r$ be the distinct divisors of $n > 1$. Then, selecting $x_i = d_{i+1}$ $(i \doteq \overline{1, r-1})$, inequality (30) becomes:

$$(x_1 + 1)\cdots(x_{r-1} + 1) \geq x_1 + \cdots + x_{r-1} + 1, \quad r \geq 2. \tag{31}$$

This inequality is well-known, and can be proved e.g. by induction upon r. Since there is equality only for $r = 2$, the equality case in (30) will be only for n being a prime.

The second proof suggests the following:

Remark 5. *All solutions to the equation $V(n) = 2\sigma(n)$ are $n = 1$ and $n =$ prime. This follows by the proof of Theorem 6 and the fact that $V(1) = 2$ and $\sigma(1) = 1$.*

Theorem 7. *When n is composite, one has*

$$V(n) \geq 2.(\sigma(n) + T(n)), \tag{32}$$

where $T(n) = \prod_{d|n} d = n^{d(n)/2}$ denotes the product of the divisors of n.

Proof. Let $d_2, \ldots, d_r$ be the divisors > 1 of n, and put $x_i = d_{i+1}$, as in the second proof of Theorem 6. Then (32) becomes

$$(x_r + 1)\cdots(x_s + 1) \geq x_1 \cdots x_s + x_1 + \cdots + x_s + 1, \tag{33}$$

where $s = r - 1$. As n is composite, $r \geq 3$, so $s \geq 2$. For $s = 2$ there is equality in (33), while for $s = 3$ there is strict inequality, as

$$\begin{aligned}(x_1+1)(x_2+1)(x_3+1) &= x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1 + x_2 + x_3 + 1 \\ &> x_1x_2x_3 + x_1 + x_2 + x_3 + 1.\end{aligned}$$

By induction upon s, it follows that (33) holds true with strict inequality for $s \geq 3$. $\square$

Remark 6. *All solutions to the equation*

$$V(n) = 2.(\sigma(n) + T(n)) \tag{34}$$

are $n = p^2$, where $p \geq 2$ is a prime.

Indeed, n prime is not a soluton, while for n composite, by the proof of Theorem 7 one can have equality only when $s = 2$, i.e., n has $r = 3$ distinct divisors. This is true only for $n = p^2$, having divisors $1, p, p^2$.

Remark 7. *Since it is well-known (see, e.g. [4]) that $\sigma(n) < n\sqrt{n} = n^{3/2}$ for $n \geq 3$, and since $T(n) = n^{d(n)/2} \geq n^{3/2}$ for composite n, we get $T(n) > \sigma(n)$ for composite n, so (32) implies*

$$V(n) > 4\sigma(n) \ \textit{for all composite } n. \tag{35}$$

Theorem 8. *For infinitely many n we have*

$$V(n+1) > V(n), \tag{36}$$

while for infinitely many m we have

$$V(m+1) < V(m). \tag{37}$$

Proof. Inequality (36) is valid for any $n = p$ being prime. Indeed, by relation (21) one can write $V(p+1) > 2(p+2) > 2(p+1) = V(p)$. On the other hand, remark that if $p \geq 3$, then $p-1$ is divisible by $1, 2$ and $p-1$, so $V(p-1) \geq 6p > 2(p+1) = V(p)$, i.e., $V(p-1) > V(p)$ for $p \geq 3$. By letting $m = p-1$, inequality (37) follows. □

Conjecture 1. *As we have seen above, $V(p+1) > V(p)$ for primes $p \geq 2$ and $V(p-1) > V(p)$ for $p \geq 3$. The inequality*

$$V(p-1) < V(p+1) \tag{38}$$

is not always true (e.g., for $p = 31, 37$, etc.). We conjecture that (38) holds true for infinitely many primes p. Also, the inequality

$$V(q-1) > V(q+1) \tag{39}$$

holds true for infinitely many primes q.

Theorem 9.

$$\lim_{n\to\infty} \frac{V(2^n)}{2^{n(n-1)/2}} = 2.\iota, \tag{40}$$

where $\iota \in (2, e)$ is a constant.

Proof. As all divisors of 2^n are $1, 2, 2^2, \ldots, 2^n$, we get

$$V(2^n) = 2.(2+1).(2^2+1)\cdots(2^n+1). \tag{41}$$

By $2.2^2 \cdots 2^n = 2^{n(n-1)/2}$ one has

$$\frac{V(2^n)}{2^{n(n-1)/2}} = 2.(1+\frac{1}{2}).(1+\frac{1}{2^2})\cdots(1+\frac{1}{2^n}). \tag{41$'$}$$

Let $x_n = (1+\frac{1}{2}).(1+\frac{1}{2^2})\cdots(1+\frac{1}{2^n})$. By the classical inequality,

$$\ln(1+x) < x \quad (x > 0), \tag{42}$$

we get

$$\ln x_n < \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n} < 1. \tag{43}$$

As the sequence (x_n) is strictly increasing, and $x_n \in (1; e)$, it will be convergent, having a limit $x_n \to \iota$. (In other words, the infinite product converges to $\prod_{n=1}^{\infty}(1+\frac{1}{2^x}) = \iota$.)

