Regular polygons, Morgan-Voyce polynomials, and Chebyshev polynomials

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Abstract: We say that a monic polynomial with integer coefficients is a polynomial if its each zero is obtained by squaring the edge or a diagonal of a regular \( n \)-gon with unit circumradius. We find connections of certain polygomials with Morgan-Voyce polynomials and further with Chebyshev polynomials of second kind.

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1 Introduction

We call the edge and diagonals of a polygon by a common name chord. Let \( G_n \) be a regular \( n \)-gon with unit circumradius. Its chords (their lengths) are

\[ e_{nk} = 2 \sin \frac{k \pi}{n}, \quad k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor. \]

We say that a monic polynomial with integer coefficients is a polynomial if its all zeros are squared chords (not necessarily squares of all chords) of some \( G_n \).

This paper is a sequel to Mustonen et al. [7, Section 2] on the polygomials

\[ A_m(x) = \sum_{k=0}^m (-1)^{m-k} \binom{m+k+1}{2k+1} x^k = \prod_{k=1}^m \left[ x - 4 \sin^2 \frac{k \pi}{2(m+1)} \right] \quad (1) \]

(for the second equation, see [7, Theorem 1]) and
\[ \hat{A}_m(x) = x^m + \sum_{k=0}^{m-1} (-1)^{m-k} \frac{2m+1}{m-k} \left( \frac{m+k}{2k+1} \right) x^k = \prod_{k=1}^{m} \left[ x - 4 \sin^2 \frac{k\pi}{2m+1} \right] \]

(for the second equation, see [7, Theorem 2]). Define that the “empty sum” is zero and the “empty product” one; then \( A_0(x) = \hat{A}_0(x) = 1 \). In [7], \( B_m = \hat{A}_m \) and \( \hat{A}_m \) has another meaning.

**Example 1.1.** In particular,
\[
A_1(x) = x - 2, \quad A_2(x) = x^2 - 4x + 3, \quad A_3(x) = x^3 - 6x^2 + 10x - 4, \\
A_4(x) = x^4 - 8x^3 + 21x^2 - 20x + 5,
\]
\[
\hat{A}_1(x) = x - 3, \quad \hat{A}_2(x) = x^2 - 5x + 5, \quad \hat{A}_3(x) = x^3 - 7x^2 + 14x - 7, \\
\hat{A}_4(x) = x^4 - 9x^3 + 27x^2 - 30x + 9.
\]

The sequence \( (A_m) \) satisfies [7, Equation (6)] the recursion
\[
A_0(x) = 1, \quad A_1(x) = x - 2, \quad A_{m+1}(x) = (x - 2)A_m(x) - A_{m-1}(x),
\]
and \( (\hat{A}_m) \) satisfies [7, Equation (11)]
\[
\hat{A}_0(x) = 1, \quad \hat{A}_1(x) = x - 3, \quad \hat{A}_{m+1}(x) = (x - 3)A_m(x) - \hat{A}_{m-1}(x).
\]
We show that
\[
\hat{A}_0(x) = 1, \quad \hat{A}_1(x) = x - 3, \quad \hat{A}_{m+1}(x) = (x - 2)\hat{A}_m(x) - \hat{A}_{m-1}(x).
\]
Thus \( (\hat{A}_m) \) follows the same recursion formula as \( (A_m) \).

For all \( k \geq 2 \),
\[
\hat{A}_k(x) \equiv _3 (x - 3)A_{k-1}(x) - A_{k-2}(x) \quad \equiv _2 (x - 3)A_{k-1}(x) + A_k(x) - (x - 2)A_{k-1}(x) = A_k(x) - A_{k-1}(x).
\]

Therefore
\[
\hat{A}_{m+1}(x) - (x - 2)\hat{A}_m(x) + \hat{A}_{m-1}(x) \equiv _5 A_{m+1}(x) - A_m(x) - (x - 2)(A_m(x) - A_{m-1}(x)) + A_{m-1}(x) - A_{m-2}(x) \\
= A_{m+1}(x) - (x - 2)A_m(x) + A_{m-1}(x) - [A_m(x) - (x - 2)A_{m-1}(x) + A_{m-2}(x)] \\
\equiv _2 0 - 0 = 0,
\]
verifying the claim.

