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The generalized *k*-Fibonacci polynomials and generalized *k*-Lucas polynomials

Merve Taştan[1], Engin Özkan[2] and Anthony G. Shannon[3]

[1] Graduate School of Natural and Applied Sciences, Erzincan Binali Yildirim University
Erzincan, Turkey
e-mail: mervetastan@gmail.com

[2] Department of Mathematics, Erzincan Binali Yildirim University
Erzincan, Turkey
e-mails: eozkanmath@gmail.com, eozkan@erzincan.edu.tr

[3] Warrane College, the University of New South Wales
Kensington, NSW 2033, Australia
e-mails: t.shannon@warrane.unsw.edu.au, tshannon38@gmail.com

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Abstract: In this paper, we define new families of Generalized Fibonacci polynomials and Generalized Lucas polynomials and develop some elegant properties of these families. We also find the relationships between the family of the generalized *k*-Fibonacci polynomials and the known generalized Fibonacci polynomials. Furthermore, we find new generalizations of these families and the polynomials in matrix representation. Then we establish Cassini's Identities for the families and their polynomials. Finally, we suggest avenues for further research.
Keywords: Generalized Fibonacci polynomials, *k*-Fibonacci numbers, Generalized Lucas polynomials, *k*-Lucas numbers.
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1 Introduction

Fibonacci number sequences have attractive properties and many applications in various branches of mathematics [3, 5, 14, 21]. Historically too there have been different approaches to defining Fibonacci and Lucas polynomials [7, 9, 11, 13, 20].

More recently, Mikkawy and Sogabe [10] defined a new family of *k*-Fibonacci numbers. Then in [12], Özkan et al defined a new family of *k*-Lucas numbers and developed some properties about them. Falcon and Plaza [4] produced some general k-Fibonacci numbers and showed how properties of these numbers can be related to elementary matrix algebra. Bolat and Köse [1] also found some important properties about k-Fibonacci numbers.

This paper is arranged in three parts. In the second part, we outline some pertinent definitions and properties. In the third part, we define the new families of Generalized k-Fibonacci polynomials and Generalized k-Lucas polynomials. These yield some neat properties of the families. We then develop the relationships between the family of Generalized k-Fibonacci polynomials and known Generalized Fibonacci polynomials and Generalized k-Lucas polynomials and known Generalized Lucas polynomials. We also establish new generalizations of these families in matrix representation and prove Cassini's Identities for the families.

2 Materials and methods

Definition 1. *Generalized Fibonacci polynomials can be defined by*

$$f_{n+1}(x) = \begin{cases} s, & n = 0 \\ sx, & n = 1 \\ xf_n(x) + f_{n-1}(x), & n \geq 2 \end{cases}.$$

If $s = 1$, *then we obtain the classical Fibonacci polynomial sequence.*

It is well known that the Fibonacci polynomials and Lucas polynomials are closely related.

Definition 2. *Generalized Lucas polynomials can be defined similarly by*

$$l_n(x) = \begin{cases} 2s, & n = 0 \\ sx, & n = 1 \\ xl_{n-1}(x) + l_{n-2}(x), & n \geq 2 \end{cases}.$$

If $s = 1$, *then we obtain the classical Lucas polynomial sequence.*

A Binet formula for $f_n(x)$ is given by

$$f_n(x) = s\frac{r_1^n - r_2^n}{r_1 - r_2}m, \quad (1)$$

where $r_1 = \frac{x + \sqrt{x^2+4}}{2}$ and $r_2 = \frac{x - \sqrt{x^2+4}}{2}$.

A Binet formula for $l_n(x)$ is then given by

$$l_n(x) = s(r_1^n + r_2^n), \quad (2)$$

where $r_1 = \frac{x + \sqrt{x^2+4}}{2}$ and $r_2 = \frac{x - \sqrt{x^2+4}}{2}$.