Inequality (43), combined with $x_n < x_{n+1}$ implies $\iota < e$. On the other hand, we will prove that $x_n > 2$ for $x \geq 3$. Indeed, $x_3 = \frac{135}{64} > 2$, and assuming $x_n > 2$ one has $x_{n+1} > x_n > 2$, so by induction we get $x_n > 2$ for all $n \geq 3$. Thus $\iota > 2$, finishing the proof of Theorem 9. □

Theorem 10. *For all $n \geq 2$ one has*

$$V(2^n) > V(2^n - 1), \tag{44}$$

and

$$V(2^n - 1) > 2^{n+2}.(n+1) \quad \textit{for } n \geq 4, \tag{45}$$

if $2^n = 1$ is composite.

Proof. If d is odd, by Euler's divisibility theorem one has $d|(2^{\varphi(d)} - 1)$. Therefore, there exists the least $r(d)$ (the order of d) such that $d|(2^{r(d)} - 1)$. It is well known (e.g., from Group theory) that $d|(2^n - 1)$ <u>iff</u> $r(d)|n$. Therefore,

$$V(2^n - 1) = \sum_{d|2^n - 1} (d+1) \leq 2^{\sum_{d|n} r(d)} \leq 2^{nd(n)},$$

as $d \leq 2^{r(d)} - 1$ and $r(d) \leq n$. By taking logarithms, by (41) it will be sufficient to prove that

$$\ln 2 + \ln(2+1) + \cdots + \ln(2^n + 1) > nd(n) \ln 2. \tag{46}$$

As $\ln(2+1)+\cdots+\ln(2^n+1)-(\ln 2+\ln 2^2+\cdots+\ln 2^n) = \ln x_n$, where (x_n) is the sequence from Theorem 9; by $x_n > 2$ we get that the left-hand side of (46) is greater than $(\ln 2)\left[2 + \frac{n(n-1)}{2}\right]$.

Therefore, one has to prove the inequality

$$nd(n) < 2 + \frac{n(n-1)}{2}. \tag{47}$$

Using again $d(n) < 2\sqrt{n}$ (as in the proof of Theorem 2), the inequality

$$2n\sqrt{n} < 2 + \frac{n(n-1)}{2} \tag{48}$$

can be ealisy proved to hold for any $n \geq 25$. However, by direct computations, (47) is true also for $9 \leq n \leq 24$. For $n < 9$ relation (44) may be verified directly.

For the proof of (45) we shall use a result due to A. Schinzel [4] to the effect that

$$P(2^n - 1) \geq 2n + 1 \quad \text{for } n \geq 13, \tag{49}$$

where $P(m)$ denotes the greatest prime factor of m.

If $2^n - 1$ is not prime, then let $1 < p < 2^n - 1$ be its greatest prime divisor. Then clearly

$$V(2^n - 1) \geq 2.2^n.(p+1) \geq 2^{n+1}.(2n+2) = 2^{n+2}.(n+1)$$

for $n \geq 13$. Put $M_n = 2^n - 1$. It is known that $M_2, M_3, M_5, M_7, M_{13}, \ldots$ are primes. For $n = 4, 6, 8, 9, 10, 11, 12$, M_n are composite and in these cases relation (45) holds true. Therefore, it is sufficient to assume $n \geq 4$ and M_n composite. □

Remark 8. *Let $\varepsilon(n) > 0$ and $\varepsilon(n) \to 0$ as $n \to \infty$. In 1977, P. Stewart [4] proved that*

$$P(2^n - 1) > \varepsilon(n).n(\log n)^2/\log\log n, \tag{50}$$

ignoring a set of integers of asymptotic density 0.

M. R. Murty and S. Wong [4] proved, assuming the ABC-conjecture, that for each fixed $\alpha > 0$ one has

$$P(2^n - 1) > n^{2-\alpha} \quad \textit{for } n \geq n(\alpha), \tag{51}$$

where $n(\alpha)$ is an integer depending only on α.

Clearly all of these results can be used for the obtained lower bounds of $V(2^n - 1)$, when $2^n - 1$ is composite.

Remark 9. *When $2^n - 1$ is prime, then $V(2^n - 1) = 2^{n+1} < V(2^n + 1)$, as $V(2^n + 1) \geq 2.(2^n + 2) = 2^{n+1} + 4$. Therefore, for Mersenne primes it is true that:*

$$V(2^n - 1) < V(2^n + 1). \tag{52}$$

Conjecture 2. *Inequality (52) holds true for infinitely many n. The reverse inequality also holds true for infinitely many numbers n.*

3 Conclusion

In the second part of this paper, more properties of $V(n)$ will be studied.

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