We are interested in connections of \( A_m \) and \( \hat{A}_m \) with well-known polynomials. We introduce in Sections 3 and 4 the Morgan-Voyce polynomials \( b_m \) and \( B_m \), and their generalizations \( B^{(r)}_m \). We see that \( A_m \) and \( \hat{A}_m \) are connected with \( B_m \) and \( B^{(r)}_m \), respectively. We also find a polynomial \( a_m \) connected with \( b_m \). In Section 5, recalling how \( b_m, B_m, \) and \( B^{(2)}_m \) reduce to the Chebyshev polynomials of second kind, we reduce also \( a_m, A_m, \) and \( \hat{A}_m \) to them. The motivation of \( b_m \) and \( B_m \) rises from a problem on a ladder network of resistances. We see in Section 6 that also \( a_m \) and \( A_m \) apply to this problem. Finally, we complete this paper with conclusions and remarks in Section 7.
2 Background

Let me first describe the background of this paper. Neeme Vaino, an amateur mathematician from Estonia, introduced [11] his “regular polynomials”

\[ R_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} r_{nk} x^{n-2k}, \]

whose coefficients are obtained from the OEIS [8] sequence A132460. Actually [5, Equation (1.5)] \( R_n \) is the Vieta–Lucas polynomial

\[ v_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}. \]

**Example 2.1.** In particular,

\[ v_1(x) = x, \quad v_2(x) = x^2 - 2, \quad v_3(x) = x^3 - 3x, \quad v_4(x) = x^4 - 4x^2 + 2, \]
\[ v_5(x) = x^5 - 5x^3 + 5x, \quad v_6(x) = x^6 - 6x^4 + 9x^2 - 2. \]

The polynomial \( \tilde{A}_m \) relates to \( v_{2m+1} \) via

\[ v_{2m+1}(x) = x \tilde{A}_m(x^2), \]

cf. [5, Theorem 3(b)].

**Example 2.2.** In particular,

\[ x \tilde{A}_2(x^2) = x(x^4 - 5x^2 + 5) = x^5 - 5x^3 + 5x = v_5(x). \]

The *Chebyshev polynomials of first kind* are defined by

\[ T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \]

**Example 2.3.** In particular,

\[ T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1, \]
\[ T_5(x) = 16x^5 - 20x^3 + 5x, \quad T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1. \]

The polynomial \( v_n \) relates to \( T_n \) [5, Equation (9.4)] via \( v_n(x) = 2T_n(x^2) \).

**Example 2.4.** In particular,

\[ 2T_4(x^2) = 2 \cdot \left[ 8 \left( \frac{x}{2} \right)^4 - 8 \left( \frac{x}{2} \right)^2 + 1 \right] = x^4 - 4x^2 + 2 = v_4(x). \]

The *Vieta–Fibonacci polynomials* are defined by

\[ V_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k}, \]

cf. [5, Equation (1.3)]. Here \( V_n \) denotes the same as \( V_{n+1} \) in [5], in order to make its degree equal to \( n \).
Example 2.5. In particular,
\[ V_1(x) = x, \quad V_2(x) = x^2 - 1, \quad V_3(x) = x^3 - 2x, \quad V_4(x) = x^4 - 3x^2 + 1, \]
\[ V_5(x) = x^5 - 4x^3 + 3x, \quad V_6(x) = x^6 - 5x^4 + 6x^2 - 1. \]

The polynomial \( A_m \) relates to \( V_{2m+1} \) via \( V_{2m+1}(x) = xA_m(x^2) \), cf. [5, Theorem 2(a)].

Example 2.6. In particular,
\[ xA_2(x^2) = x(x^4 - 4x^2 + 3) = x^5 - 4x^3 + 3x = V_5(x). \]

The Chebyshev polynomials of second kind are defined by
\[ U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x). \]

Example 2.7. In particular,
\[ U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x, \quad U_4(x) = 16x^4 - 12x^2 + 1, \]
\[ U_5(x) = 32x^5 - 32x^3 + 6x, \quad U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1. \]

The polynomial \( V_n \) relates to \( U_n \) via \( V_n(x) = U_n(\frac{x}{2}) \), cf. [5, Equation (9.3)]. Actually \( V_n(x) \) should read \( V_{n+1}(x) \) in this reference, in order to make (9.3) compatible with (1.3).

