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A formula for the number of non-negative integer solutions of $a_1x_1 + a_2x_2 + \cdots + a_mx_m = n$ in terms of the partial Bell polynomials

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Abstract: We derive a formula for the number of non-negative integer solutions of the equation $a_1x_1 + a_2x_2 + \cdots + a_mx_m = n$ in terms of the partial Bell polynomials via the Faà di Bruno's formula.

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1 Introduction

Let $a_1, a_2, \ldots, a_m$ be positive integers, and let the number of non-negative integer solutions of the linear Diophantine equation

$$a_1x_1 + a_2x_2 + \cdots + a_mx_m = n$$

be denoted by $N(a_1, a_2, \ldots, a_m; n)$. For the special case when $a_1 = a_2 = \ldots = a_m = 1$, it has been proven in [1] that

$$N(\underbrace{1, \ldots, 1}_{m}; n) = \binom{n+m-1}{m-1}.$$

When $m = 2$, then there exist explicit formulas for $N(a_1, a_2; n)$, see the paper [2]. The case when $m = 3$ has been studied by Binner [3]. Komatsu [1] has studied the general case.

We use the Faà di Bruno's formula to give the expression for $N(a_1, a_2, \ldots, a_m; n)$ in terms of the partial Bell polynomials.

Theorem 1.1. *Let* $g(x) = (1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_m}) = \sum_{s=0}^{a_1+a_2+\cdots+a_m} \theta_s x^s$. *Then we have*

$$N(a_1, a_2, \ldots, a_m; n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^k \, k! \, B_{n,k}(1! \, \theta_1, 2! \, \theta_2, \ldots, (n-k+1)! \, \theta_{n-k+1}). \qquad (1)$$

where $B_{n,k} \equiv \mathrm{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ *is the* (n,k)*-th partial exponential Bell polynomial in the variables* $x_1, x_2, \ldots, x_{n-k+1}$ *which can be computed using the recurrence* [5, p. 415]:

$$B_{n,k} = \sum_{i=1}^{n-k+1} \binom{n-1}{i-1} x_i B_{n-i,k-1}, \qquad (2)$$

where

$$\begin{aligned} B_{0,0} &= 1 \\ B_{n,0} &= 0 \ \text{ for } n \geq 1; \\ B_{0,k} &= 0 \ \text{ for } k \geq 1. \end{aligned}$$

Before proving the above result, we relate the above result to weighted integer compositions.

Relation to weighted integer compositions

Let n be a non-negative integer. Then a k-tuple of non-negative integers $(\pi_1, \pi_2, \ldots, \pi_k)$ is said to be an *integer composition* of n if $\pi_1 + \pi_2 + \cdots + \pi_k = n$. The numbers π_i's are called *parts*.

Let $f : \mathbb{N} \to \mathbb{R}$ be an arbitrary function. For each possible *part size* $s \in \mathbb{N} = \{0, 1, 2, \ldots, \}$, let $f(s)$ be the *weight* of part size s. Let $\binom{k}{n}_f$ denote the *total weight* of all f-weighted integer compositions of n with k parts, that is,

$$\binom{k}{n}_f = \sum_{\pi_1+\pi_2+\ldots+\pi_k=n} f(\pi_1) \, f(\pi_2) \cdots f(\pi_k).$$

Then, interpreting $f(s)$ ($s \in \mathbb{N}$) as indeterminates, Eger [6] proved that

$$\frac{k!}{n!} \, B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \binom{k}{n}_f,$$

where

$$\begin{aligned} f(0) &= f(n-k+2) = f(n-k+3) = \ldots = 0, \\ f(s) &= \frac{x_s}{s!}, \quad \text{for } s \in \{1, 2, \ldots, n-k+1\}. \end{aligned}$$

Our main result translates to the following result:

Theorem 1.2. *We have*

$$N(a_1, a_2, \ldots, a_n; n) = \sum_{k=0}^{n} (-1)^k \binom{k}{n}_f ,$$

where

$$f(0) = f(n-k+2) = f(n-k+3) = \ldots = 0,$$
$$f(s) = \theta_s, \quad \text{for } s \in \{1, 2, \ldots, n-k+1\},$$

where θ_s is as defined in the statement of Theorem 1.1.

