Generalized Lucas numbers of the form $3 \times 2^m$

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Abstract: For an integer $k \geq 2$, let $(L_n^{(k)})_n$ be the $k$-generalized Lucas sequence which starts with $0, \ldots, 0, 2, 1$ ($k$ terms) and each term afterwards is the sum of the $k$ preceding terms. In this paper, we look at the $k$-generalized Lucas numbers of the form $3 \times 2^m$ i.e. we study the Diophantine equation $L_n^{(k)} = 3 \times 2^m$ in positive integers $n, k, m$ with $k \geq 2$.

Keywords: $k$-generalized Lucas numbers, Linear form in logarithms, Reduction method.

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1 Introduction

Let $k \geq 2$ be an integer. We consider a generalization of Lucas sequence called the $k$-generalized Lucas sequence $L_n^{(k)}$ defined as

$$L_n^{(k)} = L_{n-1}^{(k)} + L_{n-2}^{(k)} + \cdots + L_{n-k}^{(k)} \quad \text{for all } n \geq 2,$$

with the initial conditions $L_{-k}^{(k)} = L_{-k-1}^{(k)} = \cdots = L_{-1}^{(k)} = 0$, $L_0^{(k)} = 2$ and $L_1^{(k)} = 1$. If $k = 2$, we obtain the classical Lucas sequence.
\[ L_0 = 2, \quad L_1 = 1, \quad \text{and} \quad L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad n \geq 2. \]
\[
(L_n)_{n \geq 0} = \{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, \ldots \}. \]

If \( k = 3 \), then the 3-Lucas sequence is
\[
(L^{(3)}_n)_{n \geq -1} = \{0, 2, 1, 3, 6, 10, 19, 35, 64, 118, 217, 399, 734, 1350, 2483, 4567, \ldots \}. \]

If \( k = 4 \), then the 4-Lucas sequence is
\[
(L^{(4)}_n)_{n \geq -2} = \{0, 0, 2, 1, 3, 6, 12, 22, 43, 83, 160, 308, 594, 1145, 2207, 4254, 8200, \ldots \}. \]

It is known that if \( 2 \leq n \leq k \), then
\[
L^{(k)}_n = 3 \times 2^{n-2}, \tag{2}\]
see Lemma 2 in [5]. This raises the following natural question: are there any positive integers \( n, m, k \) such that
\[
L^{(k)}_n = 3 \times 2^m? \tag{3}\]

The aim of this paper is to give an answer to this problem by proving the following result.

**Theorem 1.1.** The Diophantine equation (3) has no solution if \( n \geq k + 1 \).

Our proof of Theorem 1.1 is mainly based on linear forms in logarithms of algebraic numbers and a reduction algorithm originally introduced by Baker and Davenport [2]. Here, we use a version due to Dujella and Pethő in [6, Lemma 5(a)].

### 2 The tools

#### 2.1 Linear forms in logarithms

For any non-zero algebraic number \( \eta \) of degree \( d \) over \( \mathbb{Q} \), whose minimal polynomial over \( \mathbb{Z} \) is \( a \prod_{j=1}^d (X - \eta^{(j)}) \), we denote by
\[
h(\eta) = \frac{1}{d} \left( \log |a| + \sum_{j=1}^d \log \max \left(1, |\eta^{(j)}|\right) \right)
\]
the usual absolute logarithmic height of \( \eta \). In particular, if \( \eta = p/q \) is a rational number with \( \gcd(p, q) = 1 \) and \( q > 0 \), then \( h(\eta) = \log \max\{|p|, q\} \). The following properties of the function absolute logarithmic height \( h() \), which will be used in the next sections without special reference, are also known:
\[
\begin{align*}
    h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \tag{4} \\
    h(\eta^{\gamma \pm 1}) &\leq h(\eta) + h(\gamma), \tag{5} \\
    h(\eta^s) &\leq |s|h(\eta) \quad (s \in \mathbb{Z}). \tag{6}
\end{align*}
\]

With this notation, Matveev proved the following theorem (see [7]).
Theorem 2.1. Let \( \eta_1, \ldots, \eta_s \) be real algebraic numbers and let \( b_1, \ldots, b_s \) be nonzero rational integer numbers. Let \( d_K \) be the degree of the number field \( \mathbb{Q}(\eta_1, \ldots, \eta_s) \) over \( \mathbb{Q} \) and let \( A_j \) be a positive real number satisfying

\[
A_j = \max\{d_K h(\eta), |\log \eta|, 0.16\} \quad \text{for} \quad j = 1, \ldots, s.
\]

Assume that \( B \geq \max\{|b_1|, \ldots, |b_s|\} \). If \( \eta_1^{b_1} \cdots \eta_s^{b_s} - 1 \neq 0 \), then

\[
|\eta_1^{b_1} \cdots \eta_s^{b_s} - 1| \geq \exp(-1.3 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_K^2 (1 + \log d_K) (1 + \log B) A_1 \cdots A_s).
\]

2.2 Reduction algorithm

The following lemma can be found in [1].

