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Bi-unitary multiperfect numbers, V

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Dedicated to the memory of Prof. D. Suryanarayana

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Abstract: A divisor d of a positive integer n is called a unitary divisor if $\gcd(d, n/d) = 1$; and d is called a bi-unitary divisor of n if the greatest common unitary divisor of d and n/d is unity. The concept of a bi-unitary divisor is due to D. Surynarayana (1972). Let $\sigma^{**}(n)$ denote the sum of the bi-unitary divisors of n. A positive integer n is called a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$ for some $k \geq 3$. For $k = 3$ we obtain the bi-unitary triperfect numbers.

Peter Hagis (1987) proved that there are no odd bi-unitary multiperfect numbers. The present paper is part V in a series of papers on even bi-unitary multiperfect numbers. In parts I, II and III we determined all bi-unitary triperfect numbers of the form $n = 2^a u$, where $1 \leq a \leq 6$ and u is odd. In parts IV(a-b) we solved partly the case $a = 7$. In this paper we fix the case $a = 8$. In fact, we show that $n = 57657600 = 2^8.3^2.5^2.7.11.13$ is the only bi-unitary triperfect number of the present type.

Keywords: Perfect numbers, Triperfect numbers, Multiperfect numbers, Bi-unitary analogues.
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1 Introduction

Throughout this paper, all lower case letters denote positive integers; p and q denote primes. The letters u, v and w are reserved for odd numbers.

*Prof. Varanasi Sitaramaiah passed away on 2 October 2020.

A divisor d of n is called a unitary divisor if $\gcd(d, n/d) = 1$. If d is a unitary divisor of n, we write $d \| n$. A divisor d of n is called a *bi-unitary* divisor if $(d, n/d)^{**} = 1$, where the symbol $(a, b)^{**}$ denotes the greatest common unitary divisor of a and b. The concept of a bi-unitary divisor is due to D. Suryanarayana (cf. [8]). Let $\sigma^{**}(n)$ denote the sum of bi-unitary divisors of n. The function $\sigma^{**}(n)$ is multiplicative, that is, $\sigma^{**}(1) = 1$ and $\sigma^{**}(mn) = \sigma^{**}(m)\sigma^{**}(n)$ whenever $(m, n) = 1$. If p^α is a prime power and α is odd, then every divisor of p^α is a bi-unitary divisor; if α is even, each divisor of p^α is a bi-unitary divisor except for $p^{\alpha/2}$. Hence

$$\sigma^{**}(p^\alpha) = \begin{cases} \sigma(p^\alpha) = \frac{p^{\alpha+1}-1}{p-1} & \text{if } \alpha \text{ is odd,} \\ \sigma(p^\alpha) - p^{\alpha/2} & \text{if } \alpha \text{ is even.} \end{cases} \tag{1.1}$$

If α is even, say $\alpha = 2k$, then $\sigma^{**}(p^\alpha)$ can be simplified to

$$\sigma^{**}(p^\alpha) = \left(\frac{p^k - 1}{p - 1}\right) .(p^{k+1} + 1).$$

From (1.1), it is not difficult to observe that $\sigma^{**}(n)$ is odd only when $n = 1$ or $n = 2^\alpha$.

The concept of a bi-unitary perfect number was introduced by C. R. Wall [9]; a positive integer n is called a bi-unitary perfect number if $\sigma^{**}(n) = 2n$. C. R. Wall [9] proved that there are only three bi-unitary perfect numbers, namely $6, 60$ and 90. A positive integer n is called a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$ for some $k \geq 3$. For $k = 3$ we obtain the bi-unitary triperfect numbers.

Peter Hagis [1] proved that there are no odd bi-unitary multiperfect numbers. Our present paper is part V in a series of papers on even bi-unitary multiperfect numbers. In parts I, II and III (see [2–4]) we considered bi-unitary triperfect numbers of the form $n = 2^a u$, where $1 \leq a \leq 6$ and u is odd. In parts IV(a-b) (see [5, 6]) we solved partly the case $a = 7$. In this paper we fix the case $a = 8$. In fact, we show that $n = 57657600 = 2^8.3^2.5^2.7.11.13$ is the only bi-unitary triperfect number of the present type.