3 Generalized polynomials

Definition 3. *Let n and $k \neq 0$ be natural numbers, then there exist unique numbers m and r such that $n = mk + r$ $(0 \leq r < k)$. The generalized k-Fibonacci polynomials $f_n^{(k)}(x)$ are then defined by*

$$f_n^{(k)}(x) := s^k \left(\frac{r_1^m - r_2^m}{r_1 - r_2}\right)^{k-r} \left(\frac{r_1^{m+1} - r_2^{m+1}}{r_1 - r_2}\right)^r,$$

where $r_1 = \frac{x+ \sqrt{x^2+4}}{2}$ and $r_2 = \frac{x- \sqrt{x^2+4}}{2}$.

Also, we can find the generalized *k*-Fibonacci polynomials by matrix methods. Indeed, it is clear that

$$f_n^{k-1}(x)Q_2^n = \begin{bmatrix} F_{kn+k+1}^{(k)}(x) & F_{kn+k}^{(k)}(x) \\ F_{kn+k}^{(k)}(x) & F_{kn+k-1}^{(k)}(x) \end{bmatrix},$$

in which

$$Q_2^n = \begin{bmatrix} F_{n+1}^{(k)}(x) & F_n^{(k)}(x) \\ F_n^{(k)}(x) & F_{n-1}^{(k)}(x) \end{bmatrix}, n \geq 0.$$

We now give some values for the generalized *k*-Fibonacci polynomials in Table 1.

	k = 1	*k* = 2	*k* = 3	*k* = 4
$F_0^{(k)}(x)$	s	s^2	s^3	s^4
$F_1^{(k)}(x)$	sx	s^2x	s^3x	s^4x
$F_2^{(k)}(x)$	$sx^2 + 1$	s^2x^2	s^3x^2	s^4x^2
$F_3^{(k)}(x)$	$sx^3 + 2sx$	$s^2x^3 + s^2x$	s^3x^3	s^4x^2
$F_4^{(k)}(x)$	$sx^4 + 3sx^2 + s$	$s^2x^4 + 2s^2x^2 + s^2$	$s^3x^4 + s^3x^2$	s^4x^4
$F_5^{(k)}(x)$	$sx^5 + 4sx^3 + 3sx$	$s^2x^5 + 3s^2x^3 + 2s^2x$	$s^3x^5 + 2s^3x^3 + s^{3x}$	$s^4x^5 + s^4x^3$

Table 1. Some Generalized *k*-Fibonacci polynomials

Definition 3, we can obtain the generalized *k*-Fibonacci polynomials via the generalized Fibonacci polynomials.

$$F_n^{(k)}(x) = \left(f_m(x)\right)^{k-r} (f_{m+1}(x))^r, n = mk + r. \tag{3}$$

If $k = 1$ in equation (3), then we have that $m = n$ and $r = 0$ so $GF_n^{(1)}(x) = f_n(x)$. Throughout this paper, let $k, m \in \{1,2,3, \dots .\}$.

For k = 2, 3, 4 and n, we have the following properties which connect the generalized k-Fibonacci polynomials and the known generalized Fibonacci polynomials.