2 Proof of Theorem 1.1

Proof. In [1] it has been proved that the number of non-negative integer solutions of $a_1x_1 + a_2x_2 + \cdots + a_mx_m = n$ is equal to the coefficient of x^n in

$$\frac{1}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_m})}.$$

Let $f(x) = 1/x$ and $g(x) = (1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_m})$. Using Faà di Bruno's formula [4, p. 137] we have

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x)) \cdot B_{n,k}\left(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)\right). \tag{3}$$

Since $f^{(k)}(x) = \frac{(-1)^k\, k!}{x^{k+1}}$ and $g(0) = 1$, letting $x \to 0$ in the above equation gives

$$N(a_1, a_2, \ldots, a_m; n)\, n! = \sum_{k=1}^{n} (-1)^k\, k!\, B_{n,k}\left(g'(0), g''(0), \ldots, g^{(n-k+1)}(0)\right),$$

where $g^{(l)}(0) = \theta_l\, l!$ by the Maclaurin series expansion of $g(x)$ in Theorem 1.1. □

Example 2.1. We use our formula to calculate the number of non-negative integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 4$. Our formula should give us the result 35, of course, because we can apply the formula for $N(1, 1, 1, 1; 4)$, which is a simple binomial coefficient which is "seven over three" and that is 35.

Theorem 1.1 gives us

$$N(1, 1, 1, 1; 4) = \frac{1}{4!} \sum_{k=1}^{4} (-1)^k\, k!\, B_{4,k}(1!\,\theta_1, 2!\,\theta_2, \ldots, (4-k+1)!\,\theta_{4-k+1}),$$

where $\theta_1 = -4$, $\theta_2 = 6$, $\theta_3 = -4$ and $\theta_4 = 1$ since

$$g(x) = (1-x)^4 = 1 - 4x + 6\,x^2 - 4\,x^3 + x^4.$$

We use the following values to find the correct answer:

$$B_{4,1}(x_1, x_2, x_3, x_4) = x_4,$$
$$B_{4,2}(x_1, x_2, x_3) = 4\,x_1\,x_3 + 3\,x_2^2,$$
$$B_{4,3}(x_1, x_2) = 6\,x_1^2\,x_2,$$
$$B_{4,4}(x_1) = x_1^4,$$

which can be computed using the recurrence (2) (see [7] for a Python library for symbolic mathematics where the partial Bell polynomials are implemented).

The following calculation gives us the required answer:

$$\begin{aligned}N(1,1,1,1;4) &= \frac{1}{4!}(-1!\,B_{4,1}(1!\,\theta_1, 2!\,\theta_2, 3!\,\theta_3, 4!\,\theta_4) + 2!\,B_{4,2}(1!\,\theta_1, 2!\,\theta_2, 3!\,\theta_3)\\ &\quad - 3!\,B_{4,3}(1!\,\theta_1, 2!) + 4!\,B_{4,4}(1!\,\theta_1))\\ &= \frac{1}{4!}(-\,4!\,\theta_4 + 2!\,(4\,1!\,\theta_1\,3!\,\theta_3 + 3\,(2!\,\theta_2)^2) - 3!\,6\,(1!\,\theta_1)^2\,2!\,\theta_2 + 4!\,(1!\,\theta_1)^4)\\ &= \frac{1}{4!}(-4! + 4!\cdot 32 + 4!\cdot 36 - 4!\cdot 12\cdot 24 + 4!\cdot 4^4) = 35.\end{aligned}$$

Example 2.2. Suppose we wish to calculate the number of non-negative integer solutions of the equation

$$x_1 + 2\,x_2 + 3\,x_3 + 4\,x_4 = 10.$$

In this case,

$$g(x) = (1-x)\,(1-x^2)\,(1-x^3)\,(1-x^4) = 1 - x - x^2 + 2\,x^5 - x^8 - x^9 + x^{10}$$

and thus

$$\theta_1 = \theta_2 = -1,\ \theta_3 = \theta_4 = 0,\ \theta_5 = 2,\ \theta_6 = \theta_7 = 0,\ \theta_8 = \theta_9 = -1,\ \theta_{10} = 1.$$