For a general account on various perfect-type numbers, we refer to [7].

2 Preliminaries

We assume that the reader has parts I, II, III, IV(a-b) (see [2–6]) available. We, however, recall Lemma 2.1 from these parts because it is so important also here.

Lemma 2.1. (I) *If α is odd, then*

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} > \frac{\sigma^{**}(p^{\alpha+1})}{p^{\alpha+1}}$$

for any prime p.

(II) *For any $\alpha \geq 2\ell - 1$ and any prime p,*

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} \geq \left(\frac{1}{p-1}\right)\left(p - \frac{1}{p^{2\ell}}\right) - \frac{1}{p^\ell} = \frac{1}{p^{2\ell}}\left(\frac{p^{2\ell+1} - 1}{p - 1} - p^\ell\right).$$

(III) *If p is any prime and α is a positive integer, then*

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} < \frac{p}{p-1}.$$

Remark 2.1. (I) and (III) of Lemma 2.1 are mentioned in C. R. Wall [9]; (II) of Lemma 2.1 has been used by him [9] without explicitly stating it.

3 Bi-unitary triperfect numbers of the form $n = 2^8 u$

Let n be a bi-unitary triperfect number divisible unitarily by 2^8 so that $\sigma^{**}(n) = 3n$ and $n = 2^8.u$, where u is odd. Since $\sigma^{**}(2^8) = (2^4 - 1)(2^5 + 1) = 15.33 = 3^2.5.11 = 495$, using $n = 2^8 u$ in $\sigma^{**}(n) = 3n$, we get

$$2^8.u = 3.5.11.\sigma^{**}(u). \tag{3.1}$$

This implies that u is divisible by 3, 5 and 11. Let $u = 3^b.5^c.11^d.v$, where $(v, 2.3.5.11) = 1$. Hence we have

$$n = 2^8.3^b.5^c.11^d.v, \tag{3.1a}$$

and from (3.1),

$$2^8.3^{b-1}.5^{c-1}.11^{d-1}.v = \sigma^{**}(3^b).\sigma^{**}(5^c).\sigma^{**}(11^d).\sigma^{**}(v), \tag{3.1b}$$

where

$$v \text{ has at most five odd prime factors and } (v, 2.3.5.11) = 1. \tag{3.1c}$$

We prove the following:

Theorem 3.1. *The number $n = 57657600 = 2^8.3^2.5^2.7.11.13$ is the only bi-unitary triperfect number of the form $n = 2^8.u$, where u is odd.*

Proof. For the proof of Theorem 3.1, we need the following lemmas:

Lemma 3.1. *Let $n = 2^8.3^b.5^c.11^d.v$, where $(v, 2.3.5.11) = 1$, be as in (3.1a). If $b \geq 3$, then n cannot be a bi-unitary triperfect number.*

Proof. We assume that $b \geq 3$ and n is a bi-unitary triperfect number so that (3.1b) holds. We derive a contradiction. From Lemma 2.1, $\frac{\sigma^{**}(3^b)}{3^b} \geq \frac{112}{81}$ for $b \geq 3$, and $\frac{\sigma^{**}(5^c)}{5^c} \geq \frac{756}{625}$ for $c \geq 3$. Also, $\frac{\sigma^{**}(2^8)}{2^8} = \frac{495}{256}$. Hence from (3.1$a$), for $c \geq 3$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{112}{81}.\frac{756}{625} = 3.234 > 3,$$

a contradiction. Hence $c = 1$ or $c = 2$.

Let $c = 1$. From (3.1a) ($c = 1$), we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{112}{81}.\frac{6}{5} = 3.208333333 > 3,$$

a contradiction.

