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On the average order of the gcd-sum function over arbitrary sets of integers

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Abstract: Let $\mathbb{N}$ denote the set of all positive integers and for $j, n \in \mathbb{N}$, let (j, n) denote their greatest common divisor. For any $S \subseteq \mathbb{N}$, we define $P_S(n)$ to be the sum of those $(j, n) \in S$, where $j \in \{1, 2, 3, \ldots, n\}$. An asymptotic formula for the summatory function of $P_S(n)$ is obtained in this paper which is applicable to a variety of sets S. Also the formula given by Bordellès for the summatory function of $P_{\mathbb{N}}(n)$ can be derived from our result. Further, depending on the structure of S, the asymptotic formulae obtained from our theorem give better error terms than those deducible from a theorem of Bordellès (see Remark 4.4).

Keywords: Pillai function, gcd-sum function, Asymptotic formula, Möbius function of S, Dirichlet product, r-free integer, Semi-r-free integer, (k, r)-integer, Unitary divisor.
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1 Introduction

Let $\mathbb{N}$ denote the set of all positive integers. For $j, n \in \mathbb{N}$, let (j, n) denote their greatest common divisor (gcd).

If $S \subseteq \mathbb{N}$, then define

$$P_S(n) = \sum_{\substack{j=i \\ (j,n) \in S}}^{n} (j, n) \qquad \text{for } n \in \mathbb{N}. \tag{1.1}$$

Observe that $P_{\mathbb{N}}(n) = P(n)$, the arithmetic function studied by Pillai [9]. Possibly unaware of this work, Broughan [5] considered the same function (under a different notation) and obtained an asymptotic formula for $\sum_{n \leq x} P(n)$. Later, Bordellès [2] improved the error term in that asymptotic formula.

Also Bordellès [3] introduced a more general situation of

$$\mathcal{P}_f(n) = \sum_{j=1}^{n} f\left((j, n)\right), \tag{1.2}$$

where f is any arithmetic function and gave a proof of the Cesarò formula:

$$\mathcal{P}_f(n) = (f * \varphi)(n) \quad \text{for any } n \in \mathbb{N}, \tag{1.3}$$

in which φ is the Euler totient function and $*$ is the classical Dirichlet product of arithmetic functions. Moreover in the same paper unified asymptotic formulae for $\sum_{n \leq x} \mathcal{P}_f(n)$ are obtained for multiplicative arithmetic functions that lie in certain special classes.

A very informative survey on the gcd-sum functions by Tóth [14] and the paper on the weighted gcd-sum function (which is yet another general situation) by the same author [15] are worth to be mentioned here.

The purpose of this paper is to estimate $\sum_{n \leq x} P_S(n)$, for $S \subseteq \mathbb{N}$ which satisfy a condition; and to show that the formula of Bordellès [2] is deducible from our result. Further the formula is applicable to a variety of sets of integers such as the set of r-free integers, the set of semi-r-free integers and the set of (k, r)-integers studied by earlier researchers, in different contexts. The error terms in these asymptotic formulae are better than those deducible from a theorem of Bordellès ([3], Theorem 4, Part 4).

2 Notation and Preliminaries

For $S \subseteq \mathbb{N}$, let $\chi_S(n)$ be its *characteristic function.* (That is, $\chi_S(n) = 1$ or 0, respectively, as $n \in S$ or $n \notin S$.) Following Cohen [6], *the Möbius function of* S, denoted by $\mu_S(n)$, is defined by

$$\mu_S(n) = \sum_{d|n} \mu\left(d\right) \chi_S\left(\frac{n}{d}\right) = (\mu * \chi_S)(n) \text{ for } n \in \mathbb{N}, \tag{2.1}$$

where $\mu(n)$ is the well-known Möbius function.

Several properties of $*$ are studied in [1] (Chapter 2) some of which we use in this paper. For example, if $u(n) = 1$ for all $n \in \mathbb{N}$ and $\varepsilon_0(n) = 1$ or 0, respectively, as $n = 1$ or $n > 1$, then

$$\mu * u = \varepsilon_0 \tag{2.2}$$

and $f * \varepsilon_0 = f$ for any arithmetic function f.

It follows from (2.1) and (2.2), that

$$\mu_{\{1\}} = \mu \text{ and } \mu_{\mathbb{N}} = \varepsilon_0, \tag{2.3}$$

since $\chi_{\{1\}} = \varepsilon_0$ and $\chi_{\mathbb{N}} = u$; and that

$$\chi_S = u * \mu_S \text{ or equivalently } \chi_S(n) = \sum_{d|n} \mu_S\left(d\right) \text{ for any } S \subseteq \mathbb{N} \text{ and } n \in \mathbb{N}. \tag{2.4}$$

Also if $I(n) = n$ for all $n \in \mathbb{N}$ then $I(f * g) = If * Ig$ for arithmetic functions f and g. Further, it is clear that

$$(u * u)(n) = \tau(n), \text{the number of positive divisors of } n \in \mathbb{N}. \tag{2.5}$$

A well-known identity is

$$\varphi(n) = \sum_{d|n} d\mu\left(\frac{n}{d}\right) \quad \text{or equivalently} \quad \varphi = I * \mu. \tag{2.6}$$

Now we express below P_S as a Dirichlet product of some of the functions mentioned above.

