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A note on prime zeta function and Riemann zeta function. Corrigendum

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Abstract: In [1] the author proposed two new results concerning the prime zeta function and the Riemann zeta function but they turn out to be wrong. In the present paper we provide their correct form.

Keywords: Prime zeta function, Riemann zeta function, Prime numbers.
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1 Introduction

Let $\zeta(s)$ denote the Riemann zeta, i.e.

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s},$$

where $s \in \mathbb{C}$ ($\mathbb{C}$ is the set of complex numbers) and $\Re(s) > 1$. Let $P(s)$ be the prime zeta function:

$$P(s) = \sum_{p} \frac{1}{p^s},$$

where $s \in \mathbb{C}$ and $\Re(s) > 1$, and $p \in \mathbb{P}$ ($\mathbb{P}$ is the set of all primes).

In [1] the following two results were claimed as true:

Theorem 1 ([1]). *For integer $s > 1$, the following recurrent relation*

$$(1 - P(s))^2 = \frac{2}{\zeta(s)} - 1 + P(2s).$$

holds.

And as a Corollary:

$$P(s) = 1 - \sqrt{\frac{2}{\zeta(s)} - 1 + P(2s)}\,.$$

Theorem 2 ([1]). *Prime zeta function $P(s)$ for every fixed integer $s > 1$, could be expressed with the help of the values of Riemann zeta function $\zeta(2^k.s), k = 1, 2, 3, \dots,$ by the formula*

$$P(s) = 1 - \sqrt{\frac{2}{\zeta(s)} - \sqrt{\frac{2}{\zeta(2s)} - \sqrt{\frac{2}{\zeta(4s)} - \sqrt{\frac{2}{\zeta(8s)} \cdots}}}}. \tag{1}$$

Recently, Richard P. Brent disproved the stated above Theorems (and Corollary), [2].

2 Correct form of the theorems

Below, we introduce some denotations and give the correct form of our Theorems.

Let

$$P_m(s) \stackrel{\text{def}}{=} \sum \frac{1}{(p_{i_1} p_{i_2} \cdots p_{i_m})^s}, \tag{2}$$

where $m \geq 2$ and $p_{i_1} < p_{i_2} < \cdots < p_{i_m}$ run over $\mathbb{P}$.

$$\varepsilon(s) \stackrel{\text{def}}{=} \sum_{m=3}^{\infty} (-1)^{m-1} P_m(s). \tag{3}$$

In (2) and (3) $s \in \mathbb{C}$ and $\Re(s) > 1$. Then we provide the correct form of [1, Theorem 1] as follows

Theorem 3 (**Correct form of Theorem 1**). *For $s \in \mathbb{C}$ and $\Re(s) > 1$, the following recurrent relation*

$$(1 - P(s))^2 = \frac{2}{\zeta(s)} + 2\varepsilon(s) - 1 + P(2s) \tag{4}$$

holds.

For real $s > 1$ from (4) we obtain (since $0 < P(s) < 1$):

Corollary 1.

$$P(s) = 1 - \sqrt{\frac{2}{\zeta(s)} + 2\varepsilon(s) - 1 + P(2s)}\,. \tag{5}$$

Theorem 4 (**Correct form of Theorem 2**). *The representation*

$$P(s) = 1 - \sqrt{\frac{2}{\zeta(s)} + 2\varepsilon(s) - \sqrt{\frac{2}{\zeta(2s)} + 2\varepsilon(2s) - \sqrt{\frac{2}{\zeta(4s)} + 2\varepsilon(4s) - \sqrt{\frac{2}{\zeta(8s)} + 2\varepsilon(8s) \cdots}}}} \tag{6}$$

holds for real $s > 1$.

Since (6) follows directly from (5), it suffices to prove Theorem 3.

Proof of Theorem 3. We obviously have

$$\frac{1}{\zeta(s)} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^s},$$

where μ is the Möbius function.

Hence,
$$\frac{1}{\zeta(s)} = 1 - P(s) + P_2(s) - \varepsilon(s) \tag{7}$$
(see (2) and (3)).

Now we use that
$$P_2(s) = \frac{(P(s))^2 - P(2s)}{2}.$$
The last equality and (7) yield (4) and Theorem 3 is proved. □

3 Final comments

1) Using the well known representation:
$$P(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(\zeta(ks))$$
(see e.g. [3–5]), it is clear that we may express $\varepsilon(s)$ with the help of $\zeta(s)$ because of (4) and the fact that
$$P(2s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(\zeta(2ks)).$$
So in the right hand-side of (6) only values of Riemann's zeta function will appear.

2) Let us note, that formula (1) is an approximation of the prime zeta function $P(s)$ when $s > 1$ is a real number, since
$$\lim_{s \to +\infty} \varepsilon(s) = 0.$$

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