$$F_{2n}^{(2)}(x) = f_n^2(x),$$

$$F_{2n+1}^{(2)}(x) = f_n(x)f_{n+1}(x),$$

$$F_{2n+1}^{(2)}(x) = xF_{2n}^{(2)}(x) + F_{2n-1}^{(2)}(x),$$

$$F_{3n}^{(3)}(x) = f_n^3(x),$$

$$F_{3n+1}^{(3)}(x) = f_n^2(x)f_{n+1}(x),$$

$$F_{3n+2}^{(2)}(x) = f_n^2(x)f_{n+1}(x),$$

$$F_{3n+1}^{(3)}(x) = xF_{3n}^{(3)}(x) + F_{3n-1}^{(3)}(x),$$

$$F_{4n}^{(4)}(x) = f_n^4(x),$$

$$F_{4n+1}^{(4)}(x) = f_n^3(x)f_{n+1}(x),$$

$$F_{4n+2}^{(4)}(x) = f_n^2f_{n+1}^2(x),$$

$$F_{4n+3}^{(4)}(x) = f_n(x)f_{n+1}^3(x),$$

$$F_{4n+1}^{(4)}(x) = xF_{4n}^{(4)}(x) + F_{4n-1}^{(4)}(x).$$

Theorem 1. *For n, we have the following relation*

$$F_{kn+1}^{(k)}(x) = xF_{kn}^{(k)}(x) + F_{kn-1}^{(k)}(x).$$

Proof. By using (3), we have

$$\begin{aligned} xF_{kn}^{(k)}(x) + F_{kn-1}^{(k)}(x) &= xF_n^k + \left(F_{n-1}(x)F_n^{k-1}(x)\right) \\ &= F_n^{k-1}(x)(xF_n(x) + F_{n-1}(x)) \\ &= F_n^{k-1}(x)F_{n+1}(x) \\ &= F_{kn+1}^{(k)}(x). \end{aligned}$$

From Definition 3, we know that $n = mk + r$, where $m, k \neq 0$ natural numbers and $0 \leq r < k$.
For $k = 1,2, ...,$ let $F_{-n}^{(k)} = 0$ be.

Theorem 2. (Cassini's Identity) Let $GF_n^k(x)$ represent the generalized Gauss k-Fibonacci polynomials. For $n, k \geq 2$, the Cassini's Identity $F_n^{(k)}(x)$ is as follows:

$$F_{kn+t}^{(k)}(x)F_{kn+t-2}^{(k)}(x) - \left(F_{kn+t-1}^{(k)}(x)\right)^2 = \begin{Bmatrix} F_n^{2k-2}(x)(-1)^n(s^k), & t = 1 \\ 0 & t \neq 1 \end{Bmatrix}.$$

Proof. By using (3), we find

$$F_{kn+t}^{(k)}(x)F_{kn+t-2}^{(k)}(x) - \left(F_{kn+t-1}^{(k)}(x)\right)^2$$

$$= \left(F_{n-1}^{k-1}(x)F_{n+t}(x)F_n^{k-1}(x)\right)\left(F_{n+t-2}(x)\right) - \left(F_n^{k-1}(x)F_{n+t-1}(x)\right)^2$$

$$= \left(F_n^{k-1}(x)\right)^2\left(F_{n+t}(x)GF_{n+t-2}(x) - \left(GF_{n+t-1}(x)\right)^2\right)$$

$$= F_n^{2k-2}(x)(F_{n+t}(x)F_{n+t-2}(x)(F_{n+t-1}(x))^2).$$

for $t = 1$,

$$= F_n^{2k-2}(x)\left(F_{n+t}(x)F_{n+t-2}(x)\left(F_{n+t-1}(x)\right)^2\right)$$

$$= F_n^{2k-2}(x)(-1)^n(s^{2k}).$$

for $t \neq 1, t = m, (m \in N)$,

$$= F_n^{2k-2}(x)(F_{n+m}(x)F_{n+m-2}(x)(F_{n+m-1}(x))^2)$$

$$= F_n^{2k-2}(x)\left(F_{2n+2m-2}^{(2)}(x) - F_{2m+2n-2}^{(2)}(x)\right) = 0.$$

For *n*, we obtain an interesting relation between the known generalized Fibonacci polynomials and the generalized *k*-Fibonacci polynomials

$$F_{nk+t}^{(k)}(x) = F_n^{k-t}(x)F_{n+1}^t(x),$$

where $t = 0, 1, \ldots, k - 1$.

Definition 4. *Let n and $k \neq 0$ be natural numbers, then there exist unique numbers m and r such that $n = mk + r$ $(0 \leq r < k)$. The generalized k-Lucas polynomials $L_n^{(k)}(x)$ are defined by*

$$L_n^{(k)}(x) := s^k(r_1^m + r_2^m)^{k-r}(r_1^{m+1} + r_2^{m+1})^r,$$

where $r_1 = \frac{x+\sqrt{x^2+4}}{2}$ and $r_2 = \frac{x-\sqrt{x^2+4}}{2}$.

