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Refined enumeration of 2-noncrossing trees

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Abstract: A 2-noncrossing tree is a connected graph without cycles that can be drawn in the plane with its vertices on the boundary of circle such that the edges are straight line segments that do not cross and all the vertices are coloured black and white with no ascent (i, j), where i and j are black vertices, in a path from the root. In this paper, we use generating functions to prove a formula that counts 2-noncrossing trees with a black root to take into account the number of white vertices of indegree greater than zero and black vertices. Here, the edges of the 2-noncrossing trees are oriented from a vertex of lower label towards a vertex of higher label. The formula is a refinement of the formula for the number of 2-noncrossing trees that was obtained by Yan and Liu and later on generalized by Pang and Lv. As a consequence of the refinement, we find an equivalent refinement for 2-noncrossing trees with a white root, among other results.
Keywords: 2-noncrossing trees, Local orientation, Sources.
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1 Introduction and preliminaries

A *noncrossing tree* is a connected acyclic graph that can be drawn in the plane with its vertices on the boundary of a circle such that the edges are straight line segments that do not cross. The number of these trees on n vertices is known to be given by

$$\frac{1}{n-1}\binom{3n-3}{n-2}, \tag{1}$$

see [1, 3, 4] for details.

If the vertices of the noncrossing trees are coloured black and white such that there is no ascent (i, j), where i and j are black vertices, in a path from the root, then the noncrossing tree is called 2-*noncrossing tree*, [9]. If the root is coloured black, then the number of such trees on n vertices is given by

$$\frac{1}{5n-4}\binom{5n-4}{n-1}. \tag{2}$$

Formula (2) also counts the number of 5-ary trees with $n-1$ internal vertices. The number of 2-noncrossing trees on n vertices with a white root is given by

$$\frac{2}{5n-3}\binom{5n-3}{n-1}.$$

These trees were generalized by Pang and Lv in [7] to k-noncrossing trees. A k-*noncrossing tree* is a noncrossing tree where each node receives a label in $\{1, 2, \ldots, k\}$ such that the sum of labels along an ascent does not exceed $k+1$. The aforementioned authors showed that these trees with root labelled by k on n vertices are counted by $(2k+1)$-Catalan number,

$$\frac{1}{2k(n-1)+1}\binom{(2k+1)(n-1)}{n-1}.$$

They also showed that the formula counts $(2k+1)$-ary trees. A refinement of this formula to take into account the number of nodes of each label was obtained by Okoth and Wagner in [6].

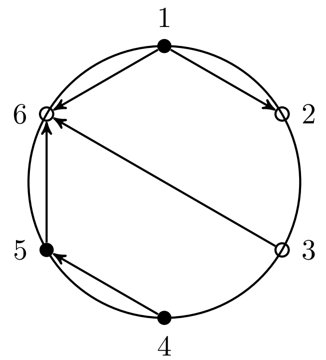


Figure 1. A 2-noncrossing tree rooted at a black vertex 1

In this paper, we consider 2-noncrossing trees with a local orientation. Here, all edges are oriented from a vertex of lower label towards a vertex of higher label, [2]. A *source* (respectively, *sink*) is a vertex with indegree (respectively, outdegree) zero. A vertex which is not a source will be referred to as *non-source* vertex. In Figure 1, we have a 2-noncrossing tree on 6 vertices rooted at a black vertex 1, and equipped with a local orientation. Vertex 3 is a white source while vertex 2 is a non-source white vertex. The number of noncrossing trees, having a local orientation, with a given number of sources and sinks was obtained by the present author and his co-author in [5]. These trees are called *locally oriented noncrossing trees* or *lnc-trees* therein. The main aim of this paper is to prove the following theorem:

Theorem 1.1 (**Main theorem**). *The number of* 2*-noncrossing trees on* n *vertices with a black root such that there are* k *black vertices and* ℓ *non-source white vertices is given by*

$$\frac{1}{n-1}\binom{2n-2}{k-1}\binom{2n-2}{\ell-1}\binom{n-1}{k+\ell-1}. \tag{3}$$

In Section 2, we prove the theorem by means of generating functions. In the proof, we make use of *butterfly decomposition* of noncrossing trees introduced by Flajolet and Noy in [3]. By a *butterfly*, we mean an ordered pair of noncrossing trees that have a common root. Let $T(z)$ and $B(z)$ be the generating function for noncrossing trees and butterflies respectively then

$$T(z) = \frac{z}{1-B} \quad \text{and} \quad B(z) = \frac{T^2}{z}.$$

Quite a number of corollaries of the main theorem follow in Section 3 and an equivalent result of Theorem 1.1, for 2-noncrossing trees with a white root, is presented in Section 4.

