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On two theorems of Vassilev-Missana

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Abstract: We show that Theorem 1 of Vassilev-Missana [this journal, 2016, 22(4), 12–15] is false, and deduce that Theorem 2 of the same paper is also false.
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1 Introduction

Theorem 1 of Vassilev-Missana [3] states that*, for all integer $s > 1$,

$$2/\zeta(s) = 1 + (1 - P(s))^2 - P(2s), \tag{1}$$

where $\zeta(s)$ is the Riemann zeta-function and $P(s)$ is the prime zeta-function [2]. We remark that there is no need for the assumption that s is an integer. If correct, the proof of [3, Theorem 1] would hold for all complex s with $\Re(s) > 1$.

In §2 we disprove Theorem 1 using a Dirichlet series argument, and in §3 we deduce that Theorem 2 is also false. Finally, in §4 we provide numerical confirmation of these conclusions.

2 Disproof of Theorem 1

Assume that $\Re(s) > 1$. Recalling that $1/\zeta(s) = \sum \mu(n)n^{-s}$, we expand each side of (1) as a Dirichlet series $\sum a_n n^{-s}$. On the right-hand side (RHS), the only terms with nonzero coefficients a_n are for integers n of the form $p^\alpha q^\beta$, where p and q are primes, $\alpha \geq 0$, and $\beta \geq 0$. However, on the left-hand side (LHS), we find $a_{30} = 2\mu(30) = -2$, since 30 has three distinct prime factors, implying that $\mu(30) = -1$. This is a contradiction, so (1) is false. □

*For later convenience, we have made a trivial re-ordering of the terms in (1).

3 Disproof of Theorem 2

Theorem 2 of [3] states that, for all integer $s > 1$,

$$P(s) = 1 - \sqrt{2/\zeta(s) - \sqrt{2/\zeta(2s) - \sqrt{2/\zeta(4s) - \sqrt{2/\zeta(8s) - \cdots}}}}. \tag{2}$$

We now show that (2) is false. The proof is by way of contradiction. Assume that (2) is correct. Replacing s by $2s$ and using the result to simplify (2), we obtain

$$1 - P(s) = \sqrt{2/\zeta(s) - (1 - P(2s))}. \tag{3}$$

Squaring both sides of (3) and simplifying gives (1), but we showed in §2 that (1) is false. This contradiction shows that (2) is false. □

4 Numerical confirmation

To confirm the theoretical arguments above, we performed a direct numerical evaluation of each side of (1) for the case $s = 2$ (and for other cases not detailed here). We used the well-known formula [2, page 188] that can be proved by Möbius inversion:

$$P(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log \zeta(ks). \tag{4}$$

For $s = 2$, the LHS of (1) is $12/\pi^2 \approx 1.216$, and the RHS is 1.223, with both values correct to 3 decimal places. Thus, LHS $\neq$ RHS. This is a contradiction, confirming that (1) is false.

Similarly, we evaluated each side of (2) at $s = 2$. We found that the LHS is $P(2) \approx 0.452$, and the RHS is 0.459, with both values correct to 3 decimals. This confirms that (2) is false.

Further details regarding the numerical computations may be found in [1].

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References

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