Lemma 2.1. $P_S = (I\mu_S) * (I\tau * \mu)$, *for any* $S \subseteq \mathbb{N}$.

Proof. First observe that $P_S(n) = \sum_{j=1}^{n} I\left((j,n)\right)\chi_S\left((j,n)\right) = \mathcal{P}_{I\chi_S}(n)$, so that, in view of (1.3), (2.4), (2.6) and (2.5),

$$\begin{aligned} P_S = \mathcal{P}_{I\chi_S} &= (I\chi_S) * \varphi = I(\mu_S * u) * (I * \mu) = I\mu_S * Iu * (Iu * \mu) \\ &= I\mu_S * I(u * u) * \mu = I\mu_S * (I\tau * \mu), \end{aligned}$$

proving the lemma. □

One can observe that if $S = \mathbb{N}$ then Lemma 2.1 gives $P = I\tau * \mu$, a result proved by Bordellès ([2], Lemma 2.1).

If $M(x) = \sum_{n \le x} \mu(n)$, then its exact order of magnitude is not known. The best estimate given by Walfisz ([16], p.191) is that

$$M(x) = O\left(x\delta(x)\right) \text{ for } x > 1, \tag{2.7}$$

where

$$\delta(x) = \exp\{-A(\log x)^{\frac{3}{5}}.(\log\log x)^{\frac{-1}{5}}\}, \tag{2.8}$$

in which A is a positive constant.

Note that $\delta(x)$ is a monotonic decreasing function.

Using (2.7), Suryanarayana and Siva Rama Prasad [13] proved that, when $x > 1$,

$$\sum_{n \le x} \frac{\mu(n)}{n^t} = \frac{1}{\zeta(t)} + O\left(\frac{\delta(x)}{x^{t-1}}\right) \text{ for } t > 1 \quad \text{([13], Lemma 2.2)} \tag{2.9}$$

and

$$\sum_{n \le x} \frac{\mu(n)\log n}{n^t} = \frac{\zeta'(t)}{\zeta^2(t)} + O\left(\frac{\delta(x)\log x}{x^{t-1}}\right) \text{ for } t > 1 \quad \text{([13], Lemma 2.3)}, \tag{2.10}$$

where $\zeta(t)$ is the Riemann-zeta function.

The classical Dirichlet divisor problem seeks the least value of θ for which the asymptotic formula

$$\sum_{n \le x} \tau(n) = x\log x + (2\gamma - 1)x + O(x^\theta) \tag{2.11}$$

holds, where γ is the Euler constant. It is known that $\frac{1}{4} \le \theta \le \frac{517}{1648}$. The lower bound for θ is due to Hardy [8] while the upper bound is obtained recently by Bourgain and Watt [4].

Now using (2.11) and the Abel's identity ([1], Theorem 4.2), it is easy to prove

$$\sum_{n\leq x} I(n)\tau(n) = \frac{1}{2}x^2\left(\log x + 2\gamma - \frac{1}{2}\right) + O\left(x^{1+\theta+\varepsilon}\right), \tag{2.12}$$

where $\varepsilon > 0$.

3 Main result

In this section we prove the theorem given below:

Theorem 3.1. *Suppose $S \subseteq \mathbb{N}$ is such that the infinite series $\sum_{n=1}^{\infty} \frac{\mu_S(n)\log n}{n}$ converges absolutely. Then for $x \geq 1$, we have*

$$\sum_{n\leq x} P_S(n) = \frac{x^2}{2\zeta(2)}\left\{\alpha_S\left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) - \beta_S\right\} + \Delta_S(x),$$

where

$$\Delta_S(x) = \frac{x^2}{2\zeta(2)}\left(\beta_S(x) - \alpha_S(x)\right) + O\left(x^{1+\theta+\varepsilon}\gamma_S(x)\right), \tag{3.1}$$

$$\alpha_S = \sum_{n=1}^{\infty} \frac{\mu_S(n)}{n}, \tag{3.2}$$

$$\beta_S = \sum_{n=1}^{\infty} \frac{\mu_S(n)\log n}{n}, \tag{3.3}$$

$$\alpha_S(x) = \sum_{n>x} \frac{\mu_S(n)}{n}, \tag{3.4}$$

$$\beta_S(x) = \sum_{n>x} \frac{\mu_S(n)\log n}{n}, \tag{3.5}$$

and

$$\gamma_S(x) = \sum_{n\leq x} \frac{|\mu_S(n)|}{n^{\theta+\varepsilon}}, \tag{3.6}$$

in which $\varepsilon > 0$.

Proof. Under the hypothesis of the theorem, note that β_S and hence α_S are both well-defined.

By Lemma 2.1, we have $P_S = f * g$, where $f = I\mu_S$ and $g = I\tau * \mu$, so that

$$\sum_{n\leq x} P_S(n) = \sum_{u\leq x} f(u)\left\{\sum_{v\leq \frac{x}{u}} g(v)\right\}. \tag{3.7}$$