Lemma 2.2. Let \( M \) be a positive integer, \( p/q \) be a convergent of the continued fraction of the irrational \( \gamma \) such that \( q > 6M \), and let \( A, C, \mu \) be some real numbers with \( A > 0 \) and \( C > 1 \). Let

\[
\varepsilon = ||\mu q|| - M \cdot ||\gamma q||,
\]

where \( || \cdot || \) denotes the distance from the nearest integer. If \( \varepsilon > 0 \), then there is no solution of the inequality

\[
0 < u\gamma - v + \mu < AC^{-w}
\]

in positive integers \( u, v \) and \( w \) with

\[
u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log C}.
\]

2.3 Properties of the \( k \)-generalized Lucas sequence

In this subsection, we recall some facts and properties of these sequences which will be used later.

We know that the characteristic polynomial of the \( k \)-generalized Lucas numbers \( (L_n^{(k)})_n \), namely

\[
\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,
\]

is irreducible over \( \mathbb{Q}[x] \) and has just one root outside the unit circle; the other roots are strictly inside the unit circle (see, for example, [8, 9, 12]). We denote by \( \alpha := \alpha(k) \) the single root, which is located between \( 2(1 - 2^{-k}) \) and \( 2 \) (see [12]). We label its roots by \( \alpha_1, \ldots, \alpha_k \) with \( \alpha := \alpha_1 \). To simplify the notation, in general, we omit the dependence on \( k \) of \( \alpha \).

For an integer \( s \geq 2 \), we define the function

\[
f_s(x) = \frac{x - 1}{2 + (s + 1)(x - 2)}.
\]

Now, we are ready to recall in the following lemmas some properties of the sequence \( (L_n^{(k)})_{n \geq -(k-2)} \), which will be used for the proof of Theorem 1.1.

Lemma 2.3. [5, p. 144]

(a) For all \( n \geq 1 \) and \( k \geq 2 \), we have

\[
\alpha^{n-1} \leq L_n^{(k)} \leq 2\alpha^n.
\]
The following "Binet-like" formula holds for all \( n \geq -(k-2) \):

\[
L_n^{(k)} = \sum_{i=1}^{k} (2\alpha_i - 1) f_k(\alpha_i)\alpha_i^{n-1}.
\]  
(9)

For all \( n \geq -(k-2) \), we have

\[
|L_n^{(k)} - (2\alpha - 1)f_k(\alpha)\alpha^{n-1}| < \frac{3}{2}.
\]  
(10)

**Lemma 2.4.** [11] For every positive integer \( n \geq 2 \), we have

\[
L_n^{(k)} \leq 3 \cdot 2^{n-2}.
\]  
(11)

Moreover, if \( n \geq k + 2 \), then the above inequality is strict.

**Lemma 2.5.** [4, pp. 89–90] For \( k \geq 2 \), let \( \alpha \) be the dominant root of \( \Psi_k(x) \), and consider the function \( f_s(x) \) defined in (7). Then:

(i) The inequalities \( 1/2 < f_k(\alpha) < 3/4 \) and \( |f_k(\alpha^{(i)})| < 1 \), for \( 2 \leq i \leq k \) hold. So the number \( f_k(\alpha) \) is not an algebraic integer.

(ii) The logarithmic height function satisfies \( h(f_k(\alpha)) < 3 \log k \).

## 3 The proof of the main result

### 3.1 An inequality for \( n \) and \( m \) in terms of \( k \)

From now on, we assume that \( n \geq k + 1 \). By Lemma 2.4 and Equation (3) we get

\[
3 \cdot 2^m = L_n^{(k)} \leq 3 \cdot 2^{n-2},
\]
so we deduce that \( m < n - 1 \). Thus, we may suppose that \( n \geq 3 \) and \( m \geq 2 \).

Now, we prove the following lemma.