Example 2.8. In particular,
\[ U_4 \left( \frac{x}{2} \right) = 16 \left( \frac{x}{2} \right)^4 - 12 \left( \frac{x}{2} \right)^2 + 1 = x^4 - 3x^2 + 1 = V_4(x). \]

To summarize, \( A_m \) relates to \( V_{2m+1} \) and further to \( U_{2m+1} \), and \( \tilde{A}_m \) relates to \( v_{2m+1} \) and further to \( T_{2m+1} \). But we will see that \( A_m \) and \( \tilde{A}_m \) have also more direct relations to well-known polynomials.

3 Morgan-Voyce polynomials

Changing in \( A_m \) all minus signs into plus, we define
\[ B_m(x) = (-1)^mA_m(-x). \] (6)

This is one of the two Morgan-Voyce polynomials \( b_m \) and \( B_m \), usually defined by the recursion pair
\[ b_0(x) = B_0(x) = 1, \quad b_{m+1}(x) = xB_m(x) + b_m(x), \quad B_{m+1}(x) = (x + 1)B_m(x) + b_m(x). \] (7)

Example 3.1. In particular,
\[ b_1(x) = x + 1, \quad b_2(x) = x^2 + 3x + 1, \quad b_3(x) = x^3 + 5x^2 + 6x + 1, \]
\[ b_4(x) = x^4 + 7x^3 + 15x^2 + 10x + 1, \]
\[ B_1(x) = x + 2, \quad B_2(x) = x^2 + 4x + 3, \quad B_3(x) = x^3 + 6x^2 + 10x + 4, \]
\[ B_4(x) = x^4 + 8x^3 + 21x^2 + 20x + 5. \]
By (6) and (2),

\[ B_0(x) = 1, \quad B_1(x) = x + 2, \quad B_{m+1}(x) = (x + 2)B_m(x) - B_{m-1}(x). \]

This recursion is well-known [9, p. 73] as a consequence of (7). Likewise,

\[ b_0(x) = 1, \quad b_1(x) = x + 1, \quad b_{m+1}(x) = (x + 2)b_m(x) - b_{m-1}(x). \]

By (6) and (1),

\[ B_m(x) = \sum_{k=0}^{m} \left( \frac{m + k + 1}{2k + 1} \right) x^k = \prod_{k=1}^{m} \left[ x + 4 \sin^2 \frac{k\pi}{2(m+1)} \right]. \]

Regarding zeros, this is well-known [9, Equation (39)] (the first of the two equations with this number) and [10, Section 6]. Similarly, by [9, Equation (40)] and [10, Section 6] (containing a typo),

\[ b_m(x) = \sum_{k=0}^{m} \left( \frac{m + k}{2k} \right) x^k = \prod_{k=1}^{m} \left[ x + 4 \sin^2 \frac{(2k - 1)\pi}{2(2m+1)} \right]. \] (8)

### 4 Counterparts of \( \tilde{A}_m \) and \( b_m \)

Equation (6) connects \( A_m \) and \( B_m \) but does not connect \( \tilde{A}_m \) and \( b_m \). Instead, \( b_m \) is connected with

\[ a_m(x) = (-1)^m b_m(-x) = (-1)^m \prod_{k=1}^{m} \left[ x + 4 \sin^2 \frac{(2k - 1)\pi}{2(2m+1)} \right]. \] (9)

This is a polynomial, since its zeros are \( e^{2\pi i (m+2)} \), \( e^{2\pi i (m+2)} \), \( e^{2\pi i (m+2)} \), \ldots, \( e^{2\pi i (m+2m-1)} \).

**Example 4.1.** In particular,

\[ a_1(x) = x - 1, \quad a_2(x) = x^2 - 3x + 1, \quad a_3(x) = x^3 - 5x^2 + 6x - 1, \quad a_4(x) = x^4 - 7x^3 + 13x^2 - 10x + 1. \]

The Morgan-Voyce polynomials have been widely generalized, see [4] and its references. André-Jeannin [1] generalizes them by the recursion

\[ B_0^{(r)}(x) = 1, \quad B_1^{(r)}(x) = x + r + 1, \quad B_{m+1}^{(r)}(x) = (x + 2)B_m^{(r)}(x) - B_{m-1}^{(r)}(x), \]

where \( r \) is a given real number. In particular,

\[ B_0^{(0)} = b_m, \quad B_1^{(1)} = B_m, \]

and the polynomials

\[ \tilde{B}_m = B_m^{(2)} \]

satisfy the recursion

\[ \tilde{B}_0(x) = 1, \quad \tilde{B}_1(x) = x + 3, \quad \tilde{B}_{m+1}(x) = (x + 2)\tilde{B}_m(x) - \tilde{B}_{m-1}(x). \] (10)
By (4) and (10), it is easy to see that
\[ \tilde{B}_m(x) = (-1)^m \tilde{A}_m(-x). \]  