To estimate the inner sum on the right of (3.7), we use (2.12), (2.9) and (2.10) to get

$$\begin{aligned}\sum_{n\leq x} g(n) &= \sum_{d\leq x} \mu(d)\left\{\sum_{t\leq \frac{x}{d}} I(t)\tau(t)\right\} \\ &= \sum_{d\leq x} \mu(d)\left\{\frac{(x/d)^2}{2}\left(\log\left(\frac{x}{d}\right) + 2\gamma - \frac{1}{2}\right) + O\left(\left(\frac{x}{d}\right)^{1+\theta+\varepsilon}\right)\right\}\end{aligned}$$

$$= \frac{x^2}{2}\left(\log x + 2\gamma - \frac{1}{2}\right)\sum_{d\le x}\frac{\mu(d)}{d^2} - \frac{x^2}{2}\sum_{d\le x}\frac{\mu(d)\log d}{d^2} + O\left(x^{1+\theta+\varepsilon}\cdot\sum_{d\le x}\frac{|\mu(d)|}{d^{1+\theta+\varepsilon}}\right)$$
$$= \frac{x^2}{2}\left(\log x + 2\gamma - \frac{1}{2}\right)\left\{\frac{1}{\zeta(2)} + O\left(\frac{\delta(x)}{x}\right)\right\}$$
$$- \frac{x^2}{2}\left\{\frac{\zeta'(2)}{\zeta^2(2)} + O\left(\frac{\delta(x)\log x}{x}\right)\right\} + O\left(x^{1+\theta+\varepsilon}\right),$$

since $\sum_{d\le x}\frac{|\mu(d)|}{d^{1+\theta+\varepsilon}} = O(1)$. Thus

$$\sum_{n\le x} g(n) = \frac{x^2}{2\zeta(2)}\left\{\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right\} + O\left(x\log x\,\delta(x)\right) + O\left(x^{1+\theta+\varepsilon}\right)$$
$$= \frac{x^2}{2\zeta(2)}\left\{\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right\} + O\left(x^{1+\theta+\varepsilon}\right). \tag{3.8}$$

Now, using (3.8) in (3.7), we get

$$\sum_{n\le x} P_S(n) = \sum_{u\le x} u\mu_S(u)\left\{\frac{(x/u)^2}{2\zeta(2)}\left(\log\left(\frac{x}{u}\right) + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) + O\left(\left(\frac{x}{u}\right)^{1+\theta+\varepsilon}\right)\right\}$$
$$= \frac{x^2}{2\zeta(2)}\left\{\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right\}\sum_{u\le x}\frac{\mu_S(u)}{u} - \frac{x^2}{2\zeta(2)}\sum_{u\le x}\frac{\mu_S(u)\log u}{u} + E_S(x), \tag{3.9}$$

where $E_S(x) = \sum_{u\le x} u\mu_S(u)R(x,u)$, in which $|R(x,u)| \le C_\varepsilon \cdot \left(\frac{x}{u}\right)^{1+\theta+\varepsilon}$ for some $C_\varepsilon > 0$, so that

$$E_S(x) = O\left(x^{1+\theta+\varepsilon}\sum_{u\le x}\frac{|\mu_S(u)|}{u^{\theta+\varepsilon}}\right) = O\left(x^{1+\theta+\varepsilon}\cdot\gamma_S(x)\right). \tag{3.10}$$

Now (3.9) and (3.10) prove the theorem, since

$$\sum_{u\le x}\frac{\mu_S(u)}{u} = \alpha_S - \alpha_S(x) \text{ and } \sum_{u\le x}\frac{\mu_S(u)\log u}{u} = \beta_S - \beta_S(x).$$ □

Corollary 3.2. ([2, Theorem 1.1]) *For* $x \ge 1$*,*

$$\sum_{n\le x} P(n) = \frac{x^2}{2\zeta(2)}\left\{\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right\} + O\left(x^{1+\theta+\varepsilon}\right).$$

Proof. In view of (2.3), the condition of Theorem 3.1 holds if $S = \mathbb{N}$. Also since $\alpha_{\mathbb{N}} = 1$, $\beta_{\mathbb{N}} = 0$, $\alpha_{\mathbb{N}}(x) = \beta_{\mathbb{N}}(x) = 0$ and $\gamma_{\mathbb{N}}(x) = 1$ for $x \ge 1$, the corollary follows. □

Recall that, for $t > 1$,

$$\zeta(t) = \sum_{n=1}^{\infty} \frac{1}{n^t} \tag{3.11}$$

and

$$\frac{1}{\zeta(t)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^t}. \tag{3.12}$$

Both the series on the right of (3.11) and (3.12) converge absolutely and, therefore, by Theorem 11.2 of [1], they can be differentiated term by term with respect to t, to get

$$\zeta'(t) = -\sum_{n=1}^{\infty} \frac{\log n}{n^t} \text{ for } t > 1 \tag{3.13}$$

and

$$\frac{\zeta'(t)}{\zeta^2(t)} = \sum_{n=1}^{\infty} \frac{\mu(n)\log n}{n^t} \text{ for } t > 1. \tag{3.14}$$

Now (2.9) and (3.12) give

$$\sum_{n>x} \frac{\mu(n)}{n^t} = O\left(\frac{\delta(x)}{x^{t-1}}\right) \text{ for } t > 1; \tag{3.15}$$

while (2.10) and (3.14) show

$$\sum_{n>x} \frac{\mu(n)\log n}{n^t} = O\left(\frac{\delta(x)\log x}{x^{t-1}}\right) \text{ for } t > 1. \tag{3.16}$$

4 Application to some special subsets of $\mathbb{N}$

In the rest of this paper $n \in \mathbb{N}$ with $n > 1$ is of the form $n = \prod_{i=1}^{l} p_i^{\alpha_i}$, where $p_1, p_2, \ldots, p_l$ are distinct primes and integers α_i are ≥ 1 for $1 \leq i \leq l$.