2 Proof of the main theorem

We consider decomposition of 2-noncrossing trees. There are two cases to consider:
Case I: The root of the 2-noncrossing tree is black.

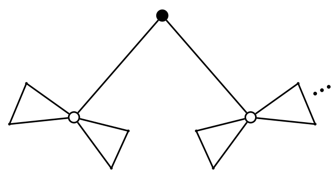


Figure 2. Decomposition of a 2-noncrossing tree with a black root.

Here, all the children of the root are white since there are no black-black ascents. The functional equation satisfied by these trees is

$$T = \frac{z}{1 - \frac{S^2}{z}},$$

where $T(z)$ and $S(z)$ are the generating functions for the number of 2-noncrossing trees with black and white roots, respectively.

Case II: Root of the 2-noncrossing tree is white. In this case, children of the root are either black or white.

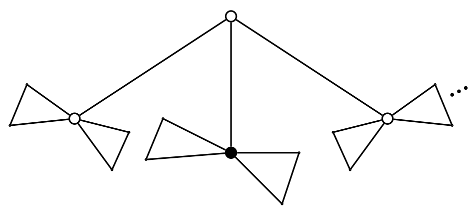
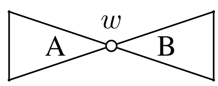


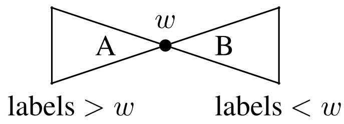
Figure 3. Decomposition of a 2-noncrossing tree with a white root.

For the trees with white children, a butterfly is represented as:



Here, a butterfly is equivalent to two 2-noncrossing trees with white roots. Thus it is represented as S^2/z.

For the trees with black children, a butterfly is represented as:



Part A is a 2-noncrossing tree with a black root. Part B is different: since the root of B has the largest label now, the ascent rule does not apply to edges between the root and its children. Therefore, B can be considered as a 2-noncrossing tree with a white root which has been recoloured. So the butterflies are represented by ST/z and we get the functional equation satisfied by these trees as

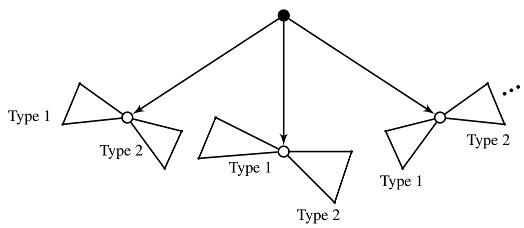
$$S = \frac{z}{1 - \left(\frac{S^2}{z} + \frac{ST}{z}\right)}.$$

Now, let u mark the number of white sources and v mark the number of black vertices (of any kind).

We will need to distinguish two types of 2-noncrossing trees with a white root now:

- Type 1: root has lowest label (label 1)
- Type 2: root has highest label (label n)

Let the generating functions of trees of Types 1 and 2 be S_1 and S_2, respectively. The root is not counted as a source. In the decomposition of trees with a black root we get:



We have one additional black vertex (the root) and every white source must be a source in one of the smaller trees in the decomposition. So we get

$$T = \frac{zv}{1 - \frac{S_1 S_2}{z}}.$$

We now consider trees with a white root. For those trees of Type 1, the decomposition is:

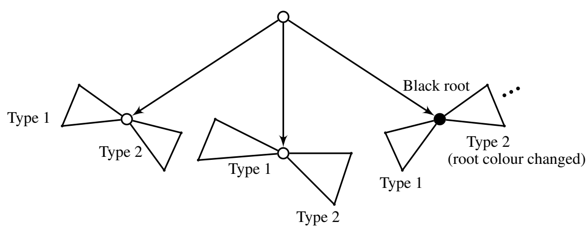


Figure 4. Decomposition of a 2-noncrossing tree with a white root of Type 1.