Using SymPy, [7] we can compute

$$\begin{aligned}B_{10,1}(x_1, x_2, \ldots, x_{10}) &= x_{10},\\ B_{10,2}(x_1, x_2, \ldots, x_9) &= 10x_1x_9 + 45x_2x_8 + 120x_3x_7 + 210x_4x_6 + 126x_5^2,\\ B_{10,3}(x_1, x_2, \ldots, x_8) &= 45x_1^2x_8 + 360x_1x_2x_7 + 840x_1x_3x_6 + 1260x_1x_4x_5\\ &\quad + 630x_2^2x_6 + 2520x_2x_3x_5 + 1575x_2x_4^2 + 2100x_3^2x_4,\\ B_{10,4}(x_1, x_2, \ldots, x_7) &= 120x_1^3x_7 + 1260x_1^2x_2x_6 + 2520x_1^2x_3x_5 + 1575x_1^2x_4^2\\ &\quad + 3780x_1x_2^2x_5 + 12600x_1x_2x_3x_4 + 2800x_1x_3^3 + 3150x_2^3x_4 + 6300x_2^2x_3^2,\\ B_{10,5}(x_1, x_2, \ldots, x_6) &= 210x_1^4x_6 + 2520x_1^3x_2x_5 + 4200x_1^3x_3x_4 + 9450x_1^2x_2^2x_4\\ &\quad + 12600x_1^2x_2x_3^2 + 12600x_1x_2^3x_3 + 945x_2^5,\\ B_{10,6}(x_1, x_2, \ldots, x_5) &= 252x_1^5x_5 + 3150x_1^4x_2x_4 + 2100x_1^4x_3^2 + 12600x_1^3x_2^2x_3 + 4725x_1^2x_2^4,\\ B_{10,7}(x_1, x_2, x_3, x_4) &= 210x_1^6x_4 + 2520x_1^5x_2x_3 + 3150x_1^4x_2^3,\\ B_{10,8}(x_1, x_2, x_3) &= 120x_1^7x_3 + 630x_1^6x_2^2,\\ B_{10,9}(x_1, x_2) &= 45x_1^8x_2,\\ B_{10,10}(x_1) &= x_1^{10}.\end{aligned}$$

Using Theorem 1.1, similar computation to Example 2.1 gives us 23 as the answer.

Example 2.3. Suppose we wish to calculate the number of non-negative integer solutions of the equation

$$x_1 + 2\,x_2 + 3\,x_3 + 4\,x_4 = 8.$$

In this case, $g(x)$ is the same as in the previous example since the left-hand sides of both equations are identical, and thus

$$\theta_1 = \theta_2 = -1,\ \theta_3 = \theta_4 = 0,\ \theta_5 = 2,\ \theta_6 = \theta_7 = 0,\ \theta_8 = \theta_9 = -1,\ \theta_{10} = 1.$$

Using SymPy, we can compute

$$\begin{aligned}
B_{8,1}(x_1, x_2, \ldots, x_8) &= x_8,\\
B_{8,2}(x_1, x_2, \ldots, x_7) &= 8x_1x_7 + 28x_2x_6 + 56x_3x_5 + 35x_4^2,\\
B_{8,3}(x_1, x_2, \ldots, x_6) &= 28x_1^2x_6 + 168x_1x_2x_5 + 280x_1x_3x_4 + 210x_2^2x_4 + 280x_2x_3^2,\\
B_{8,4}(x_1, x_2, \ldots, x_5) &= 56x_1^3x_5 + 420x_1^2x_2x_4 + 280x_1^2x_3^2 + 840x_1x_2^2x_3 + 105x_2^4,\\
B_{8,5}(x_1, x_2, x_3, x_4) &= 70x_1^4x_4 + 560x_1^3x_2x_3 + 420x_1^2x_2^3,\\
B_{8,6}(x_1, x_2, x_3) &= 56x_1^5x_3 + 210x_1^4x_2^2,\\
B_{8,7}(x_1, x_2) &= 28x_1^6x_2,\\
B_{8,8}(x_1) &= x_1^8.
\end{aligned}$$

Using Theorem 1.1, similar computation to Example 2.1 gives 15 as the answer.