To show the richness of the sets $S \subseteq \mathbb{N}$ for which Theorem 3.1 is applicable, first we make a brief study of the M-free integers introduced by Rieger [10].

Let M be a set of positive integers with the minimal element r, where $r > 1$. A number $n \geq 1$ is said to be *M-free* if $\alpha_i \notin M$ for $i = 1, 2, \ldots, l$. The set of all *M-free integers* will be denoted by Q_M.

Clearly $1 \in Q_M$ for every $M \subseteq \mathbb{N}$. Also χ_{Q_M} is a multiplicative function (that is, $\chi_{Q_M}(ab) = \chi_{Q_M}(a).\chi_{Q_M}(b)$ whenever $(a, b) = 1$). Then, by (2.1), μ_{Q_M} is a multiplicative function. Further for any prime p and $\alpha \in \mathbb{N}$ we have

$$\begin{aligned} \mu_{Q_M}(p^\alpha) &= \chi_{Q_M}(p^\alpha) - \chi_{Q_M}(p^{\alpha-1}) \\ &= \begin{cases} -1 & \text{if } \alpha \in M^* = \{\alpha \in \mathbb{N} : \alpha \in M \text{ and } \alpha - 1 \notin M\} \\ 1 & \text{if } \alpha \in M^{**} = \{\alpha \in \mathbb{N} : \alpha \notin M \text{ and } \alpha - 1 \in M\} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.1}$$

Hence, for $n > 1$, the value $\mu_{Q_M}(n)$ is non-zero if and only if (shortly, iff) n can be written as

$n = n^*.n^{**}$, where $n^* = \prod_{\alpha_i \in M^*} p_i^{\alpha_i}$ and $n^{**} = \prod_{\alpha_i \in M^{**}} p_i^{\alpha_i}$ which are such that $(n^*, n^{**}) = 1$ (since $M^* \cap M^{**} = \varnothing$). Also in this case

$$\mu_{Q_M}(n) = (-1)^{\omega(n^*)} \cdot 1^{\omega(n^{**})} = (-1)^{\omega(n^*)}, \tag{4.2}$$

where $\omega(m)$ is the number of distinct prime factors of m.

Notice that unless the elements of M are known explicity, we cannot find M^* and M^{**}; and thereby we cannot determine those n for which $\mu_{Q_M}(n) \neq 0$. Therefore we take some special sets for M and the corresponding Q_M below.

4.1 The set of r-free integers

Suppose $A = \{r, r+1, r+2, \ldots\}$, where $r \in \mathbb{N}$ and $r > 1$. Then $n > 1$ is in Q_A iff $1 \leq \alpha_i \leq r-1$ for $i = 1, 2, \ldots, l$. In other words, an integer $n > 1$ is in Q_A iff p^r is not a divisor of n for any prime p. Such numbers are called *r-free integers* in the literature. In fact, 2-free integers are well-known as *square-free integers*. Clearly n is square-free iff $\mu^2(n) = 1$. Thus Q_A is the set of all r-free integers.

For this set A, we find $A^* = \{\alpha \in \mathbb{N} : \alpha \in A \text{ and } \alpha - 1 \notin A\} = \{r\}$ and $A^{**} = \{\alpha \in \mathbb{N} : \alpha \notin A \text{ and } \alpha - 1 \in A\} = \varnothing$, so that $n^* = \prod_{\alpha_i \in A^*} p_i^{\alpha_i} = a^r$, where $a = p_1 p_2 \cdots p_l$ is square-free and $n^{**} = 1$. Therefore $\mu_{Q_A}(n)$ is non-zero iff $n = a^r$, for some square-free a. Also, by (4.2), for such n, $\mu_{Q_A}(n) = (-1)^{\omega(a)} = \mu(a)$.

Hence by (3.12) and (3.14), we get

$$\alpha_{Q_A} = \sum_{a=1}^{\infty} \frac{\mu(a)}{a^r} = \frac{1}{\zeta(r)}; \tag{4.3}$$

and

$$\beta_{Q_A} = r \cdot \sum_{a=1}^{\infty} \frac{\mu(a) \log a}{a^r} = r \cdot \frac{\zeta'(r)}{\zeta^2(r)}. \tag{4.4}$$

Also, by (3.15) and (3.16), we have

$$\alpha_{Q_A}(x) = \sum_{a > x^{1/r}} \frac{\mu(a)}{a^r} = O\left(\frac{\delta(x^{1/r})}{x^{1-\frac{1}{r}}}\right)$$

and

$$\beta_{Q_A}(x) = r \cdot \sum_{a > x^{1/r}} \frac{\mu(a) \log a}{a^r} = O\left(\frac{\delta(x^{1/r}) \log x}{x^{1-\frac{1}{r}}}\right),$$

so that

$$x^2 \cdot |\beta_{Q_A}(x) - \alpha_{Q_A}(x)| = O\left(x^{1+\frac{1}{r}} \delta(x^{1/r}) \log x\right). \tag{4.5}$$