Let $c = 2$. Since $\sigma^{**}(5^2) = 26 = 2.13$, from (3.1$b$) ($c = 2$), we get after simplification,

$$2^7.3^{b-1}.5.11^{d-1}.v = 13.\sigma^{**}(3^b).\sigma^{**}(11^d).\sigma^{**}(v). \tag{3.1d}$$

From (3.1*d*), $13|v$. Let $v = 13^e.w$, where $(w, 2.3.5.11.13) = 1$. Hence from (3.1*a*),

$$n = 2^8.3^b.5^2.11^d.13^e.w, \tag{3.2a}$$

and from (3.1*d*),

$$2^7.3^{b-1}.5.11^{d-1}.13^{e-1}.w = \sigma^{**}(3^b).\sigma^{**}(11^d).\sigma^{**}(13^e).\sigma^{**}(w), \tag{3.2b}$$

where

$$w \text{ has at most four odd prime factors and } (w, 2.3.5.11.13) = 1. \tag{3.2c}$$

By Lemma 2.1, for $d \geq 3$, $\frac{\sigma^{**}(11^d)}{11^d} \geq \frac{15984}{14641}$. Hence for $d \geq 3$, from (3.2*a*),

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{112}{81}.\frac{26}{25}.\frac{15984}{14641} = 3.03333 > 3,$$

a contradiction.

Let $d = 2$ (already $c = 2$). We have $\sigma^{**}(11^2) = 122 = 2.61$. Taking $d = 2$ in (3.2*b*), we get after simplification,

$$2^6.3^{b-1}.5.11.13^{e-1}.w = 61.\sigma^{**}(3^b).\sigma^{**}(13^e).\sigma^{**}(w). \tag{3.3}$$

From (3.3), $61|w$. Let $w = 61^f.w'$. Hence from (3.2*a*) $(d = 2)$, we get

$$n = 2^8.3^b.5^2.11^2.13^e.61^f.w', \tag{3.3a}$$

and from (3.3),

$$2^6.3^{b-1}.5.11.13^{e-1}.61^{f-1}.w' = \sigma^{**}(3^b).\sigma^{**}(13^e).\sigma^{**}(61^f).\sigma^{**}(w'), \tag{3.3b}$$

where

$$w' \text{ has at most three odd prime factors and } (w', 2.3.5.11.13.61) = 1. \tag{3.3c}$$

When $b \geq 7$, we have $\frac{\sigma^{**}(3^b)}{3^b} \geq \frac{9760}{6561}$; using this, from (3.3*a*), for $b \geq 7$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{9760}{6561}.\frac{26}{25}.\frac{122}{121} = 3.016149146 > 3,$$

a contradiction. Thus $b \geq 7$ cannot hold. Hence $3 \leq b \leq 6$. We prove that none of these choices for b is admissible.

Let $b = 3$. We have $\sigma^{**}(3^3) = \frac{3^4-1}{2} = 40 = 2^3.5$. Hence by taking $b = 3$ in (3.3*a*) and (3.3*b*), we get

$$n = 2^8.3^3.5^2.11^2.13^e.61^f.w', \tag{3.3d}$$

and

$$2^3.3^2.11.13^{e-1}.61^{f-1}.w' = \sigma^{**}(13^e).\sigma^{**}(61^f).\sigma^{**}(w'), \tag{3.3e}$$

where

$$w' \text{ cannot have not more than one odd prime factor.} \tag{3.3f}$$

From (3.3*d*), we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{40}{27}.\frac{26}{25}.\frac{122}{121} = 3.003787879 > 3,$$

a contradiction. So, $b = 3$ is not admissible.

Let $b=4$. We have $\sigma^{**}(3^4)=\left(\frac{3^2-1}{2}\right).(3^3+1)=4.28=2^4.7$. Taking $b=4$ in (3.3*b*), we get after simplification

$$2^2.3^3.5.11.13^{e-1}.61^{f-1}.w'=7.\sigma^{**}(13^e).\sigma^{**}(61^f).\sigma^{**}(w'). \tag{3.3g}$$

Comparing powers of 2 on both sides of (3.3*g*), we find that $w'=1$ and so 7 cannot divide the left hand side of (3.3*g*). This contradiction proves that $b=4$ is not admissible.