Also, we can find the generalized *k*-Lucas polynomials by matrix methods. Indeed, it is clear that

$$L_n^{k-1}(x)\varphi_{n,2} = \begin{bmatrix} L_{kn+k+1}^{(k)}(x) & L_{kn+k}^{(k)}(x) \\ L_{kn+k}^{(k)}(x) & L_{kn+k-1}^{(k)}(x) \end{bmatrix},$$

in which

$$\varphi_{n,2} = \varphi_{n-1,2}Q_2 = \begin{bmatrix} l_{n+1}(x) & l_n(x) \\ l_n(x) & l_{n-1}(x) \end{bmatrix}, Q_2 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}, \varphi_{0,2} = \begin{pmatrix} x & 2 \\ 2 & -x \end{pmatrix},$$

$n \geq 0$.

□

From Definition 4, we can obtain the Generalized k-Lucas polynomials via the Generalized Fibonacci polynomials.

$$L_n^{(k)}(x) = \left(l_m(x)\right)^{k-r}(l_{m+1}(x))^r \,, n = mk + r. \tag{4}$$

If $k = 1$ in equation (4), then we have that $m = n$ and $r = 0$ so $GL_n^{(1)}(x) = l_n(x)$. Throughout this paper, let $k, m \in \{1,2,3, \dots.\}$.

For k = 2, 3, 4 and n, we have the following properties which connect the Generalized k-Fibonacci polynomials and the known Generalized Fibonacci polynomials.

$$L_{2n}^{(2)}(x) = l_n^2(x),$$

$$L_{2n+1}^{(2)}(x) = l_n(x)l_{n+1}(x),$$

$$L_{2n+1}^{(2)}(x) = xL_{2n}^{(2)}(x) + L_{2n-1}^{(2)}(x),$$

$$L_{3n}^{(3)}(x) = L_n^3(x),$$

$$L_{3n+1}^{(3)}(x) = l_n^2(x)l_{n+1}(x),$$

$$L_{3n+2}^{(2)}(x) = l_n^2(x)l_{n+1}(x),$$

$$L_{3n+1}^{(3)}(x) = xL_{3n}^{(3)}(x) + L_{3n-1}^{(3)}(x),$$

$$L_{4n}^{(4)}(x) = l_n^4(x),$$

$$L_{4n+1}^{(4)}(x) = l_n^3(x)l_{n+1}(x),$$

$$L_{4n+2}^{(4)}(x) = l_n^2 l_{n+1}^2(x),$$

$$L_{4n+3}^{(4)}(x) = l_n(x)l_{n+1}^3(x),$$

$$L_{4n+1}^{(4)}(x) = xL_{4n}^{(4)}(x) + L_{4n-1}^{(4)}(x).$$

Theorem 3. *For n, we have the following relation*

$$L_{kn+1}^{(k)}(x) = xL_{kn}^{(k)}(x) + L_{kn-1}^{(k)}(x).$$

Proof. By using (4), we have

$$\begin{aligned} xL_{kn}^{(k)}(x) + L_{kn-1}^{(k)}(x) &= xL_n^k + \left(l_{n-1}(x)l_n^{k-1}(x)\right) \\ &= l_n^{k-1}(x)(xl_n(x) + l_{n-1}(x)) \\ &= l_n^{k-1}(x)l_{n+1}(x) \\ &= L_{kn+1}^{(k)}(x). \end{aligned}$$ □

From Definition 4, we know that $n = mk + r$, where m, $k \neq 0$ natural numbers and $0 \leq r < k$. For $k = 1,2, \dots$, let $L_{-n}^{(k)} = 0$.

Theorem 4. (Cassini's Identity) Let $GL_n^{(k)}(x)$ represent the generalized Gauss *k*-Lucas polynomials. For $n, k \geq 2$, the Cassini's Identity $L_n^{(k)}(x)$ is as follows:

$$L_{kn+t}^{(k)}(x)L_{kn+t-2}^{(k)}(x) - \left(L_{kn+t-1}^{(k)}(x)\right)^2 = \begin{Bmatrix} L_n^{2k-2}(x)(-1)^{n+1}(x^2+4)(s^{2k}), & t=1 \\ 0 & t \neq 1 \end{Bmatrix}.$$