Further $\gamma_{Q_A}(x) = \sum_{a \leq x^{1/r}} \frac{|\mu(a)|}{a^{r(\theta+\varepsilon)}}$ in which $r(\theta + \varepsilon) \leq 3\left(\frac{517}{1648} + \varepsilon\right) < 1$ for sufficiently small $\varepsilon > 0$ in case $r = 2$ or 3; and that $r(\theta + \varepsilon) > 1$ if $r \geq 4$. Therefore

$$\gamma_{Q_A}(x) = \begin{cases} O\left(x^{\frac{1}{r}-\theta-\varepsilon}\right), & \text{if } r = 2 \text{ or } 3 \\ O(1), & \text{if } r \geq 4. \end{cases} \tag{4.6}$$

Hence, by (4.5) and (4.6), we find

$$\Delta_{Q_A}(x) = \begin{cases} O\left(x^{1+\frac{1}{r}}\delta(x^{\frac{1}{r}})\log x\right) + O\left(x^{1+\frac{1}{r}}\right), & \text{if } r = 2 \text{ or } 3 \\ O\left(x^{1+\frac{1}{r}}\delta(x^{\frac{1}{r}})\log x\right) + O\left(x^{1+\theta+\varepsilon}\right), & \text{if } r \geq 4 \end{cases}$$

$$= \begin{cases} O\left(x^{1+\frac{1}{r}}\right), & \text{if } r = 2 \text{ or } 3 \\ O\left(x^{1+\theta+\varepsilon}\right), & \text{if } r \geq 4. \end{cases} \tag{4.7}$$

In view of (4.4), the condition of Theorem 3.1 holds for $S = Q_A$. Hence by (4.3), (4.4) and (4.7) we have a new asymptotic formula given below:

Corollary 4.1. *For $x \geq 1$,*

$$\sum_{n \leq x} P_{Q_A}(n) = \frac{x^2}{2\zeta(2)\zeta(r)}\left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} - r\frac{\zeta'(r)}{\zeta(r)}\right) + \Delta_{Q_A}(x),$$

where $\Delta_{Q_A}(x)$ is as in (4.7).

Note that

$$P_{Q_A}(n) = \sum_{\substack{j=i \\ (j,n) \text{ is } r\text{-free}}}^{n} (j, n).$$

4.2 The set of semi-r-free integers

Suppose $B = \{r\}$, where $r \in \mathbb{N}$ and $r > 1$. Then $n > 1$ is in Q_B iff $\alpha_i \neq r$ for $i = 1, 2, \ldots, l$. In other words, $n \in Q_B$ iff p^r is not a unitary divisor of n for any prime p. (Recall that a divisor d of n is said to be *unitary* if $\left(d, \frac{n}{d}\right) = 1$.) Such n is called a *semi-r-free integer* in [12]. Thus Q_B is the set of all *semi-r-free integers*.

For this set B, we note $B^* = \{\alpha \in \mathbb{N} : \alpha \in B \text{ and } \alpha - 1 \notin B\} = \{r\}$, while $B^{**} = \{\alpha \in \mathbb{N} : \alpha \notin B \text{ and } \alpha - 1 \in B\} = \{r+1\}$, so that $n^* = \prod_{\alpha_i \in B^*} p_i^{\alpha_i} = a^r$ and $n^{**} = \prod_{\alpha_i \in B^{**}} p_i^{\alpha_i} = b^{r+1}$, where a and b are both square-free. Thus $\mu_{Q_B}(n) \neq 0$ iff $n = a^r b^{r+1}$, where a and b are both square-free; and $(a, b) = 1$. For such n, we have, by (4.2), that $\mu_{Q_B}(n) = (-1)^{\omega(a)}\mu^2(b) = \mu(a)\mu^2(b)$.

Hence

$$\alpha_{Q_B} = \sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{\mu(a)\mu^2(b)}{a^r b^{r+1}} = \left(\sum_{a=1}^{\infty} \frac{\mu(a)}{a^r}\right)\left(\sum_{b=1}^{\infty} \frac{\mu^2(b)}{b^{r+1}}\right) = \frac{1}{\zeta(r)}\frac{\zeta(r+1)}{\zeta(2r+2)}, \tag{4.8}$$

by (3.12) and the fact that $\sum_{n=1}^{\infty} \frac{\mu^2(n)}{n^t} = \frac{\zeta(t)}{\zeta(2t)}$, which can be proved by Euler product representation theorem ([1], Theorem 11.6).

Also using this fact, Theorem 11.12 of [1], (3.12) and (3.14), we get

$$\begin{aligned}
\beta_{Q_B} &= \sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{\mu(a)\mu^2(b)}{a^r b^{r+1}}\left\{r\log a+(r+1)\log b\right\}\\
&= r\left(\sum_{a=1}^{\infty}\frac{\mu(a)\log a}{a^r}\right)\left(\sum_{b=1}^{\infty}\frac{\mu^2(b)}{b^{r+1}}\right)+(r+1)\left(\sum_{a=1}^{\infty}\frac{\mu(a)}{a^r}\right)\left(\sum_{b=1}^{\infty}\frac{\mu^2(b)\log b}{b^{r+1}}\right)\\
&= r\frac{\zeta'(r)}{\zeta^2(r)}\frac{\zeta(r+1)}{\zeta(2r+2)}-(r+1)\frac{1}{\zeta(r)}\cdot\frac{d}{dr}\left(\frac{\zeta(r+1)}{\zeta(2r+2)}\right)\\
&= \frac{\zeta(r+1)}{\zeta(r)\zeta(2r+2)}\left\{r\frac{\zeta'(r)}{\zeta(r)}-(r+1)\frac{\zeta'(r+1)}{\zeta(r+1)}+(2r+2)\frac{\zeta'(2r+2)}{\zeta(2r+2)}\right\}. \qquad (4.9)
\end{aligned}$$

