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Inequalities for generalized divisor functions

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Abstract: We offer inequalities to $\sigma_a(n)$ as a function of the real variable a. Monotonicity and convexity properties to this and related functions are proved, too. Extensions and improvements of known results are provided.

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1 Introduction

Let $n \geq 1$ be a positive integer, and a be a real variable. The sum of a-th powers of divisors of n is defined by

$$\sigma_a(n) = \sum_{d|n} d^a, \tag{1}$$

where d runs through all distinct positive divisors of n. Particularly, $\sigma_1(n) = \sigma(n)$ is the sum of divisors of n, and $\sigma_0(n) = d(n) =$ number of distinct divisors of n. Remark that

$$\sigma_{-1}(n) = \sum_{d|n} \frac{1}{d} = \frac{1}{n} \cdot \sum_{d|n} \frac{n}{d} = \frac{1}{n} \sum_{d|n} d = \frac{\sigma(n)}{n}.$$

Similarly,

$$\sigma_{-a}(n) = \frac{\sigma_a(n)}{n^a} \tag{2}$$

for any real number a. It is well-known that $(\sigma_a(n))_n$ is a multiplicative function of the natural variable n, i.e.,

$$\sigma_a(n \cdot m) = \sigma_a(n) \cdot \sigma_a(m) \tag{3}$$

for any $n, m \geq 1$; $(n, m) = 1$.

In other words, if $n = \prod_{i=1}^{r} p_i^{a_i}$ $(r \geq 1)$ is the prime factorization of n, then

$$\sigma_a(n) = \prod_{i=1}^{r} \sigma_a\left(p_i^{a_i}\right) = \prod_{i=1}^{r} \frac{p_i^{a(a_i+1)} - 1}{p_i^a - 1}. \tag{4}$$

The unitary sum of divisor function $\sigma_a^*(n)$ is defined as

$$\sigma_a^* = \sum_{d|n,(d,n/d)=1} d^a, \tag{5}$$

i.e., the sum of a-th powers of the unitary divisors of n, where $d \mid n$ is a unitary divisor of n, if $(d, n/d) = 1$. It is well-known also (see e.g. [2, 3, 6]) that $(\sigma_a^*(n))_n$ is a multiplicative function of n, i.e., fot the above prime factorization of n one has

$$\sigma_a^*(n) = \prod_{i=1}^{r} \sigma_a^*(p_i^{a_i}) = \prod_{i=1}^{r} \left(p_i^{a \cdot a_i} + 1\right). \tag{6}$$

There are many known inequalities for $\sigma_a(n)$ and $\sigma_a^*(n)$, when $a = 1$ or $a = k =$ positive integer. For example, a result of Sándor–Tóth [2] states that

$$\frac{\sigma_k(n)}{d(n)} > n^{k/2} \tag{7}$$

for $n > 1$, $k \geq 1$ integers. A result by the author [5] states that

$$\frac{\sigma_k(n)}{\sigma_k^*(n)} < \frac{d(n)}{d^*(n)}. \tag{8}$$

In what follows, we will obtain extensions and refinements of (7), (8); and many related inequalities will be offered. For other inequalities, see [7].

2 Monotonocity and convexity properties

Theorem 1. *The applications $f, g : \mathbb{R} \to \mathbb{R}_+$, defined by $f(a) = \sigma_a(n)$ and $g(a) = \sigma_a^*(n)$ are log-convex functions for any fixed integer $n \geq 1$.*

Proof. The log-convexity of the function $f(a)$ means that the function $F(a) = \log f(a)$ is convex. It is well-known, that for continuous functions, a function is convex iff it is Jensen-convex. As $f(a)$ is sum of continuous functions, clearly it is continuous, too. Thus, we have to prove that $\log f(a)$ is Jensen-convex, i.e.,

$$\log f\left(\frac{a+b}{2}\right) \leq \frac{\log f(a) + \log f(b)}{2} \tag{9}$$

or equivalently

$$f^2\left(\frac{a+b}{2}\right) \leq f(a) \cdot f(b), \tag{10}$$

where a, b are real numbers.