Proof. By using (4), we find

$$L_{kn+t}^{(k)}(x)L_{kn+t-2}^{(k)}(x) - \left(L_{kn+t-1}^{(k)}(x)\right)^2$$

$$= \left(L_{n-1}^{k-1}(x)L_{n+t}(x)L_n^{k-1}(x)\right)\left(L_{n+t-2}(x)\right) - \left(L_n^{k-1}(x)L_{n+t-1}(x)\right)^2$$

$$= \left(L_n^{k-1}(x)\right)^2\left(L_{n+t}(x)GL_{n+t-2}(x) - \left(GL_{n+t-1}(x)\right)^2\right)$$

$$= L_n^{2k-2}(x)(L_{n+t}(x)L_{n+t-2}(x)(L_{n+t-1}(x))^2).$$

for $t = 1$,

$$= L_n^{2k-2}(x)\left(L_{n+t}(x)L_{n+t-2}(x)\left(L_{n+t-1}(x)\right)^2\right)$$

$$= L_n^{2k-2}(x)(-1)^{n+1}(x^2+4)(s^{2k}).$$

for $t \neq 1, t = m, (m \in N)$,

$$= L_n^{2k-2}(x)(L_{n+m}(x)L_{n+m-2}(x)(L_{n+m-1}(x))^2)$$

$$= L_n^{2k-2}(x)\left(L_{2n+2m-2}^{(2)}(x) - L_{2m+2n-2}^{(2)}(x)\right) = 0.$$

Theorem 5. For any integer *n*, we have

$$(x^2+4)F_{2n}^{(2)}(x) + 4s^2(-1)^n = \begin{cases} s^2(r_1^n + r_2^n)^2, & if\ n\ is\ even \\ s^2(r_1^n - r_2^n)^2, & if\ n\ is\ odd \end{cases}.$$

Proof. From the Binet's formula of generalized Fibonacci polynomials

$$F_{2n}^{(2)}(x) = f_{2n}^2(x) = \frac{s^2}{(r_1^n - r_2^n)^2}\{r_1^{2n} - 2(r_1r_2)^n + r_2^{2n}\}.$$

If *n* is even, then we get $(x^2+4)F_{2n}^{(2)}(x) + 4s^2 = (sr_1^n + sr_2^n)^2$. If *n* is odd, then we have $(x^2+4)F_{2n}^{(2)}(x) - 4s^2 = (sr_1^n - sr_2^n)^2$.

Let us denote $(sr_1^n - sr_2^n)^2$ by $l_n(x)$. So, we obtain the following equation.

$$(x^2+4)F_{2n}^{(2)}(x) + 4s^2(-1)^n = l_{2n}^2(x).$$ □

Theorem 6. We have the following equations

$$L_{2n}^{(2)}(x) = 4F_{2n+2}^{(2)}(x) - 2xF_{2n+1}^{(2)}(x) + x^2F_{2n}^{(2)}(x)$$

$$L_{2n+2}^{(2)}(x) = x^2F_{2n+2}^{(2)}(x) + 4x^2F_{2n+1}^{(2)}(x) + 4x^2F_{2n}^{(2)}(x).$$

Proof.

$$L_{2n}^{(2)}(x) = l_n^2(x)$$
$$F_{2n+2}^{(2)}(x) = f_{n+1}^2(x)$$
$$F_{2n+1}^{(2)}(x) = f_n(x) f_{n+1}(x)$$
$$F_{2n}^{(2)}(x) = f_n^2(x)$$

The proof is easily seen using the above equations. □

For *n*, we obtain an interesting relation between the known generalized Lucas polynomials and the generalized *k*-Lucas polynomials

$$L_{nk+t}^{(k)}(x) = L_n^{k-t}(x) L_{n+1}^t(x),$$

where $t = 0, 1, \ldots, k-1$.

3 Concluding comments

The relations among the generalized Lucas sequences make these neat relations seem obvious since the first column of this table consists of the factorial numbers (Sloane A000142), but they do take the interested reader further afield [18]. For instance, by inserting alternating positive and negative signs into this series one gets a divergent series which can lead into Borel summation techniques [2].