(11)

**Example 4.2.** In particular,
\[ \tilde{B}_1(x) = x + 3, \quad \tilde{B}_2(x) = x^2 + 5x + 5, \quad \tilde{B}_3(x) = x^3 + 7x^2 + 14x + 7, \quad \tilde{B}_4(x) = x^4 + 9x^3 + 27x^2 + 30x + 9. \]

5 Chebyshev polynomials of second kind

It can be shown [4, Equations (4.2–4)] that
\[ B_m(x) = U_m \left( \frac{x + 2}{2} \right), \]  

(12)

\[ b_m(x) = U_m \left( \frac{x + 2}{2} \right) - U_{m-1} \left( \frac{x + 2}{2} \right), \]  

(13)

\[ \tilde{B}_m(x) = U_m \left( \frac{x + 2}{2} \right) + U_{m-1} \left( \frac{x + 2}{2} \right). \]  

(14)

**Example 5.1.** In particular,
\[ U_1 \left( \frac{x + 2}{2} \right) = 2 \frac{x + 2}{2} = x + 2 = B_1(x), \]  
\[ U_2 \left( \frac{x + 2}{2} \right) = 4 \left( \frac{x + 2}{2} \right)^2 - 1 = x^2 + 4x + 3 = B_2(x), \]  
\[ U_2 \left( \frac{x + 2}{2} \right) - U_1 \left( \frac{x + 2}{2} \right) = x^2 + 3x + 1 = b_2(x), \]  
\[ U_2 \left( \frac{x + 2}{2} \right) + U_1 \left( \frac{x + 2}{2} \right) = x^2 + 5x + 5 = \tilde{B}_2(x). \]

It is easy to see that
\[ U_m(-x) = (-1)^m U_m(x). \]  

(15)

Now,
\[ A_m(x) \overset{(6)}{=} (-1)^m B_m(-x) \overset{(12)}{=} (-1)^m U_m \left( \frac{-x + 2}{2} \right) \overset{(15)}{=} (-1)^m U_m \left( \frac{x - 2}{2} \right) = U_m \left( \frac{x - 2}{2} \right), \]  
\[ a_m(x) \overset{(9)}{=} (-1)^m b_m(-x) \overset{(13)}{=} (-1)^m \left[ U_m \left( \frac{-x + 2}{2} \right) - U_{m-1} \left( \frac{-x + 2}{2} \right) \right] \overset{(15)}{=} (-1)^m \left[ (-1)^m U_m \left( \frac{x - 2}{2} \right) - (-1)^{m-1} U_{m-1} \left( \frac{x - 2}{2} \right) \right] = U_m \left( \frac{x - 2}{2} \right) + U_{m-1} \left( \frac{x - 2}{2} \right) \]  
and
\[ \tilde{A}_m(x) \overset{(11)}{=} (-1)^m \tilde{B}_m(-x) \overset{(14)}{=} (-1)^m \left[ U_m \left( \frac{-x + 2}{2} \right) + U_{m-1} \left( \frac{-x + 2}{2} \right) \right] \overset{(15)}{=} (-1)^m \left[ (-1)^m U_m \left( \frac{x - 2}{2} \right) + (-1)^{m-1} U_{m-1} \left( \frac{x - 2}{2} \right) \right] = U_m \left( \frac{x - 2}{2} \right) - U_{m-1} \left( \frac{x - 2}{2} \right). \]
Example 5.2. In particular,

\[ U_1 \left( \frac{x - 2}{2} \right) = 2 \frac{x - 2}{2} = x - 2 = A_1(x), \]

\[ U_2 \left( \frac{x - 2}{2} \right) = 4 \left( \frac{x - 2}{2} \right)^2 - 1 = x^2 - 4x + 3 = A_2(x), \]

\[ U_2 \left( \frac{x - 2}{2} \right) + U_1 \left( \frac{x - 2}{2} \right) = x^2 - 3x + 1 = a_2(x), \]

\[ U_2 \left( \frac{x - 2}{2} \right) - U_1 \left( \frac{x - 2}{2} \right) = x^2 - 5x + 5 = \tilde{A}_2(x). \]