Let $b=5$. We have $\sigma^{**}(3^5)=\frac{3^6-1}{2}=13.28=2^2.7.13$. Taking $b=5$ in (3.3*b*), we get after simplification

$$2^4.3^4.5.11.13^{e-2}.61^{f-1}.w'=7.\sigma^{**}(13^e).\sigma^{**}(61^f).\sigma^{**}(w'). \tag{3.3h}$$

From (3.3*h*), we see that $7|w'$. Let $w'=7^g.w''$; using this in (3.3*a*), we have

$$n=2^8.3^5.5^2.11^2.13^e.61^f.7^g.w'',$$

so that

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{495}{256}.\frac{364}{243}.\frac{26}{25}.\frac{122}{121}=3.0371633>3,$$

a contradiction. Thus $b=5$ is not admissible.

Let $b=6$. We have $\sigma^{**}(3^6)=\left(\frac{3^3-1}{2}\right).(3^4+1)=13.82=2.13.41$. Taking $b=6$ in (3.3*b*), we obtain after simplification,

$$2^5.3^5.5.11.13^{e-2}.61^{f-1}.w'=41.\sigma^{**}(13^e).\sigma^{**}(61^f).\sigma^{**}(w'). \tag{3.3i}$$

From (3.3*i*), it follows that $41|w'$. Let $w'=41^g.w''$. Hence from (3.3*a*) ($b=6$),

$$n=2^8.3^6.5^2.11^2.13^e.61^f.41^g.w'', \tag{3.3j}$$

and from (3.3*i*),

$$2^5.3^5.5.11.13^{e-2}.61^{f-1}.41^{g-1}.w''=\sigma^{**}(13^e).\sigma^{**}(61^f).\sigma^{**}(41^g).\sigma^{**}(w''), \tag{3.3k}$$

where

$$w'' \text{ has at most two odd prime factors and } (w'',2.3.5.11.13.61.41)=1. \tag{3.3\ell}$$

By Lemma 2.1, we have $\frac{\sigma^{**}(13^e)}{13^e}\geq\frac{30772}{28561}$ for $e\geq 3$. Hence from (3.3*j*), for $e\geq 3$, we have

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{495}{256}.\frac{1066}{729}.\frac{26}{25}.\frac{122}{121}.\frac{30772}{28561}=3.194368571>3,$$

a contradiction.

Thus $e\leq 2$. From (3.3*k*), $e\geq 2$. Hence $e=2$. We have $\sigma^{**}(13^2)=170=2.5.17$. Taking $e=2$ in (3.3*k*), we get

$$2^4.3^5.11.61^{f-1}.41^{g-1}.w''=17.\sigma^{**}(61^f).\sigma^{**}(41^g).\sigma^{**}(w''). \tag{3.3m}$$

From (3.3*m*), $17|w''$. Let $w''=17^h.w'''$. It follows from (3.3*j*),

$$n=2^8.3^6.5^2.11^2.13^e.61^f.41^g.17^h.w''', \tag{3.4a}$$

and from (3.3m) ($e=2$),

$$2^4.3^5.11.61^{f-1}.41^{g-1}.17^{h-1}.w'''=\sigma^{**}(61^f).\sigma^{**}(41^g).\sigma^{**}(17^h).\sigma^{**}(w'''), \tag{3.4b}$$

where w''' has no more than one odd prime factor and is prime to 2.3.5.11.13.61.41.17.

By Lemma 2.1, for $h \geq 3$, $\frac{\sigma^{**}(17^h)}{17^h} \geq \frac{88452}{83521}$. Hence for $h \geq 3$, from (3.4a), we have
$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{1066}{729}.\frac{26}{25}.\frac{122}{121}.\frac{170}{169}.\frac{88452}{83521} = 3.158471032 > 3,$$
a contradiction. Hence $h = 1$ or $h = 2$.

Let $h = 1$. From (3.4a), we have
$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{1066}{729}.\frac{26}{25}.\frac{122}{121}.\frac{170}{169}.\frac{18}{17} = 3.157828283 > 3,$$
a contradiction.