In order to prove (10), we apply the Cauchy–Bunyakovsky inequality (see [1])

$$\left(\sum_{i=1}^{r} x_i y_i\right)^2 \le \left(\sum_{i=1}^{r} x_i^2\right) \cdot \left(\sum_{i=1}^{r} y_i^2\right) \tag{11}$$

for $x_i = d_i^{a/2}$, $y_i = d_i^{b/2}$, where $1 = d_1 < d_2 < \cdots < d_r = n$ are the distinct divisors of n. As

$$\sum_{i=1}^{n} d_i^a = f(a), \quad \sum_{i=1}^{r} d_i^b = f(b), \quad \sum_{i=1}^{r} d_i^{(a+b)/2} = f((a+b)/2),$$

by (11) we get relation (10).

Applying the same inequality to $x_i = (d_i^*)^{a/2}$, $y_i(d_i^*)^{b/2}$, where $1 \le d_1^* < d_2^* < \cdots < d_r^* = n$ are the distinct unitary divisors of n, we get

$$g^2\left(\frac{a+b}{2}\right) \le g(a) \cdot g(b),$$

i.e., the function $g(a)$ will be log-convex, too. □

Remark 1. *As there is equality in* (11) *only if* (x_i) *and* (y_i) *are proportional, i.e.* $x_i/y_i = \lambda$ $(i = 1, 2, \ldots, r)$ $\lambda =$ *constant, clearly there is an equality in* (10) *only for* $a = b$.

Corollary 1.

$$\left[\sigma_{(a+b)/2}(n)\right]^2 \le \sigma_a(n)\sigma_b(n) \le \left[\frac{\sigma_a(n) + \sigma_b(n)}{2}\right]^2. \tag{12}$$

The sequence of general term

$$t_k = \frac{\sigma_k(n)}{\sigma_{k-1}(n)} \quad (k \ge 1)$$

is strictly increasing for any fixed $n > 1$.

– Indeed, the second inequality of (12) *follows by* $xy \le \left(\frac{x+y}{2}\right)^2$, *where* $x = \sigma_a(n)$, $y = \sigma_b(n)$.

For $a = k-1$, $b = k+1$ *we set from* (12) *that* $(\sigma_k(n))^2 < \sigma_{k-1}(n) \cdot \sigma_{k+1}(n)$; *i.e.* $t_k < t_{k+1}$.

Corollary 2.

$$n^{a/2} \le \frac{\sigma_a(n)}{d(n)} \le \frac{\sigma_{2a}(n)}{\sqrt{d(n)}} \tag{13}$$

for any $a \in \mathbb{R}$.

Indeed, let $b = -a$ *in* (12). *By relation* (2) *we get the left-hand side of* (13).

Let now $a \to a + b$, $b \to a - b$ *in the left-hand side of* (12). *We get the inequality*

$$(\sigma_a(n))^2 \le \sigma_{a+b}(n) \cdot \sigma_{a-b}(n). \tag{14}$$

By letting $b = a$ *in* (14), *we get the right-hand side of* (13).

Remark 2. *All the above inequalities* (12)–(14) *hold true also for* $\sigma_a^*(n)$ *in plane of* $\sigma_a(n)$, *etc.*

Theorem 2. *The functions* $F, G : (0, \infty) \to \mathbb{R}$ *defined by*

$$F(a) = \sigma_a(n)/_n^{a/2}; \quad G(a) = \sigma_a^*(n)/_n^{a/2} \tag{15}$$

are strictly increasing functions. The functions $F_1, G_1 : (-\infty, 0) \to \mathbb{R}$ *with the same definitions as above, are strictly decreasing.*

Proof. Let p be a prime number. As $F(p^\alpha) = \sigma_a(p^\alpha)/p^{a\alpha/2} = s(a)$, we will prove first that $s(a)$ is strictly increasing. Then, as

$$F(a) = \prod_{i=1}^{r} \frac{\sigma_a(p_i^{\alpha_i})}{p_i^{a\alpha_i/2}},$$

$F(a)$ will be strictly increasing as the product of strictly increasing positive functions.