Further, by (3.15) and (3.16), we have

$$\begin{aligned}
\alpha_{Q_B}(x) &= \sum_{a^r b^{r+1}>x}\frac{\mu(a)\mu^2(b)}{a^r b^{r+1}} = \sum_{b=1}^{\infty}\frac{\mu^2(b)}{b^{r+1}}\left\{\sum_{a>\left(\frac{x}{b^{r+1}}\right)^{1/r}}\frac{\mu(a)}{a^r}\right\}\\
&= O\left(\sum_{b=1}^{\infty}\frac{\mu^2(b)}{b^{1+\frac{1}{r}}}\cdot\frac{\delta(x^{1/r})b^{1+\frac{1}{r}}}{x^{1-\frac{1}{r}}}\right) = O\left(\frac{\delta(x^{1/r})}{x^{1-\frac{1}{r}}}\cdot\sum_{b=1}^{\infty}\frac{\mu^2(b)}{b^{1+\frac{1}{r}}}\right) = O\left(\frac{\delta(x^{1/r})}{x^{1-\frac{1}{r}}}\right) \qquad (4.10)
\end{aligned}$$

and

$$\begin{aligned}
\beta_{Q_B}(x) &= \sum_{a^r b^{r+1}>x}\frac{\mu(a)\mu^2(b)}{a^r b^{r+1}}\left\{r\log a+(r+1)\log b\right\}\\
&= r\sum_{b=1}^{\infty}\frac{\mu^2(b)}{b^{r+1}}\left\{\sum_{a>\left(\frac{x}{b^{r+1}}\right)^{1/r}}\frac{\mu(a)\log a}{a^r}\right\}+(r+1)\sum_{b=1}^{\infty}\frac{\mu^2(b)\log b}{b^{r+1}}\left\{\sum_{a>\left(\frac{x}{b^{r+1}}\right)^{1/r}}\frac{\mu(a)}{a^r}\right\}\\
&= O\left(\sum_{b=1}^{\infty}\frac{\mu^2(b)}{b^{r+1}}\cdot\frac{\delta(x^{1/r})\log x}{(x/b^{r+1})^{1-\frac{1}{r}}}\right)+O\left(\sum_{b=1}^{\infty}\frac{\mu^2(b)\log b}{b^{r+1}}\cdot\frac{\delta(x^{1/r})}{(x/b^{r+1})^{1-\frac{1}{r}}}\right)\\
&= O\left(\frac{\delta(x^{1/r})\log x}{x^{1-\frac{1}{r}}}\right), \qquad (4.11)
\end{aligned}$$

so that, by (4.10) and (4.11), we get

$$x^2|\beta_{Q_B}(x)-\alpha_{Q_B}(x)| = O\left(x^{1+\frac{1}{r}}\delta(x^{1/r})\log x\right). \qquad (4.12)$$

Also

$$\begin{aligned}
\gamma_{Q_B}(x) &= \sum_{a\le x^{1/r}}\frac{|\mu(a)|}{a^{r(\theta+\varepsilon)}}\left(\sum_{b\le\left(\frac{x}{a^r}\right)^{1/r+1}}\frac{1}{b^{(r+1)(\theta+\varepsilon)}}\right)\\
&= \begin{cases} O\left(\sum_{a\le x^{1/r}}\frac{|\mu(a)|}{a^{r(\theta+\varepsilon)}}\cdot\left(\frac{x}{a^r}\right)^{\frac{1}{r+1}-\theta-\varepsilon}\right), & \text{if } r=2\\ O\left(\sum_{a\le x^{1/r}}\frac{|\mu(a)|}{a^{r(\theta+\varepsilon)}}\right), & \text{if } r=3\\ O(1), & \text{if } r\ge 4\end{cases}\\
&= \begin{cases} O\left(x^{\frac{1}{r}-\theta-\varepsilon}\right), & \text{if } r=2 \text{ or } 3\\ O(1), & \text{if } r\ge 4.\end{cases} \qquad (4.13)
\end{aligned}$$

Now (4.12) and (4.13) give

$$\Delta_{Q_B}(x) = \begin{cases} O\left(x^{1+\frac{1}{r}}\right), & \text{if } r = 2 \text{ or } 3 \\ O\left(x^{1+\theta+\varepsilon}\right), & \text{if } r \geq 4. \end{cases} \tag{4.14}$$

Here the condition of Theorem 3.1 holds for $S = Q_B$ in view of (4.9). Therefore using (4.8), (4.9) and (4.14) in Theorem 3.1 we get another asymptotic formula given below:

Corollary 4.2. *For* $x \geq 1$,

$$\sum_{n \leq x} P_{Q_B}(n) = \frac{\zeta(r+1)}{2\zeta(2)\zeta(r)\zeta(2r+2)} x^2 \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} - F(r)\right) + \Delta_{Q_B}(x),$$

where $F(r) = r\frac{\zeta'(r)}{\zeta(r)} - (r+1)\frac{\zeta'(r+1)}{\zeta(r+1)} + (2r+2)\frac{\zeta'(2r+2)}{\zeta(2r+2)}$ *and* $\Delta_{Q_B}(x)$ *is as given in* (4.14).