Let $h = 2$. Since $\sigma^{**}(17^2) = 290 = 2.5.29$, taking $h = 2$ in (3.4b), we see that 5 divides its right hand side but 5 does not divide its left hand side. Thus $b = 6$ cannot occur.

This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let* $n = 2^8.3.5^c.11^d.v$, *where* $(v, 2.3.5.11) = 1$. *Then* n *cannot be a bi-unitary triperfect number.*

Proof. We assume that $n = 2^8.3.5^c.11^d.v$ is a bi-unitary triperfect number. Hence n satisfies (3.1b) and (3.1c). From Lemma 2.1, for $c \geq 3$, $\frac{\sigma^{**}(5^c)}{5^c} \geq \frac{756}{625}$. Hence we have
$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{4}{3}.\frac{756}{625} = 3.1185 > 3,$$
a contradiction. Hence $c = 1$ or $c = 2$.

Let $c = 1$. Then $n = 2^8.3.5.11^d.v$, so that
$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{4}{3}.\frac{6}{5} = 3.09375 > 3,$$
a contradiction.

Let $c = 2$. Taking $c = 2$ (and $b = 1$) in (3.2a) and (3.2b), we obtain
$$n = 2^8.3.5^2.11^d.13^e.w, \qquad (3.5a)$$
and
$$2^5.5.11^{d-1}.13^{e-1}.w = \sigma^{**}(11^d).\sigma^{**}(13^e).\sigma^{**}(w), \qquad (3.5b)$$
where
$$w \text{ has not more than three odd prime factors.} \qquad (3.5c)$$
By Lemma 2.1, for $d \geq 5$, $\frac{\sigma^{**}(11^d)}{11^d} \geq \frac{1947386}{1771561}$; and for $e \geq 3$, $\frac{\sigma^{**}(13^e)}{13^e} \geq \frac{30772}{28561}$. Hence when $d \geq 5$ and $e \geq 3$, from (3.5a),
$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{4}{3}.\frac{26}{25}.\frac{1947386}{1771561}.\frac{30772}{28561} = 3.175525149 > 3,$$
a contradiction.

Let $d \geq 5$. Then $e = 1$ or $e = 2$.

If $d \geq 5$ and $e = 1$, from (3.5a) we have
$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{4}{3}.\frac{26}{25}.\frac{1947386}{1771561}.\frac{14}{13} = 3.174080416 > 3,$$
a contradiction.

Let $d \geq 5$ and $e = 2$. We have $\sigma^{**}(13^2) = 170 = 2.5.17$. Taking $e = 2$ in (3.5b), we obtain
$$2^4.11^{d-1}.13.w = 17.\sigma^{**}(11^d).\sigma^{**}(w). \qquad (3.5d)$$

From (3.5d), $17|w$. Let $w = 17^f.w'$. Hence from (3.5a) and (3.5d), we obtain

$$n = 2^8.3.5^2.11^d.13^2.17^f.w' \quad (d \geq 5), \tag{3.6a}$$

and

$$2^4.11^{d-1}.13.17^{f-1}.w' = \sigma^{**}(11^d).\sigma^{**}(17^f).\sigma^{**}(w'), \tag{3.6b}$$

where

$$w' \text{ has not more than two odd prime factors.} \tag{3.6c}$$

By Lemma 2.1, for $f \geq 3$, $\frac{\sigma^{**}(17^f)}{17^f} \geq \frac{88452}{83521}$. Hence from (3.6$a$), for $f \geq 3$ and $d \geq 5$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{4}{3}.\frac{26}{25}.\frac{1947386}{1771561}.\frac{170}{169}.\frac{88452}{83521} = 3.139839369 > 3,$$

a contradiction. Hence $f = 1$ or $f = 2$ (under $c = 2, e = 2, d \geq 5$).

Let $f = 1$. From (3.6a) $(f = 1)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{4}{3}.\frac{26}{25}.\frac{1947386}{1771561}.\frac{170}{169}.\frac{18}{17} = 3.139200411 > 3,$$

a contradiction.