By (4) one has

$$\log s(a) = \log\left(p^{a(\alpha+1)} - 1\right) - \log(p^a - 1) - \frac{a\alpha}{2}\log p = S(a).$$

One has for the derivative of $S(a)$ that

$$S'(a) = \frac{(\alpha+1)p^{a(\alpha+1)}\log p}{p^{a(\alpha+1)} - 1} - \frac{p^a \log p}{p^a - 1} - \frac{\alpha}{2}\log p.$$

By letting $p^a = x$, after some elementary computations, we get

$$S'(a) \cdot \frac{2(x-1)\cdot(x^{\alpha+1}-1)}{\log p} = x^{\alpha+2}\cdot(\alpha) - x^{\alpha+1}\cdot(\alpha+2) + x\cdot(\alpha+2) - \alpha = M(x).$$

Now, remark that $M(1) = 0$,

$$M'(x) = (\alpha+2)\cdot\left[\alpha\cdot x^{\alpha+1} - (\alpha+1)\cdot x^\alpha + 1\right] = (\alpha+2)\cdot N(x).$$

Here $N(1) = 0$ and $N'(x) = (\alpha+1)\cdot x^{\alpha-1}\cdot(x-1) > 0$, as $x = p^a > 1$, $\alpha \geq 1$. Therefore, $N(x)$ is strictly increasing, implying $N(x) > N(1) = 0$, so $M'(x) > 0$. Thus, finally, we get $M(x) > M(1) = 0$, so $S'(a) > 0$, and thus $S(a)$ is strictly increasing. This means that $s(a)$ is strictly increasing, and the first part of Theorem 2 is proved.

For the second part, remark that $\sigma_a^*(p^\alpha) = p^{a\alpha} + 1$, and it will be sufficient to consider the monotonicity of

$$\log(p^{a^\alpha} + 1) - \frac{a\alpha}{2}\log p = h(a).$$

As

$$\frac{h'(a)}{\alpha\log p} = \frac{p^{a\alpha}}{p^{a\alpha}+1} - \frac{1}{2} = \frac{x^\alpha - 1}{2(x^\alpha+1)},$$

where $x = p^a > 1$ for $a > 0$. Clearly $x^\alpha - 1 > 0$, so $h'(a) > 0$, and the proof of second part of the theorem follows. For $a < 0$ we get $0 < x < 1$, and all can be replaced for F_1 and G_1, will be strictly decreasing. □

Corollary 3.

$$\frac{\sigma_a(n)}{d(n)} > n^{a/2} \text{ for } n > 1,\ a \neq 0. \tag{16}$$

As

$$\lim_{a\to 0}\frac{\sigma_a(p^\alpha)}{p^{a\alpha/2}} = \lim_{a\to 0}\frac{p^{a(\alpha+1)} - 1}{p^a - 1} = \alpha + 1$$

(by L'Hospital's rule) and as

$$F(a) > \lim_{a\to 0} f(a) = (\alpha_1+1)\cdots(\alpha_r+1) = d(n),$$

relation (16) *follows from the first part of Theorem 2. From* $F_1(a) > \lim_{a\to 0} F_1(a)$ *for* $a < 0$, *we get the same inequalities.*

Corollary 4.

$$\frac{\sigma_a^*(n)}{d(n)} > n^{a/2} \text{ for } n > 1,\ a \neq 0. \tag{17}$$

This follows in a similar manner, from the second part of Theorem 2, first for $G(a)$, then for $G_1(a)$.

Theorem 3. *The function $H : (0, \infty) \to \mathbb{R}$, defined by*

$$H(a) = \sigma_a(n)/\sigma_a^*(n)\ (n > 1 \text{ fixed})$$

is strictly decreasing. The function $H_1 : (-\infty, 0) \to \mathbb{R}$ with the same definition is strictly increasing.

Proof. For $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ one has

$$H(n) = \sigma_a(p_1^{\alpha_1}) \cdots \sigma_a(p_r^{\alpha_r})/\sigma_a^*(\sigma_a^{p_1}) \cdots \sigma_a^*(p_r^{\alpha_1}) = f_1(a) \cdots f_r(a);$$

where $f_i(a) = \sigma_a(p_i^{\alpha_i})/\sigma_a^*(p_i^{\alpha_i})$, so it will be sufficient to prove that

$$k(a) = \sigma_a(p^\alpha)/\sigma_a^*(p^\alpha) = [p^{a(\alpha+1)} - 1]/(p^a - 1)(p^{a\alpha} + 1)$$

will be strictly decreasing for fixed prime $p > 1$. One has

$$\log k(a) = \log(p^{a(\alpha+1} - 1) - \log(p^a - 1) - \log(p^{a\alpha+1} + 1) = K(a).$$