Suggestions for further study follow from [17] where an alternative, but not unrelated, approach defined a generalized Fibonacci polynomial by means of

$$u_n(x) = \sum_{k=0}^{n} u_{n-k} \frac{n!}{k!} x^k , \tag{5}$$

with P_0 (see below) set as unity for notational convenience, and $\{u_n\}$ and $\{v_n\}$ are generalized arbitrary order *r* number sequences defined formally by

$$\begin{aligned} u_n &= \sum_{j=1}^{r} (-1)^{j+1} P_j u_{n-j}, && n > 0, \\ u_n &= 1, && n = 0, \\ u_n &= 0, && n < 0, \end{aligned}$$

and

$$\begin{aligned} v_n &= \sum_{j=1}^{r} (-1)^{j+1} P_j v_{n-j}, && n \geq 0, \\ v_n &= \sum_{j=1}^{r} \alpha_j^n, && 0 \leq n < r, \\ v_n &= 0, && n < 0, \end{aligned}$$

where the P_j are arbitrary integers and α_j are the roots, assumed distinct of the associated auxiliary equation

$$0 = x^r - \sum_{j=1}^{r} (-1)^{j+1} P_j x^{r-j},$$

which is associated with the homogeneous arbitrary order linear recurrence relations which the generalized Fibonacci numbers satisfy. For example, when $r = 2$, we have

$$u_n = P_1 u_{n-1} - P_2 u_{n-2}$$

or $\{u_n\} \equiv \{(1, P_1; P_1, P_2)\}$ in Horadam's notation [8]. These are the Lucas fundamental numbers [4] and the $\{v_n\} \equiv \{(2, P_1; P_1, P_2)\}$ correspond to the Lucas primordial numbers which can be readily shown to satisfy

$$v_n = \sum_{j=1}^{r} \alpha_j^n, \text{ for all } n \geq 0.$$

When $P_1 = -P_2 = 1$, we get the Fibonacci numbers $\{u_n\} \equiv \{F_{n+1}\}$ and the ordinary Lucas numbers $\{u_n\} \equiv \{L_n\}$, the principal properties of which can be found in Hoggatt [6]. It was found that the first few generalized Fibonacci polynomial examples of (5) are

$$\begin{aligned} u_0(x) &= 1 \\ &= u_0, \end{aligned}$$

$$\begin{aligned} u_1(x) &= x + P_1 \\ &= u_0 x + u_1, \end{aligned}$$

$$\begin{aligned} u_2(x) &= x^2 + 2P_1 x + 2(P_1^2 - P_2) \\ &= u_0 x^2 + 2u_1 x + 2u_2, \end{aligned}$$

$$\begin{aligned} u_3(x) &= x^3 + 3P_1 x^2 + 6(P_1^2 - P_2)x + 6(P_1^3 - 2P_1 P_2 + P_3) \\ &= u_0 x^3 + 3u_1 x^2 + 6u_2 x + 6u_3, \end{aligned}$$

which can be compared with the corresponding cases of $f_n(x)$ in Definition 1 above to find corresponding generalizations to relate $u_n(x)$ and $f_n(x)$ and their properties, since it is proved in [16] that any polynomial can be expressed in terms of the Fibonacci polynomial in (5).

We give some values for the generalized *k*-Lucas polynomials in Table 2, from which Tables 3 and 4 are drawn in turn.

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	1							
1	1	1						
2	1	2	2					
3	1	3	6	6				
4	1	4	12	24	24			
5	1	5	20	60	120	120		
6	1	6	30	120	360	720	720	
7	1	7	42	210	840	2520	5040	5040

Table 2. Polynomial Coefficients [Sloane [7] A122851]

Sequences	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**Sloane** [19]
Diagonal	1	1	2	3	6	11	24	51	A122852 [15]
Row	1	2	5	16	65	326	1957	13700	A000522 [8]

Table 3. Diagonal and row sequences

↓ Partial column									**Sloane**
0	1	2	3	4	5	6	7	8	0! x A000027
1	1	3	6	10	15	21	28		1! x A230364
2	2	8	20	40	70	112			2! x A000292
3	6	30	90	210	420				3! x A000332
4	24	144	504	1344					4! x A000389

Table 4. Partial column sequences

From there one can develop properties corresponding to those in present paper, where we have defined new families of generalized k-Fibonacci polynomials and generalized k-Lucas polynomials, and proved the corresponding Cassini Identities for these families.

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