Let $f = 2$ (along with $c = 2$, $e = 2$, $d \geq 5$). We have $\sigma^{**}(17^2) = 290 = 2.5.29$. Taking $f = 2$ in (3.6b), we see that 5 divides its right hand side but 5 is not a factor of its left hand side. Hence $f = 2$ is not admissible.

Thus when $c = 2$, we must have $1 \leq d \leq 4$. We now show that none of these choices of d are admissible.

When $d = 1, 3, 4$, we have $3|\sigma^{**}(11^d)$. It now follows from (3.5b) that 3 is a factor of its right hand side but it is not so with respect to its left hand side.

It remains to examine the case $d = 2$. Let $d = 2$. We have $\sigma^{**}(11^2) = 122 = 2.61$. Taking $d = 2$ in (3.5b), we get after simplification

$$2^4.5.11.13^{e-1}.w = 61.\sigma^{**}(13^e).\sigma^{**}(w). \tag{3.6d}$$

From (3.6d), $61|w$. Let $w = 61^f.w'$. From (3.5a) and (3.6d), we obtain

$$n = 2^8.3.5^2.11^2.13^e.61^f.w', \tag{3.7a}$$

and

$$2^4.5.11.13^{e-1}.61^{f-1}.w' = \sigma^{**}(13^e).\sigma^{**}(61^f).\sigma^{**}(w'), \tag{3.7b}$$

where

$$w' \text{ has at most two odd prime factors and } (w', 2.3.5.11.13.61) = 1. \tag{3.7c}$$

We show that if n is as in (3.7a), then $7 \nmid n$. On the contrary we assume that $7|n$ and obtain a contradiction. Suppose that $7|n$. Let $w' = 7^g.w''$, where w'' is prime to 2.3.5.7.11.13.61. From (3.7a) and (3.7b), we obtain

$$n = 2^8.3.5^2.11^2.13^e.61^f.7^g.w'', \tag{3.8a}$$

and

$$2^4.5.11.13^{e-1}.61^{f-1}.7^g.w'' = \sigma^{**}(13^e).\sigma^{**}(61^f).\sigma^{**}(7^g).\sigma^{**}(w'), \tag{3.8b}$$

where

$$w'' \text{ has at most one odd prime factor and } (w'', 2.3.5.7.11.13.61) = 1. \tag{3.8c}$$

Let $g = 1$. We have $\sigma^{**}(7) = 8 = 2^3$. Taking $g = 1$ in (3.8*b*), we see that 2^5 divides its right hand side whereas 2^4 unitarily divides its left hand side. Hence $g \geq 2$.

Let $g = 2$. We have $\sigma^{**}(7^2) = 50 = 2.5^2$. Taking $g = 2$ in (3.8*b*), it follows that 5^2 divides its right hand side but 5 is a unitary divisor of its left hand side. Hence we may assume that $g \geq 3$.

From Lemma 2.1, for $g \geq 3$, $\frac{\sigma^{**}(7^g)}{7^g} \geq \frac{2752}{2401}$. From (3.8*a*), we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{4}{3}.\frac{26}{25}.\frac{122}{121}.\frac{2752}{2401} = 3.098618 > 3,$$

a contradiction. Thus $7 \nmid n$ in (3.7*a*) and (3.7*b*).

We will obtain a contradiction when $d = 2$ by examining the factors of $\sigma^{**}(13^e)$ in (3.7*b*).

If e is odd or $4|e$, then $7|\sigma^{**}(13^e)$. From (3.7*b*), it follows that $7|w'$ and consequently $7|n$. But we proved that $7 \nmid n$. Hence we may assume that $e = 2k$, where k is odd.

First we show that $k = 1$ is not admissible (so that $k \geq 3$).