One has

$$\frac{K'(a)}{p^a \log p} = \frac{(a+1) \cdot p^{a\alpha}}{p^{a(\alpha+1)} - 1} - \frac{1}{p^a - 1} - \frac{\alpha p^{a(\alpha+1)}}{p^{a\alpha} + 1},$$

and after some elementary computations we can write

$$\begin{aligned} &K'(a) \cdot \frac{\left[p^{a(\alpha+1)} - 1\right](p^a - 1)(p^{a\alpha} + 1)}{p^a \log p} \\ &= (\alpha+1)x^\alpha \cdot (x-1) \cdot (x^\alpha + 1) - (x^\alpha + 1) \cdot (x^{\alpha+1} - 1) - \alpha \cdot x^{\alpha-1} \cdot (x-1) \cdot (x^{\alpha+1} - 1) \\ &= R(x), \end{aligned}$$

where $x = p^a > 1$. Now, $R(x)$ can be written as $R(x) = \alpha \cdot x^{\alpha+1} - x^{2\alpha} - \alpha \cdot x^{\alpha-1} + 1$. We will prove that $-R(x) = x^{2\alpha} - \alpha \cdot x^{\alpha+1} + \alpha \cdot x^{\alpha-1} - 1 \geq 0$. One has

$$-R'(x) = \alpha \cdot x^{\alpha-2} \cdot \left[2x^{\alpha+1} - (\alpha+1)x^2 + \alpha - 1\right].$$

Let $U(x) = 2x^{\alpha+1} - (\alpha+1)x^2 + \alpha - 1$. Here $U(1) = 0$ and $U'(x) = 2(\alpha+1) \cdot x \cdot (x^{\alpha-1} - 1) \geq 0$ as $x > 1$ and $\alpha - 1 \geq 0$. Thus $U(x) > U(1) = 0$, so we get $R'(x) < 0$ implying $R(x) < R(1) = 0$. Thus we have proved that $K'(a) < 0$. As $\frac{k'(a)}{k(a)} = K'(a)$, this implies finally that $k'(a) < 0$, i.e., $k(a)$ is strictly decreasing. For $a < 0$ one has $0 < x < 1$, and we get that $H_1(x)$ is strictly increasing. □

Corollary 5. *For any $a \neq 0$ one has*

$$\frac{\sigma_a(n)}{\sigma_a^*(n)} < \frac{d(n)}{d^*(n)} \tag{18}$$

Indeed, as

$$H(a) < \lim_{a \to 0+} H(a) = \frac{\alpha + 1}{2},$$

by Theorem 3 we can write

$$\frac{\sigma_a(n)}{\sigma_a^*(n)} < \frac{\alpha_1 + 1}{2} \cdots \frac{\alpha_r + 1}{2} = \frac{d(n)}{d^*(n)}.$$

From $H_1(a) < \lim_{a \to 0-} H_1(a)$ we get the same inequality.

Remark 3. *Inequality* (18) *extends* (8) *from positive integers k to all real numbers $a \neq 0$. Finally, in this context, we will prove:*

Theorem 4. *The function $T(a) : (0, \infty) \to \infty$, defined by*

$$T(a) = \frac{\sigma_a(n)}{\sigma_a^*(\gamma(n))}, \tag{19}$$

(where $\gamma(n)$ is the "core" of n) is strictly increasing. The function $T_1(a) : (-\infty, 0) \to \mathbb{R}$ with the same definition is strictly decreasing.