6 Revisiting the ladder network

We show that \( a_m \) and \( A_m \) apply to the same ladder network problem [2, 6] as \( b_m \) and \( B_m \). To begin, we present a recursion pair for \( a_m \) and \( A_m \). Since

\[ a_{m+1}(x) \overset{(9)}{=} (-1)^{m+1} b_{m+1}(-x) \overset{(7)}{=} (-1)^{m+1} \left[ (-x)B_m(-x) + b_m(-x) \right] \]

\[ = x(-1)^m B_m(-x) - (-1)^m b_m(-x) \overset{(6)}{=} xA_m(x) - a_m(x) \]

and

\[ A_{m+1}(x) \overset{(6)}{=} (-1)^{m+1} B_{m+1}(-x) \overset{(7)}{=} (-1)^{m+1} \left[ (-x + 1)B_m(-x) + b_m(-x) \right] \]

\[ = (x - 1)(-1)^m B_m(-x) - (-1)^m b_m(-x) \overset{(6),(9)}{=} (x - 1)A_m(x) - a_m(x), \]

we have

\[ a_0(x) = A_0(x) = 1, \quad a_{m+1}(x) = xA_m(x) - a_m(x), \quad A_{m+1}(x) = (x - 1)A_m(x) - a_m(x). \] (16)

We use the figures and notations of Hoggatt and Bicknell [2, Section 1]. Instead of \( x \), we let \(-x\) denote the resistance of each component in the upper sidepiece of the ladder. It is reasonable to require that \(-x > 0\), i.e., \( x < 0 \). However, it is not complete nonsense to accept also nonpositive resistances, because we may think that the voltage across these components can be increased externally. Anyway, whether or not to accept nonpositive resistances, it does not effect on the following calculations.

We proceed as in [2, p. 148] but write \( R(x) = R \) and \( Z_m(x) = Z_n \). Then

\[ R(x) = Z_m(x) - x, \]

\[ \frac{1}{Z_{m+1}(x)} = \frac{1}{Z_m(x) - x} + 1 = \frac{Z_m(x) - x + 1}{Z_m(x) - x}, \]

and

\[ Z_{m+1}(x) = \frac{Z_m(x) - x}{Z_m(x) - x + 1}. \] (17)

We show that

\[ Z_m(x) = \frac{a_m(x)}{A_m(x)} \] (18)

satisfies (17). Since
\[ Z_{m+1}(x) = \frac{a_{m+1}(x)}{A_{m+1}(x)} = \frac{x A_m(x) - a_m(x)}{(x - 1) A_m(x) - a_m(x)} \]

\[ = \frac{a_m(x) - x A_m(x)}{a_m(x) - (x - 1) A_m(x)} = \frac{\frac{a_m(x)}{A_m(x)} - x}{\frac{a_m(x)}{A_m(x)} - x + 1} \]

\[ Z_m(x) - x = \frac{Z_m(x) - x}{x} \]

the claim follows.

7 Conclusions and remarks

The polynomial \( A_m \) is connected with \( B_m \) via the equation (6). The Morgan-Voyce polynomial \( b_m \) defines by (9) the polynomial \( a_m \). The polynomial \( \tilde{A}_m \) has the connection (11) with the generalized Morgan-Voyce polynomial \( \tilde{B}_m = B_m^{(2)} \). Since these Morgan-Voyce polynomials reduce to Chebyshev polynomials of second kind via (12), (13), and (14), also the above-mentioned polynomials reduce to them.

More generally, we define

\[ A_m^{(r)}(x) = (-1)^m B_m^{(r)}(-x). \]

In particular,

\[ A_m^{(0)} = a_m, \quad A_m^{(1)} = A_m, \quad A_m^{(2)} = \tilde{A}_m. \]

If \( r \in \{0, 1, 2\} \), then \( A_m^{(r)} \) is a polynomial. Is it a polynomial also for some other appropriate values of \( r \)? To answer, we should find the zeros of \( B_m^{(r)} \). I did not find them from the literature. According to Horadam [3, p. 348], André-Jeannin [1] has given them, but actually he [1, p. 231] considered only the cases \( r = 0, 1, 2 \).

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References