Assume that $k = 1$. Then $e = 2$, and we have $\sigma^{**}(13^2) = 170 = 2.5.17$. Taking $e = 2$ in (3.7*b*), we obtain

$$2^3.11.13.61^{f-1}.w' = 17.\sigma^{**}(61^f).\sigma^{**}(w'). \tag{3.8d}$$

Hence $17|w'$ so that we may assume that $w' = 17^g.w''$, where w'' is prime to 2.3.5.11.13.17.61. From (3.7*a*) ($e = 2$) and (3.8*d*), we get

$$n = 2^8.3.5^2.11^2.13^2.61^f.17^g.w'', \tag{3.9a}$$

and

$$2^3.11.13.61^{f-1}.17^{g-1}.w'' = \sigma^{**}(61^f).\sigma^{**}(17^g).\sigma^{**}(w''); \tag{3.9b}$$

also,

$$w'' = 1 \text{ or an odd prime power relatively prime to 3.5.11.13.61.17.} \tag{3.9c}$$

By examining the factors of $\sigma^{**}(17^g)$ we will obtain a contradiction to (3.9*b*). This will force us to assume that $k > 1$.

If g is odd or $4|g$, we have $3|\sigma^{**}(17^g)$. From (3.9*b*) it follows that 3 is a factor of its right hand side whereas 3 is not a factor of its left hand side. We may assume that $g = 2\ell$, where ℓ is odd. If $\ell = 1$, then $g = 2$. Note that $\sigma^{**}(17^2) = 290$. Thus, in (3.9*b*) we see that 5 divides its right hand side but 5 cannot be a factor of its left hand side. Thus $\ell \geq 3$. We have

$$\sigma^{**}(17^g) = \left(\frac{17^\ell - 1}{16}\right).(17^{\ell+1} + 1) \quad (\ell \text{ odd and } \ell \geq 3).$$

We note the following:

(1) $16|17^\ell - 1$ but $32 \nmid 17^\ell - 1$, since ℓ is odd. Hence $\frac{17^\ell-1}{16}$ is odd and > 1, since $\ell \geq 3$.

(2) $11|17^\ell - 1 \iff 10|\ell$; $13|17^\ell - 1 \iff 6|\ell$ and $61|17^\ell - 1 \iff 60|\ell$. So, in order that $17^\ell - 1$ is divisible by 11 or 13 or 61, ℓ must be even. Since ℓ is odd, $17^\ell - 1$ is not divisible by 11 or 13 or 61; trivially not divisible by 17.

Thus $\frac{17^\ell-1}{16}$ is odd > 1 and not divisible by 11 or 13 or 17 or 61. From (3.9*b*) it follows that each prime factor of $\frac{17^\ell-1}{16}|\sigma^{**}(17^g)$ is a prime factor of w''. Let $p|\frac{17^\ell-1}{16}$. Then $p|w''$.

Consider $17^{\ell+1} + 1$, where ℓ is odd. We have

(3) $11|17^{\ell+1}+1 \iff \ell+1=5u$. Since $5u$ is odd and $\ell+1$ is even, $11 \nmid 17^{\ell+1}+1$. Similarly, $13|17^{\ell+1}+1 \iff \ell+1=3u$. Hence $13 \nmid 17^{\ell+1}+1$.

(4) $61|17^{\ell+1}+1 \iff \ell+1=30u$. Thus $61|17^{\ell+1}+1$ implies that $17^{30}+1|17^{\ell+1}+1$. Since $5^2|17^{30}+1|17^{\ell+1}+1$, it follows from (3.9*b*) that 5 divides its left hand side. But this is not possible. Hence $61 \nmid 17^{\ell+1}+1$.

Thus $\frac{17^{\ell+1}+1}{2}$ is odd, >1 and not divisible by 11 or 13 or 17 or 61. From (3.9*b*), each prime factor of $\frac{17^{\ell+1}+1}{2}$ should divide w''. Let $q|\frac{17^{\ell+1}+1}{2}$. Then $q|w''$. From (3.9*b*), neither $17^{\ell}-1$ nor $17^{\ell+1}+1$ is divisible by 3. Hence $\frac{17^{\ell}-1}{16}$ and $\frac{17^{\ell+1}+1}{2}$ are relatively prime so that $p \neq q$. It follows that w'' is divisible by two distinct odd primes and this violates (3.9*c*).