**Lemma 3.1.** If \( (n,k,m) \) is a nontrivial solution in integers of Equation (3) with \( k \geq 2 \) and \( n \geq k + 1 \), then the inequalities

\[
m \leq n < 4 \times 10^{12}k^4 \log^3 k
\]  
(12)

hold.

**Proof.** Combining (3) with (10), one gets

\[
|3 \times 2^m - (2\alpha - 1)f_k(\alpha)\alpha^{n-1}| < \frac{3}{2}.
\]  
(13)

Notice that \( \alpha > 1 \), \( 2^k > k + 1 \) and \( 2^k > (k + 1)(2 - (2 - 2^{-k+1})) > (k + 1)(2 - \alpha) \).

Thus, \( (2\alpha - 1)f_k(\alpha)\alpha^{n-1} \) is positive. Now, we divide both sides of the above inequality by \( (2\alpha - 1)f_k(\alpha)\alpha^{n-1} \) to obtain the following inequality

\[
|3 \cdot 2^m \cdot \alpha^{-(n-1)} \cdot ((2\alpha - 1)f_k(\alpha))^{-1} - 1| < \frac{3}{\alpha^{n-1}},
\]  
(14)

where we used the facts \( 2 + (k + 1)(\alpha - 2) < 2 \) and \( 1/(2\alpha - 1) < 1/2 \).
In order to prove inequalities (12), we will apply Theorem 2.1. To this end, we take
\[ t := 3, \quad \eta_1 := 2, \quad \eta_2 := \alpha, \quad \eta_3 := 3((2\alpha - 1)f_k(\alpha))^{-1}, \]
and
\[ b_1 := m, \quad b_2 := -(n - 1), \quad b_3 := 1. \]

We put
\[ \Lambda := 3 \cdot 2^m \cdot \alpha^{(n-1)} \cdot ((2\alpha - 1)f_k(\alpha))^{-1} - 1 \]
and inequality (14) becomes
\[ |\Lambda| < 3 \alpha n - 1. \]  
(15)

For these choices, the field \( K := \mathbb{Q}(\alpha) \) contains \( \eta_1, \eta_2, \eta_3 \) and has \( d_K = k \). Since \( h(\eta_1) = \log 2 \) and \( h(\eta_2) = (\log \alpha)/k < (\log 2)/k \), it follows that
\[ \max\{kh(\eta_1), |\log \eta_1|, 0.16\} = k \log 2 := A_1 \]
and
\[ \max\{kh(\eta_2), |\log \eta_2|, 0.16\} = \log 2 := A_2. \]

On the other hand, we use the estimate (ii) of Lemma 2.5 and the properties (5), (6) to deduce that for all \( k \geq 2 \)
\[ h(\eta_3) \leq h(2\alpha - 1) + h(f_k(\alpha)) + h(3) < \log 3 + 3 \log k + \log 3 < 7 \log k, \]
so we get
\[ \max\{kh(\eta_3), |\log \eta_3|, 0.16\} = 7k \log k := A_3. \]

As \( m < n - 1 \), we can take \( B := n - 1 \).

Before applying Theorem 2.1, it remains us to prove that \( \Lambda \neq 0 \). Assume the contrary, i.e., \( \Lambda = 0 \), this imply that
\[ 3 \cdot 2^m = \frac{(2\alpha - 1)(\alpha - 1)}{2 + (k + 1)(\alpha - 2)} \alpha^{n-1}. \]
If we conjugate the above relation by the automorphism of Galois \( \sigma : \alpha \rightarrow \alpha_i \) \((i > 1)\) and then taking absolute values, we get
\[ 3 \cdot 2^m = \left| \frac{(2\alpha_i - 1)(\alpha_i - 1)}{2 + (k + 1)(\alpha_i - 2)} \alpha_i^{n-1} \right|. \]
But the above relation is not possible since its left-hand side is greater than or equal to 12, while its right-hand side is smaller than \( 6/(k - 1) < 8 \) as \( |\alpha_i| < 1 \) and
\[ |2 + (k + 1)(\alpha_i - 2)| \geq (k + 1) |\alpha_i - 2| - 2 > k - 1. \]
Thus, \( \Lambda \neq 0 \). Therefore, applying Theorem 2.1 to get a lower bound for \( |\Lambda| \) and comparing this with inequality (16), we get
\[ (n - 1) \log \alpha - \log 3 < 4.82 \times 10^{14} k^4 \log k(1 + \log k)(1 + \log(n - 1)). \]
Since \( 1 + \log k < 2 \log k, 1 + \log(n - 1) < 2 \log(n - 1) \) and \( 1/\log \alpha < 2 \) for \( k \geq 3 \) and \( n \geq 4 \), we conclude that
\[ \frac{n - 1}{\log(n - 1)} < 4 \times 10^{12} k^4 \log^2 k. \]  
(17)
We know that the function \( x \mapsto x / \log x \) is increasing for all \( x > e \), so it is easy to check that the inequality \( x / \log x < A \) implies \( x < 2A \log A \), whenever \( A \geq 3 \).

