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A short remark on an arithmetic function

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Abstract: An explicit form of A. Shannon's arithmetic function δ is given. A possible application of it is discussed for representation of the well-known arithmetic functions ω and Kronecker's delta-function $\delta_{m,s}$.
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1 Introduction

45 years ago, in [7], A. G. Shannon introduced the following, back then new, arithmetic function:

$$\delta(m,s) = \begin{cases} 1, & \text{if } m \mid s \\ 0, & \text{otherwise} \end{cases}. \tag{1}$$

Here, we give an explicit representation of (1).

2 Preliminaries

For the needs of the next section, we introduce two functions.

Let $\mathbb{N}$ be the set of all positive integers. If $n > 1$ and $n \in \mathbb{N}$, then n has the form

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

that is called a canonical factorization of n, where $k, \alpha_1, \ldots, \alpha_k \in \mathbb{N}, n > 1$, $p_1, \ldots, p_k$ are different primes. For n it is defined the set-theoretical function (see [1])

$$\underline{\text{set}}(n) = \{p_1, p_2, \ldots, p_k\},$$

and the arithmetic function (see [2])

$$\overline{\text{sg}}(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ 0, & \text{otherwise} \end{cases}.$$

3 Main result

Having in mind that the natural number m divides the natural number s if and only if (iff) each divisor of m is a divisor of s and the degree with this divisor participate in m is smaller or equal to the degree with this divisor participate in s, we can represent this assertion in the following predicate logical form:

$$P(m, s) = \text{“}(\forall p \in \underline{\text{set}}(m))(\deg_m(p) \leq \deg_s(p))\text{”},$$

where $\deg_m(p)$ is the degree with which the prime number p participates in the natural number m.

It is clear that if the predicate $P(m, s)$ is true, then $\underline{\text{set}}(m) \subseteq \underline{\text{set}}(s)$.

Now, we give an arithmetic form of predicate $P(m, s)$. It is

$$P(m, s) = \prod_{p \,\in\, \underline{\text{set}}(m)} \overline{\text{sg}}(\deg_m(p) - \deg_s(p)).$$

Really,

$$\overline{\text{sg}}(\deg_m(p) - \deg_s(p)) = \begin{cases} 1, & \text{if } \deg_m(p) \leq \deg_s(p) \\ 0, & \text{otherwise} \end{cases}.$$

Therefore,

$$P(m, s) = \begin{cases} 1, & \text{if } (\forall p \in \underline{\text{set}}(m))(\deg_m(p) \leq \deg_s(p)) \\ 0, & \text{otherwise} \end{cases}.$$

Hence, giving an arithmetic form of the predicate $P(m, s)$, we proved the following theorem.

Theorem 1. *For every two natural numbers m and s:*

$$\delta(m,s) = \prod_{p\,\in\,\underline{\mathrm{set}}(m)} \overline{\mathrm{sg}}(\deg_m(p) - \deg_s(p)). \qquad (2)$$

The δ-function can be used for representation, e.g., of the prime omega function $\omega(n)$ giving the number of the distinct prime divisors of $n \in \mathbb{N}, n > 1$ and $\omega(1) = 0$.

Theorem 2. *For each natural number $n \geq 2$:*

$$\omega(n) = \sum_{p \leq n} \delta(p,n),$$

where the variable p represents a prime number.

Proof. Let $n \geq 2$ be given. Then for each p $(2 \leq p \leq n)$, from (2) we obtain:

$$\delta(p,n) = \begin{cases} 1, & \text{if } p \mid n \\ 0, & \text{otherwise} \end{cases}.$$

Hence, $\sum\limits_{i=2}^{n-1} \delta(p,n)$ is the number of the divisors of the natural number n. □

4 Conclusion

The delta symbol was obviously chosen because of a connection with the Kronecker delta $\delta_{m,s}$:

$$\delta_{m,s} = \delta(m,s)\delta(s,m),$$

since

$$\delta(m,s)\delta(s,m) = 1$$

iff $m \mid s$ and $s \mid m$; that is, iff $m = s$; otherwise

$$\delta(m,s)\delta(s,m) = 0.$$

The $\delta(m,s)$ was used in solving a problem of Morgan Ward [8] for a generalization of the Staudt–Clausen problem [3]. Further, Mollie Horadam [4] defined the function

$$e(n) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases}.$$

Note that $e(n) = \delta(n,1),\ n \geq 1$, and

$$e(n) = \sum_{d|n} \mu(d),$$

where $\mu(d)$ is the Möbius function. Of more immediate relevance to this paper is that $e(n)$ acts as an identity element for $\delta(n,n)$. Popken [6] defined the convolution product of two arithmetical functions $f(n)$ and $g(n)$ as

$$f(n) * g(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

For notational convenience for convolutions associated with the delta functions, we shall consider the first term in the parentheses as the one indexed, so that

$$e(n) * \delta(n, n) = \sum_{d|n} e(d)\delta\left(\frac{n}{d}, n\right) = e(1)\delta(n, n) + 0 = \delta(n, n),$$

as required.

The use of notation can itself be mathematically creative and stimulate further enquiry which sometimes leads to patterns of importance [5].

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