Example 2.4. Suppose we wish to calculate the number of non-negative integer solutions of the equation

$$x_1 + 2\,x_2 + 3\,x_3 + 4\,x_4 = 12.$$

In this case, $g(x)$ is the same as in the previous example and thus

$$\theta_1 = \theta_2 = -1,\ \theta_3 = \theta_4 = 0,\ \theta_5 = 2,\ \theta_6 = \theta_7 = 0,\ \theta_8 = \theta_9 = -1,\ \theta_{10} = 1.$$

Using SymPy, we can compute

$$\begin{aligned}
B_{12,1}(x_1, x_2, \ldots, x_{12}) &= x_{12},\\
B_{12,2}(x_1, x_2, \ldots, x_{11}) &= 12x_1x_{11} + 66x_{10}x_2 + 220x_3x_9 + 495x_4x_8 + 792x_5x_7 + 462x_6^2,\\
B_{12,3}(x_1, x_2, \ldots, x_{10}) &= 66x_1^2x_{10} + 660x_1x_2x_9 + 1980x_1x_3x_8 + 3960x_1x_4x_7 + 5544x_1x_5x_6\\
&\quad + 1485x_2^2x_8 + 7920x_2x_3x_7 + 13860x_2x_4x_6\\
&\quad + 8316x_2x_5^2 + 9240x_3^2x_6 + 27720x_3x_4x_5 + 5775x_4^3,\\
B_{12,4}(x_1, x_2, \ldots, x_9) &= 220x_1^3x_9 + 2970x_1^2x_2x_8 + 7920x_1^2x_3x_7 + 13860x_1^2x_4x_6 + 8316x_1^2x_5^2\\
&\quad + 11880x_1x_2^2x_7 + 55440x_1x_2x_3x_6 + 83160x_1x_2x_4x_5 + 55440x_1x_3^2x_5\\
&\quad + 69300x_1x_3x_4^2 + 13860x_2^3x_6 + 83160x_2^2x_3x_5 + 51975x_2^2x_4^2\\
&\quad + 138600x_2x_3^2x_4 + 15400x_3^4,\\
B_{12,5}(x_1, x_2, \ldots, x_8) &= 495x_1^4x_8 + 7920x_1^3x_2x_7\\
&\quad + 18480x_1^3x_3x_6 + 27720x_1^3x_4x_5 + 41580x_1^2x_2^2x_6\\
&\quad + 166320x_1^2x_2x_3x_5 + 103950x_1^2x_2x_4^2 + 138600x_1^2x_3^2x_4 + 83160x_1x_2^3x_5\\
&\quad + 415800x_1x_2^2x_3x_4 + 184800x_1x_2x_3^3 + 51975x_2^4x_4 + 138600x_2^3x_3^2,
\end{aligned}$$

$$
\begin{aligned}
B_{12,6}(x_1, x_2, \ldots, x_7) &= 792x_1^5x_7 + 13860x_1^4x_2x_6 + 27720x_1^4x_3x_5 + 17325x_1^4x_4^2 \\
&\quad + 83160x_1^3x_2^2x_5 + 277200x_1^3x_2x_3x_4 + 61600x_1^3x_3^3 + 207900x_1^2x_2^3x_4 \\
&\quad + 415800x_1^2x_2^2x_3^2 + 207900x_1x_2^4x_3 + 10395x_2^6, \\
B_{12,7}(x_1, x_2, \ldots, x_6) &= 924x_1^6x_6 + 16632x_1^5x_2x_5 + 27720x_1^5x_3x_4 + 103950x_1^4x_2^2x_4 \\
&\quad + 138600x_1^4x_2x_3^2 + 277200x_1^3x_2^3x_3 + 62370x_1^2x_2^5, \\
B_{12,8}(x_1, x_2, \ldots, x_5) &= 792x_1^7x_5 + 13860x_1^6x_2x_4 + 9240x_1^6x_3^2 + 83160x_1^5x_2^2x_3 + 51975x_1^4x_2^4, \\
B_{12,9}(x_1, x_2, x_3, x_4) &= 495x_1^8x_4 + 7920x_1^7x_2x_3 + 13860x_1^6x_2^3, \\
B_{12,10}(x_1, x_2, x_3) &= 220x_1^9x_3 + 1485x_1^8x_2^2, \\
B_{12,11}(x_1, x_2) &= 66x_1^{10}x_2, \\
B_{12,12}(x_1) &= x_1^{12}.
\end{aligned}
$$

Using Theorem 1.1, similar computation to Example 2.1 gives 34 as the answer.

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