Note that

$$P_{Q_B}(n) = \sum_{\substack{j=i \\ (j,n) \text{ is semi-}r\text{-free}}}^{n} (j,n).$$

4.3 The set of (k, r)-integers

Let $r, k \in \mathbb{N}$ be such that $2 \leq r < k$. Suppose $C = \{\alpha \in \mathbb{N} : \alpha \geq r \text{ and } \alpha \equiv j \pmod{k}$ for some j with $r \leq j \leq k-1\}$.

Now $n > 1$ is in Q_C iff for each i $(1 \leq i \leq l)$ we have either $\alpha_i < r$ or $\alpha_i \equiv v_i \pmod{k}$ for some v_i with $0 \leq v_i \leq r-1$ in which case we can write n as

$$n = \prod_{\substack{i=1 \\ \alpha_i \geq r}}^{l} p_i^{ku_i+v_i} \cdot \prod_{\substack{i=1 \\ \alpha_i < r}}^{l} p_i^{\alpha_i},$$

where $u_i \in \mathbb{N}$. Thus $n \in Q_C$ iff $n = a^k.b.c$, where $a = \prod_{\alpha_i \geq r} p_i^{u_i}$, $b = \prod_{\alpha_i \geq r} p_i^{v_i}$ and $c = \prod_{\alpha_i < r} p_i^{\alpha_i}$. Here $(ab, c) = 1$; and b, c are both r-free giving bc is r-free. Hence $n \in Q_C$ iff n is of the form $n = a^k \cdot m$, where $a \in \mathbb{N}$ and $m = bc \in Q_A$ (the set of r-free integers). Such numbers are called (k, r)-*integers* in [11]; and the same numbers were considered by Cohen [7], under a different notation. Since (∞, r)-integers are r-free integers, the notion of a (k, r)-free integer may be regarded as a generalization of an r-free integer. Thus Q_C is the set of all (k, r)-integers.

For this set C, the set $C^* = \{\alpha \in \mathbb{N} : \alpha \in C \text{ and } \alpha - 1 \notin C\} = \{\alpha \in \mathbb{N} : \alpha \equiv r \pmod{k}\}$ and $C^{**} = \{\alpha \in \mathbb{N} : \alpha \notin C \text{ and } \alpha - 1 \in C\} = \{\alpha \in \mathbb{N} : \alpha \equiv 0 \pmod{k}\}$. Therefore, by (4.2), for $n > 1$, writing $\alpha_i = ku_i + r$ if $\alpha_i \in C^*$ and $\alpha_i = kv_i$ if $\alpha_i \in C^{**}$, we have $n = a^k b^r c^k$, where $a = \prod_{\alpha_i \in C^*} p_i^{u_i}$, $b = \prod_{\alpha_i \in C^*} p_i$ and $c = \prod_{\alpha_i \in C^{**}} p_i^{v_i}$. Also for such n the value of $\mu_{Q_C}(n)$ is non-zero and is given by $\mu_{Q_C}(n) = \mu(b)$, since b is square-free.

Hence

$$\alpha_{Q_C} = \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \frac{\mu(b)}{a^k b^r c^k} = \left(\sum_{a=1}^{\infty} \frac{1}{a^k}\right) \left(\sum_{b=1}^{\infty} \frac{\mu(b)}{b^r}\right) \left(\sum_{c=1}^{\infty} \frac{1}{c^k}\right) = \frac{\zeta^2(k)}{\zeta(r)} \tag{4.15}$$

and

$$
\begin{aligned}
\beta_{Q_C} &= \sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\frac{\mu(b)\left\{k\log a + r\log b + k\log c\right\}}{a^k b^r c^k} \\
&= k\left(\sum_{a=1}^{\infty}\frac{\log a}{a^k}\right)\left(\sum_{b=1}^{\infty}\frac{\mu(b)}{b^r}\right)\left(\sum_{c=1}^{\infty}\frac{1}{c^k}\right) + r\left(\sum_{a=1}^{\infty}\frac{1}{a^k}\right)\left(\sum_{b=1}^{\infty}\frac{\mu(b)\log b}{b^r}\right)\left(\sum_{c=1}^{\infty}\frac{1}{c^k}\right) \\
&\quad + k\left(\sum_{a=1}^{\infty}\frac{1}{a^k}\right)\left(\sum_{b=1}^{\infty}\frac{\mu(b)}{b^r}\right)\left(\sum_{c=1}^{\infty}\frac{\log c}{c^k}\right) \\
&= -k\frac{\zeta'(k)\zeta(k)}{\zeta(r)} + r\frac{\zeta^2(k)\zeta'(r)}{\zeta^2(r)} - k\frac{\zeta(k)\zeta'(k)}{\zeta(r)} \\
&= \frac{\zeta(k)}{\zeta^2(r)}\left\{r\zeta(k)\zeta'(r) - 2k\zeta(r)\zeta'(k)\right\}, \qquad (4.16)
\end{aligned}
$$

wherein we used (3.11), (3.12), (3.13) and (3.14).