In this decomposition, there are two types of butterflies which correspond to $\frac{S_1S_2}{z}$ and $\frac{S_2T}{z}$, respectively. Each black vertex or white source must already occur in one of the smaller trees of the decomposition. So we get,

$$S_1 = \frac{z}{1 - \left(\frac{S_1S_2}{z} + \frac{S_2T}{z}\right)}.$$

For these trees of Type 2, the root has highest label. So this only changes the orientation of the edges from the root. Thus, if w is a white root of a butterfly, then w becomes a source if B consists only of the root. So the equation changes to

$$S_2 = \frac{z}{1 - \left(uS_1 + \frac{S_1(S_2-z)}{z} + \frac{S_2T}{z}\right)}.$$

Here, uS_1 means that B is only one node and $\frac{S_1(S_2-z)}{z}$ implies that B has more than one node. So we have a system

$$T = \frac{zv}{1 - \frac{S_1S_2}{z}}, \tag{4}$$

$$S_1 = \frac{z}{1 - \left(\frac{S_1S_2}{z} + \frac{S_2T}{z}\right)}, \tag{5}$$

$$S_2 = \frac{z}{1 - \left(uS_1 + \frac{S_1(S_2-z)}{z} + \frac{S_2T}{z}\right)}, \tag{6}$$

which we solve.

First we substitute $T = vz(1+Y)$. Then Equation (4) gives

$$1 - \frac{S_1S_2}{z} = \frac{vz}{T} = \frac{1}{1+Y}$$

and thus

$$S_1S_2 = \frac{zY}{1+Y} \tag{7}$$

or

$$S_2 = \frac{zY}{S_1(1+Y)}. \tag{8}$$

We substitute Equations (7) and (8) in Equation (5) to give

$$\begin{aligned} S_1 &= \frac{z}{1 - \left(\frac{Y}{1+Y} + \frac{YT}{S_1(1+Y)}\right)} = \frac{z}{1 - \left(\frac{Y}{1+Y} + \frac{vzY}{S_1}\right)} \\ &= \frac{z}{\frac{1}{1+Y} - \frac{vzY}{S_1}} = \frac{zS_1(1+Y)}{S_1 - vzY(1+Y)}. \end{aligned}$$

Since $S_1 \neq 0$, it follows that

$$S_1 - vzY(1+Y) = z(1+Y)$$

and finally

$$S_1 = vzY(1+Y) + z(1+Y) = z(1+Y)(1+vY). \tag{9}$$

Since $S_2 = \frac{zY}{S_1(1+Y)}$ (by Equation (8)), we have

$$S_2 = \frac{Y}{(1+Y)^2(1+vY)}. \tag{10}$$

Plugging Equations (9) and (10) and $T = zv(1+Y)$ into Equation (6), we obtain

$$\begin{aligned}
\frac{Y}{(1+Y)^2(1+vY)} &= \frac{z}{1 - uz(1+Y)(1+vY) - \frac{Y}{1+Y} + z(1+Y)(1+vY) - \frac{vY}{(1+Y)(1+vY)}} \\
&= \frac{z}{\left(1 - \frac{Y}{1+Y} - \frac{vY}{(1+Y)(1+vY)}\right) - (u-1)z(1+Y)(1+vY)} \\
&= \frac{z}{\frac{1}{(1+Y)(1+vY)} - (u-1)z(1+Y)(1+vY)}.
\end{aligned}$$

So,

$$\frac{Y}{(1+Y)^3(1+vY)^2} - \frac{(u-1)zY}{1+Y} = z.$$

Thus

$$\frac{Y}{(1+Y)^3(1+vY)^2} = z\frac{1+uY}{1+Y}$$

or

$$Y = z(1+Y)^2(1+vY)^2(1+uY).$$

This is the right format for applying Lagrange Inversion Formula, (See [8, Theorem 5.4.2] for details). We extract the coefficient of z^n in the generating function $T(z)$ as follows:

$$\begin{aligned}
[z^n]T &= v[z^{n-1}]Y \\
&= v \cdot \frac{1}{n-1}[y^{n-2}]((1+y)^2(1+vy)^2(1+uy))^{n-1} \\
&= v \cdot \frac{1}{n-1}[y^{n-2}](1+y)^{2n-2}(1+vy)^{2n-2}(1+uy)^{n-1} \\
&= v \cdot \frac{1}{n-1}[y^{n-2}]\sum_{j\geq 0}\sum_{i\geq 0}\binom{2n-2}{j}\binom{n-1}{i}v^j y^j u^i y^i (1+y)^{2n-2} \\
&= \frac{1}{n-1}\sum_{j\geq 0}\sum_{i\geq 0}[y^{n-j-i-2}]\binom{2n-2}{j}\binom{n-1}{i}v^{j+1}u^i(1+y)^{2n-2} \\
&= \frac{1}{n-1}\sum_{j\geq 0}\sum_{i\geq 0}\binom{2n-2}{j}\binom{n-1}{i}v^{j+1}u^i\binom{2n-2}{n-j-i-2} \\
&= \frac{1}{n-1}\sum_{k\geq 1}\sum_{i\geq 0}\binom{2n-2}{k-1}\binom{n-1}{i}v^k u^i\binom{2n-2}{n-k-i-1}
\end{aligned}$$

Thus the number of 2-noncrossing trees with a black root, n vertices in total, k of which are black and i white sources is given by

$$\frac{1}{n-1}\binom{2n-2}{k-1}\binom{n-1}{i}\binom{2n-2}{n-k-i-1}.$$

Setting $n-k-i = \ell$, we obtain the required result.