Thus, taking \( A := 4 \times 10^{12} k^4 \log^2 k \), inequality (17) and as \( 30 + 4 \log k + 2 \log \log k \) for all \( k \geq 3 \), we get

\[
n - 1 < 2(4 \times 10^{12} k^4 \log^2 k) \log(4 \times 10^{12} k^4 \log^2 k) < (8 \times 10^{12} k^4 \log^2 k)(30 + 4 \log k + 2 \log \log k) < 4 \times 10^{14} k^4 \log^3 k.\]

\[
3.2 \text{ The case } 2 \leq k \leq 170
\]

In this subsection, we study the cases when \( k \in [2, 170] \). We prove the following lemma.

**Lemma 3.2.** The Diophantine equation (3) has no solution when \( k \in [2, 170] \) and \( n \geq k + 1 \).

**Proof.** Let

\[
\Gamma = m \log 2 - (n - 1) \log \alpha - \log \left( \frac{(2\alpha - 1)f_k(\alpha)/3}{3} \right).
\]

Then \( e^{\Gamma} - 1 = \Lambda \), where \( \Lambda \) is defined by (15). Therefore, (16) can be rewritten as

\[
|e^{\Gamma} - 1| < \frac{3}{\alpha^{n-1}}.
\]

Notice that \( \Gamma \neq 0 \) since \( \Lambda \neq 0 \), so we distinguish the following two cases.

- First, if \( \Gamma > 0 \), then \( e^{\Gamma} - 1 > 0 \). Using the fact that \( x \leq e^x - 1 \) for all \( x \in \mathbb{R} \), inequality (19) gives

\[
0 < \Gamma < \frac{3}{\alpha^{n-1}}.
\]

Replacing \( \Gamma \) in the above inequality by its formula (18), dividing both sides of the resulting inequality by \( \log \alpha \) and using the fact that \( 1 / \log \alpha < 2 \) for all \( k \geq 2 \), we get

\[
0 < m \left( \frac{\log 2}{\log \alpha} \right) - n + \left(1 - \frac{\log((2\alpha - 1)f_k(\alpha)/3)}{\log \alpha} \right) < 6 \cdot \alpha^{-(n-1)}.
\]

Putting

\[
\gamma := \frac{\log 2}{\log \alpha}, \quad \mu := 1 - \frac{\log((2\alpha - 1)f_k(\alpha)/3)}{\log \alpha}, \quad A := 6, \quad \text{and} \quad C := \alpha,
\]

the above inequality (20) yields

\[
0 < m \gamma - n + \mu < AC^{-(n-1)}.
\]

It is clear that \( \gamma \) is an irrational number because \( \alpha > 1 \) is a unit in \( O_K \), the ring of integers of \( K \). So \( \alpha \) and 2 are multiplicatively independent.

For each \( k \in [2, 170] \), we find a good approximation of \( \alpha \) and a convergent \( p_\ell/q_\ell \) of the continued fraction of \( \gamma \) such that \( q_\ell > 6M \), where \( M = [4 \times 10^{14} k^4 \log^3 k] \), which is an upper bound on \( m \) from Lemma 12. After doing this, we use Lemma 2.2 to Inequality (20). A computer search with Mathematica revealed that the maximum value of \( \left[ \log(Aq/\varepsilon)/\log C \right] \) over all \( k \in [2, 170] \) is 175, which according to Lemma 2.2, is an upper bound on \( n - 1 \). Hence, we deduce that the possible solutions \((n, k, m)\) of Equation (3) for which \( k \in [2, 170] \) and \( \Gamma > 0 \) have \( n \leq 176 \), therefore \( m \leq 175 \), since \( m < n \).
• Now, we consider the case $\Gamma < 0$. It is easy to see that $2/\alpha^{n-1} < 1/2$ holds for all $k \geq 2$ and $n \geq 3$. Thus, inequality (19) implies $|e^{\Gamma} - 1| < 1/2$ and therefore $e^{\Gamma} < 2$. As $\Gamma < 0$, we get $0 < |\Gamma| \leq e^{\Gamma} - 1 = e^{\Gamma} |e^{\Gamma} - 1| < \frac{6}{\alpha^{n-1}}$. Similarly, as the case when $\Gamma > 0$, we get