Also, by (3.15)

$$
\begin{aligned}
\alpha_{Q_C}(x) &= \sum_{u^k b^r > x}\frac{\mu(b)}{u^k b^r} = \sum_{u=1}^{\infty}\frac{1}{u^k}\left\{\sum_{b>\left(\frac{x}{u^k}\right)^{1/r}}\frac{\mu(b)}{b^r}\right\} \\
&= O\left(\sum_{u=1}^{\infty}\frac{1}{u^k}\frac{\delta(x^{1/r})}{(x/u^k)^{1-\frac{1}{r}}}\right) = O\left(\frac{\delta(x^{1/r})}{x^{1-\frac{1}{r}}}\cdot\sum_{u=1}^{\infty}\frac{1}{u^{k/r}}\right) \\
&= O\left(\frac{\delta(x^{1/r})}{x^{1-\frac{1}{r}}}\right), \qquad (4.17)
\end{aligned}
$$

because $2 \leq r < k$ implies that the series in the order term is convergent.

Again, using (3.15) and (3.16), we find that

$$
\begin{aligned}
\beta_{Q_C}(x) &= \sum_{u^k b^r > x}\frac{\mu(b)\log(u^k b^r)}{u^k b^r} \\
&= k\sum_{u^k b^r > x}\frac{\mu(b)\log u}{u^k b^r} + r\sum_{u^k b^r > x}\frac{\mu(b)\log b}{u^k b^r} \\
&= k\sum_{u=1}^{\infty}\frac{\log u}{u^k}\left(\sum_{b>\left(\frac{x}{u^k}\right)^{1/r}}\frac{\mu(b)\log b}{b^r}\right) + r\sum_{u=1}^{\infty}\frac{1}{u^k}\left(\sum_{b>\left(\frac{x}{u^k}\right)^{1/r}}\frac{\mu(b)}{b^r}\right) \\
&= O\left(\sum_{u=1}^{\infty}\frac{\log u}{u^k}\frac{\delta(x^{1/r})}{(x/u^k)^{1-\frac{1}{r}}}\right) + O\left(\sum_{u=1}^{\infty}\frac{1}{u^k}\frac{\delta(x^{1/r})\log x}{(x/u^k)^{1-\frac{1}{r}}}\right) \\
&= O\left(\frac{\delta(x^{1/r})}{x^{1-\frac{1}{r}}}\sum_{u=1}^{\infty}\frac{\log u}{u^{k/r}}\right) + O\left(\frac{\delta(x^{1/r})\log x}{x^{1-\frac{1}{r}}}\sum_{u=1}^{\infty}\frac{1}{u^{k/r}}\right) \\
&= O\left(\frac{\delta(x^{1/r})\log x}{x^{1-\frac{1}{r}}}\right), \qquad (4.18)
\end{aligned}
$$

because $2 \leq r < k$ implies that both the series in the order terms are convergent.

Hence

$$x^2 \cdot |\beta_{Q_C}(x) - \alpha_{Q_C}(x)| = O\left(x^{1+\frac{1}{r}}\delta(x^{1/r})\log x\right). \tag{4.19}$$

Further

$$\gamma_{Q_C}(x) = \sum_{u^k b^r \le x} \frac{|\mu(b)|}{u^{k(\theta+\varepsilon)} b^{r(\theta+\varepsilon)}} = \sum_{u \le x^{1/k}} \frac{1}{u^{k(\theta+\varepsilon)}} \left(\sum_{b \le \left(\frac{x}{u^k}\right)^{1/k}} \frac{|\mu(b)|}{b^{r(\theta+\varepsilon)}} \right)$$

$$= \begin{cases} O\left(x^{\frac{1}{r}-\theta-\varepsilon}\right), & \text{if } r = 2 \text{ or } 3 \\ O(1), & \text{if } r \ge 4. \end{cases} \tag{4.20}$$

Now (4.19) and (4.20) give

$$\Delta_{Q_C}(x) = \begin{cases} O\left(x^{1+\frac{1}{r}}\right), & \text{if } r = 2 \text{ or } 3 \\ O(x^{1+\theta+\varepsilon}), & \text{if } r \ge 4. \end{cases} \tag{4.21}$$

In view of (4.16), the condition of Theorem 3.1 holds if $S = Q_C$. Therefore using (4.15), (4.16) and (4.21) in Theorem 3.1, we get yet another asymptotic formula.

Corollary 4.3. *For* $x \ge 1$,

$$\sum_{n \le x} P_{Q_C}(n) = \frac{\zeta(k).x^2}{2\zeta(2)\zeta(r)} \left\{ \zeta(k)\left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) - \frac{1}{\zeta(r)}\left(r\zeta'(r)\zeta(k) - 2k\zeta(r)\zeta'(k)\right) \right\}$$
$$+ \Delta_{Q_C}(x),$$

where $\Delta_{Q_C}(x)$ *is as in* (4.21).

Note that

$$P_{Q_C}(n) = \sum_{\substack{j=1 \\ (j,n) \text{ is a } (k,r)\text{-integer}}}^{n} (j, n).$$

Remark 4.4. Any $f \in \{I\chi_{Q_A}, I\chi_{Q_B}, I\chi_{Q_C}\}$ lies in the class of multiplicative functions discussed in the case 4 of Theorem 4 in [3], wherein asymptotic formula with error term $O(x^2)$ is given for $\sum_{n \le x} \mathcal{P}_f(n)$. That is, the asymptotic formulae established in Corollaries 4.1, 4.2 and 4.3 are deducible from case 4 of Theorem 4 in [3], but with error terms $O(x^2)$ in each case. Observe that the error terms obtained in this paper are better than those in [3].

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