3 Consequences of the main theorem

In this section, we obtain several corollaries of the main theorem. If we sum over all ℓ in Equation (3), we obtain

$$\frac{1}{n-1}\binom{2n-2}{k-1}\binom{3n-3}{n-k-1}, \tag{11}$$

as the number of 2-noncrossing trees on n vertices with with a black root and exactly k black vertices. Since formula (3) is symmetric in k and ℓ, Equation (11) gives the number of 2-noncrossing trees on n vertices with a black root and exactly k non-source white vertices.

Remark 1. *We remark that setting $k = 1$ in Equation* (11)*, we recover the formula* (1) *for the number of noncrossing trees on n vertices. Since the root is always a source, then setting $k = 1$ and $\ell = n - i$, we rediscover the formula*

$$\frac{1}{n-1}\binom{2n-2}{n-i-1}\binom{n-1}{i-1}$$

for the number of noncrossing trees on n vertices with i sources. This formula was first obtained by Okoth and Wagner in [5].

The following corollary follows by seting $\ell = n - k - i$ in Equation (3) and summing over all k:

Corollary 3.0.1. *There are*

$$\frac{1}{n-1}\binom{n-1}{i}\binom{4n-4}{n-i-2} \tag{12}$$

2*-noncrossing trees on n vertices with i white sources such that vertex* 1*, coloured black, is identified as the root.*

Corollary 3.0.2. *The mean and variance of the number of white sources in* 2*-noncrossing trees on n vertices with a black root are equal to $(n-2)/5$ and $(16n^2 - 44n + 24)/(125n - 150)$ respectively.*

Proof. From Equations (2) and (12), the probability that a 2-noncrossing tree with a black root on n vertices has i white sources is given by

$$\frac{\binom{n-1}{i}\binom{4n-4}{n-i-2}}{\binom{5n-5}{n-2}}. \tag{13}$$

Now multiplying Equation (13) by i and summing over all i, we obtain the mean. To obtain variance, we multiply Equation (13) by i^2 and sum over all i. We then subtract square of the mean. $\square$

Corollary 3.0.3. *The number of* 2*-noncrossing trees on n vertices with a black root such that there are a total of r black vertices and non-source white vertices is given by*

$$\frac{1}{n-1}\binom{n-1}{r-1}\binom{4n-4}{r-2}. \tag{14}$$

Proof. The result follows by setting $k = r - \ell$ and summing over all ℓ in Equation (3), or setting $i = n - r$ in Equation (12). $\square$

Setting $r = n$, we find a generalized Catalan number,

$$\frac{1}{n-1}\binom{4n-4}{n-2}.$$

This formula counts the number of 2-noncrossing trees on n vertices in which a vertex is either a non-source white vertex or a black vertex. In the next corollary, we find a structure counted by the famous Catalan numbers.

Corollary 3.0.4. *The number of* 2*-noncrossing trees on* n *vertices with a black root such that there are either* $n-1$ *non-source white vertices or,* $n-1$ *black vertices and one non-source white vertex is given by the* $(n-1)$*-th Catalan number,*

$$\frac{1}{n-1}\binom{2n-2}{n-2}.$$

Proof. The formula follows by setting $k = 1$ and $\ell = n - 1$, or $\ell = 1$ and $k = n - 1$ in Equation (3). $\square$

Corollary 3.0.5. *The mean and variance of the number of black vertices or the number of non-source white vertices are equal to* $(2n+1)/5$ *and* $(24n^2-66n+36)/(125n-150)$*, respectively. The covariance of the two equals* $(-16n^2 + 44n - 24)/(125n - 150)$.