Proof. If $n = p_1^{\alpha_1} \cdots p_1^{\alpha_r}$, then $\gamma(n) = p_1 \cdots p_r$; so $\sigma_a^*(\gamma(n)) = (p_1^a + 1) \cdots (p_r^a + 1)$. Thus, it will be sufficient to prove that the function $z(a) = \sigma_a(p^\alpha)/p^a + 1$ will be strictly increasing. As

$$\log z(a) = \log(p^{a(\alpha+1)} - 1) - \log(p^a - 1) - \log(p^a + 1) = Z(a),$$

one has

$$Z'(a) = \frac{(\alpha + 1) \cdot p^{a(\alpha+1)} \log p}{p^{a(\alpha+1)} - 1} - \frac{p^a \log p}{p^a - 1} - \frac{p^a \log p}{p^a + 1},$$

and after some elementary computations (which we omit here) we can find that

$$\frac{(p^{a(\alpha+1} - 1)(p^a - 1)(p^a + 1)Z'(a)}{\log p} = x^2 \cdot [(\alpha - 1) \cdot x^{\alpha+1} - (\alpha + 1) \cdot x^{\alpha-1} + 2],$$

where $x = p^\alpha > 1$. Let $q(x) = (\alpha - 1) \cdot x^{\alpha+1} - (\alpha + 1)x^{\alpha-1} + 2$. We have $q(1) = 0$ and $q'(x) = (\alpha^2 - 1) \cdot x^{\alpha-2} \cdot (x^2 - 1) \geq 0$ as $x > 1$ and $\alpha \geq 1$. Thus $q(x) > q(1) = 0$ and this yields $Z'(a) > 0$, so $z(a)$ will be strictly increasing. For $a < 0$ we have $0 < x < 1$, and we get that $T_1(a)$ is strictly decreasing. □

Corollary 6. *For any $a \neq 0$ and $n > 1$ one has*

$$\frac{d(n)}{d^*(n)} < \frac{\sigma_a(n)}{\sigma_a^*(\gamma(n))}. \tag{20}$$

This follows by

$$\lim_{a \to 0} T(a) = \frac{d(n)}{d^*(a)}$$

and Theorem 4.

Remark 4. *Inequality* (20) *offers a counterpart to* (18).

3 Applications of other inequalities for sums

The classical Chebyshev inequality (see [1]) states that if (x_i) and (y_i) $(i = 1, 2, \dots, r)$ are two sequences with the same (reversed) type of monotonicity, then

$$\frac{x_1 y_1 + \cdots + x_r y_r}{r} \underset{(\leq)}{\geq} \frac{x_1 + \cdots + x_r}{r} \cdots \frac{y_1 + \cdots + y_r}{r}. \tag{21}$$

Letting $x_i = d_i^a$, $y_i = d_i^b$, where $1 = d_1 < d_2 < \cdots < d_r = n$ are all divisors of n, we get the following:

$$d(n) \cdot \sigma_{a+b}(n) \geq \sigma_a(n)\sigma_b(n) \quad \text{for } a \cdot b \geq 0, \tag{22}$$

$$d(n) \cdot \sigma_{a+b}(n) \leq \sigma_a(n)\sigma_b(n) \quad \text{for } a \cdot b \leq 0. \tag{23}$$

Indeed, for $a \cdot b \geq 0$, the sequences (d_i^a) and (d_i^b) will have the same type of monotonicity, and for $a \cdot b \leq 0$, the reversed one.

From the left-hand side of (12), combined with (23), we get:

$$\left[\sigma_{(a+b)/2}(n)\right]^2 \leq \sigma_a(n)\sigma_b(n) \leq d(n) \cdot \sigma_{a+b} \quad \text{for } a \cdot b \geq 0. \tag{24}$$

Particularly, by letting $\frac{a+b}{2} = c$, the weakest part of (24) offers $(\sigma_c(n))^2 \leq d(n)\sigma_{2c}(n)$, which is the right-hand side of (13). Thus, (24) offers an improvement of right-hand side of (13) for $a \cdot b \geq 0$.

The Milne's inequality (see [1,4]) states that if (x_i) and (y_i) are positive r-tuples, then

$$\sum_{i=1}^{r}(x_i + y_i) \cdot \sum_{i=1}^{r} \frac{x_i y_i}{x_i + y_i} \leq \sum_{i=1}^{r} x_i \cdot \sum_{i=1}^{r} y_i \tag{25}$$

with equality if and only if (x_i) and (y_i) are proportional.