$$0 < (n-1)\gamma - m + \mu < AC^{-(n-1)},$$

where

$$\gamma := \frac{\log \alpha}{\log 2}, \quad \mu := \frac{(2\alpha - 1)f_k(\alpha)/3}{\log 2}, \quad A := 9, \quad C := \alpha.$$

In this case, we also took $M := [4 \times 10^{14}k^4 \log^3 k]$ which is an upper bound of $n - 1$ by Lemma 3.1, and we applied Lemma 2.2 to inequality (22). In this case, with the help of Mathematica, we found that the maximum value of $[\log(Aq/\varepsilon)/\log C]$ is 174. Thus, the possible solutions $(n, k, m)$ of Equation (3) in the range $k \in [2, 170]$ and $\Gamma < 0$ give $n \leq 175$, so $m \leq 174$.

Finally, using Mathematica we compared $L^{(k)}_n$ and $3 \times 2^m$ for the range $3 \leq n \leq 175$ and $2 \leq m \leq 174$, with $m < n$ and found that Equation (3) has no solution in this range. \hfill \Box

3.3 The case $k > 170$

In this subsection, we analyze the case $k > 170$.

**Lemma 3.3.** The Diophantine equation (3) has no solution when $k > 170$ and $n \geq k + 1$.

**Proof.** For $k > 170$, we have $n < 4 \times 10^{14}k^4 \log^3 k < 2^{k/2}$. In [5, p. 150], it was proved that

$$(2\alpha - 1)f_k(\alpha)\alpha^{n-1} = 3 \cdot 2^{n-2} + 3 \cdot 2^{n-1} \eta + \frac{\delta}{2} + \eta \delta,$$

where

$$|\eta| < \frac{2k}{2k} \quad \text{and} \quad |\delta| < \frac{2^{n+2}}{2^{k/2}}.$$

Thus, from the above equality and (13), we get

$$|3 \cdot 2^n - 3 \cdot 2^{n-2}| = \left| 3 \cdot 2^n - (2\alpha - 1)f_k(\alpha)\alpha^{n-1} + 3 \cdot 2^{n-1} \eta + \frac{\delta}{2} + \eta \delta \right|$$

$$< \frac{3}{2} + \frac{3k \cdot 2^n}{2k} + \frac{2^{n+1}}{2^{k/2}} + \frac{2^{n+3}}{2^{3k/2}}.$$

For $k > 170$, we get $4k/2^k < 1/2^{k/2}$, $8/(3 \cdot 2^{k/2}) < 3/2^{k/2}$ and $32k/(3 \cdot 2^{3k/2}) < 1/2^{k/2}$. Thus, we obtain

$$|3 \cdot 2^n - 3 \cdot 2^{n-2}| < 18 \cdot \frac{2^{n-2}}{2^{k/2}}.$$

As $n \geq k + 1$, we have $1/2^{n-1} < 1/2^{k/2}$ and factoring out $3 \cdot 2^{n-2}$ in the right-hand side of the above inequality, we get

$$|2^{m-n+2} - 1| < \frac{6}{2^{k/2}}.$$  

Moreover, since $m < n$, we have that $m - n + 2 \leq 1$, then it follows from (23) that

$$\frac{1}{2} < 1 - 2^{m-n+2} < \frac{6}{2^{k/2}}.$$
So, \(2^{k/2} < 12\), which is impossible as \(k > 170\). Hence, we have shown that there are no solutions \((n, k, m)\) to Equation (3) with \(k > 170\).

Thus, this completes the proof of Theorem 1.1.

4 Conclusion

In this paper, we prove that there are no positive integers \(m, n, k\) such that a \(k\)-generalized Lucas number has the form \(3 \times 2^m\) for \(n \geq k + 1\), i.e., the Diophantine equation \(L_n^{(k)} = 3 \times 2^m\) has no solution in positive integers \(n, k, m\) with \(k \geq 2\) and \(n \geq k + 1\).

References