Proof. From Theorem 1.1 and Equation (2), the probability of a 2-noncrossing tree to have k black vertices and ℓ non-source white vertices is given by

$$\frac{\binom{2n-2}{k-1}\binom{2n-2}{\ell-1}\binom{n-1}{k+\ell-1}}{\binom{5n-5}{n-2}}. \tag{15}$$

Now, multiplying Equation (15) by ℓ and summing over all ℓ and k, we obtain the mean. To obtain the variance, we multiply the said equation by ℓ^2 and sum over all ℓ and k, and then subtract the square of the mean. For covariance, we multiply Equation (15) by $k\ell$ and sum over all k and ℓ. We then subtract the square of the mean. $\square$

Corollary 3.0.6. *The mean and variance of the number of non-source white vertices in* 2*-noncrossing trees with a black root and* n *vertices,* k *of which are black, are given by* $(2n - 2k + 1)/3$*, and* $(4n^2 - 2k^2 - 8n - 2nk + 2k + 4)/(27n - 36)$ *respectively.*

Proof. From Theorem 1.1 and Equation (11), the probability of a 2-noncrossing tree with a black root on n vertices, k of which are black to have ℓ non-source white vertices is given by

$$\frac{\binom{2n-2}{\ell-1}\binom{n-1}{k+\ell-1}}{\binom{3n-3}{n-k-1}}. \tag{16}$$

Now, multiplying Equation (16) by ℓ and summing over all ℓ, we obtain the mean. To obtain the variance, we multiply equation (16) by ℓ^2 and sum over all ℓ. We then subtract the square of the mean. $\square$

4 Enumeration of trees with a white root

The following theorem gives the number of 2-noncrossing trees with a white root such that the number of non-source white vertices and black vertices are simultaneously given.

Theorem 4.1. *The number of* 2*-noncrossing trees on* n *vertices with a white root such that there are* k *black vertices and* ℓ *non-source white vertices is given by*

$$\frac{1}{2n-1}\binom{2n-1}{k}\binom{2n-1}{\ell}\binom{n-2}{k+\ell-1}. \tag{17}$$

Proof. 2-noncrossing trees with a white root (here the root has the lowest possible label) has the generating function S_1, given in the proof of the main theorem. From Equation (9), we have that $S_1 = z(1+Y)(1+vY)$ where Y satisfies the functional equation $Y = z(1+Y)^2(1+vY)^2(1+uY)$. As mentioned in Section 2, u marks white sources and v marks black vertices. We have,

$$\begin{aligned}[z^n]S_1 &= [z^{n-1}](1+vY+Y+vY^2)\\ &= v[z^{n-1}]Y + [z^{n-1}]Y + v[z^{n-1}]Y^2. \end{aligned} \tag{18}$$

By Lagrange inversion we have,

$$v[z^{n-1}]Y = \frac{1}{n-1}\sum_{k\geq 1}\sum_{i\geq 0}\binom{2n-2}{k-1}\binom{n-1}{i}v^k u^i\binom{2n-2}{n-k-i-1}. \tag{19}$$

We also have

$$[z^{n-1}]Y = \frac{1}{n-1}\sum_{k\geq 0}\sum_{i\geq 0}\binom{2n-2}{k}\binom{n-1}{i}v^k u^i\binom{2n-2}{n-k-i-2} \tag{20}$$

and

$$v[z^{n-1}]Y^2 = \frac{2}{n-1}\sum_{k\geq 1}\sum_{i\geq 0}\binom{2n-2}{k-1}\binom{n-1}{i}v^k u^i\binom{2n-2}{n-k-i-2}. \tag{21}$$

Now, from Equations (18), (19), (20) and (21), we have that the number of 2-noncrossing trees with a white root and n vertices, k of which are black and i of which are white sources, is given by

$$\begin{aligned}[z^n v^k u^i]S_1 &= \frac{1}{n-1}\binom{2n-2}{k-1}\binom{n-1}{i}\binom{2n-2}{n-k-i-1}\\ &+ \frac{1}{n-1}\binom{2n-2}{k}\binom{n-1}{i}\binom{2n-2}{n-k-i-2}\\ &+ \frac{2}{n-1}\binom{2n-2}{k-1}\binom{n-1}{i}\binom{2n-2}{n-k-i-2}.\end{aligned}$$

After simple algebraic manipulations we obtain

$$[z^n v^k u^i]S_1 = \frac{1}{2n-1}\binom{2n-1}{k}\binom{n-2}{i}\binom{2n-1}{n-k-i-1}.$$

Now, upon setting $n-k-i-1=\ell$ we obtain the desired result. □

Remark 2. *Following a similar trend as in Section 2, we can obtain equivalent results such as mean number of black vertices, non-source white vertices, white sources among many interesting corollaries, for* 2*-noncrossing trees with white roots.*

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