Apply now the Cauchy–Bunyakovsky inequality (11) for $x_i = \sqrt{a_i^2 + b_i^2}$, $y_i = \frac{a_i b_i}{\sqrt{a_i^2 + b_i^2}}$. We get

$$\left(\sum_{i=1}^{r} a_i b_i\right)^2 \leq \sum_{i=1}^{r}(a_i^2 + b_i^2) \cdot \sum_{i=1}^{r} \frac{a_i^2 b_i^2}{a_i^2 + b_i^2}. \tag{26}$$

Now, Milne's inequality (25) applied for $x_i = a_i^2$, $y_i = b_i^2$ and combined with (26) gives

$$\left(\sum_{i=1}^{r} a_i b_i\right)^2 \leq \sum_{i=1}^{r}(a_i^2 + b_i^2) \cdot \sum_{i=1}^{r} \frac{a_i^2 b_i^2}{a_i^2 + b_i^2} \leq \left(\sum_{i=1}^{r} a_i^2\right)\left(\sum_{i=1}^{r} b_i^2\right) \tag{27}$$

where a_i, b_i are real numbers and $a_i^2 + b_i^2 \neq 0$. This is in fact a refinement of Cauchy–Bunyakovsky inequality. Let now $a_i = d_i^{a/2}$, $b_i = d_i^{b/2}$, where $1 = d_1 < d_2 < \cdots < d_r = n$ are the distinct divisors of n. We get the following refinement of the left-hand side of (12):

$$\left(\sigma_{(a+b)/2}(n)\right)^2 \leq A(a, b, n) \leq \sigma_a(n) \cdot \sigma_b(n), \tag{28}$$

where

$$A(a, b, n) = (\sigma(a) + \sigma(b)) \cdot \sum_{d|n} \frac{d^{a+b}}{d^a + d^b}.$$

The Pólya–Szegő inequality (see [1]) states that if $0 < a \leq x_i \leq A$ and $0 < b \leq y_i \leq B$ $(i = 1, 2, \dots, r)$.

Then

$$\left(\sum_{i=1}^{r} x_i^2\right)^{1/2}\left(\sum_{i=1}^{r} y_i^2\right)^{1/2} \le \frac{1}{2}\left(\sqrt{\frac{AB}{ab}}+\sqrt{\frac{ab}{AB}}\right)\sum_{i=1}^{r} x_i y_i. \tag{29}$$

Let $x_i = d_i^{a/2}$, $y_i = d_i^{b/2}$ (d_i the divisors of n). After some elementary computations, we get:

$$\sigma_a(n)\sigma_b(n) \le \frac{1}{4}\cdot\frac{\left(n^{\frac{a+b}{2}}+1\right)^2}{n^{(a+b)/2}}\cdot\left(\sigma_{(a+b)/2}(n)\right)^2 \quad \text{if} \quad a\cdot b>0, \tag{30}$$

$$\sigma_a(n)\sigma_b(n) \le \frac{1}{4}\cdot\frac{\left(n^{a/2}+n^{b/2}\right)^2}{n^{(a+b)/2}}\cdot\left(\sigma_{(a+b)/2}(n)\right)^2 \quad \text{if} \quad a\cdot b<0. \tag{31}$$

These can complemenet the right-hand side of inequality (28).

Finally, the discrete version of Zagier's inequality (see [1]) states that

$$\frac{\left(\sum_{i=1}^{r} x_i^2\right)\left(\sum_{i=1}^{r} y_i^2\right)}{\max\left\{\sum_{i=1}^{r} x_i, \sum_{i=1}^{r} y_i\right\}} \le \sum_{i=1}^{r} x_i y_i, \tag{32}$$

where $0 < x_i, y_i \le 1$, where both of (x_i) and (y_i) are decreasing sequences.

For $x_i = d_i^{a/2}$, $y_i = d_i^{b/2}$ ($d_i =$ divisors of n), we get from (32):

$$\frac{\sigma_a(n)\cdot\sigma_b(n)}{\max\left\{\sigma_{a/2}(n), \sigma_{b/2}(n)\right\}} \le \sigma_{(a+b)/2}(n) \quad \text{for} \quad a,b<0. \tag{33}$$

Letting $a = -A$, $b = -B$ and using (2) and (33), we get:

$$\frac{\sigma_A(n)\cdot\sigma_B(n)}{\sigma_{B/2}(n)} \le n^{\frac{A}{2}}\cdot\sigma_{\frac{A+B}{A}}(n) \quad \text{for} \quad A\ge B>0. \tag{34}$$

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