Hence $k=1$ is not possible. So we may assume that $k \geq 3$ and $e=2k$, where k is odd and ≥ 3. We have

$$\sigma^{**}(13^e)=\left(\frac{13^k-1}{12}\right).(13^{k+1}+1).$$

We now prove that

(I) $\frac{13^k-1}{12}$ is divisible by an odd prime $p|w'$ and $p>293$;

(II) $\frac{13^{k+1}+1}{2}$ is divisible by an odd prime $q|w'$ and $q>293$,

where w' is given in (3.7*a*) and (3.7*b*).

Proof of (I). Let

$$S_{13}=\{p|13^k-1\ :\ p\in[3,293]-\{3,61\}\ \text{ and }\ ord_p 13\ \text{ is odd}\}.$$

If S_{13} is non-empty, the statement in (I) follows from Lemma 2.5 (a) of Part IV(a), see [5]. We may assume that S_{13} is empty. Since $p \nmid 13^k-1$ if $ord_p 13$ is even, it follows that $p \nmid 13^k-1$ if $p\in[3,293]-\{3,61\}$. The same is true with respect to $\frac{13^k-1}{12}$. We shall now discuss the divisibility of 13^k-1 by $p\in\{3,61\}$.

We have $3|13^k-1$. Further, $9|13^k-1$ implies that $3|\frac{13^k-1}{12}|\sigma^{**}(13^e)$ so that 3 is a factor of the left hand side of (3.7*b*). This cannot happen. Thus $3\|13^k-1$. Hence $\frac{13^k-1}{12}$ is not divisible by 3. Also, since k is odd, $4\|13^k-1$ so that $\frac{13^k-1}{12}$ is odd, >1 and not divisible by 3.

We have $61|13^k-1$ if and only if $3|k$; this implies that $13^3-1|13^k-1$. But $13^3-1=2^2.3^2.61$. Hence $3^2|13^k-1$ and so $3|\frac{13^k-1}{12}|\sigma^{**}(13^e)$. From (3.7*b*) it follows that 3 is a factor of its left hand side but this is false. Hence $61 \nmid 13^k-1$.

Thus $\frac{13^k-1}{12}>1$, is odd and not divisible by any prime in $[3,293]$. Let $p|\frac{13^k-1}{12}$. Then $p>293$. From (3.7*b*), it is clear that $p|w'$.

This completes the proof of (I).

Proof of (II). Let

$$T_{13}=\{q|13^{k+1}+1\ : q\in[3,293]-\{5,17\}\text{ and } s=\frac{1}{2}ord_q 13 \text{ is even}\}.$$

If T_{13} is non-empty, (II) follows immediately from Lemma 2.5 (b) of Part IV(a), see [5]. We may assume that T_{13} is empty. Since $q \nmid 13^{k+1}+1$ when $s=\frac{1}{2}ord_q 13$ is odd, it follows that $13^{k+1}+1$ is not divisible by any prime in $[3,293]-\{5,17\}$.

We may note that $5|13^{k+1}+1 \iff k+1=2u \iff 17|13^{k+1}+1$. Hence if $5 \nmid 13^{k+1}+1$, then $17 \nmid 13^{k+1}+1$. In this case, $13^{k+1}+1$ is not divisible by any prime in $[3,293]$. Hence if $q|13^{k+1}+1$, then $q>293$. Also, from (3.7*b*), it is clear that $q|w'$. Thus (II) holds.

We may assume that $5|13^{k+1}+1$. Then also $17|13^{k+1}+1$. We wish to prove that $13^{k+1}+1$ is not divisible by 5 and 17 alone. On the contrary, assume that this is not the case so that

$$\frac{13^{k+1}+1}{2} = 5^{\alpha}.17^{\beta}.$$

If $\alpha \geq 2$, then $5^2|13^{k+1}+1|\sigma^{**}(13^e)$. From (3.7*b*), it follows that 5^2 is a factor of its left hand side. But this cannot happen. Therefore $\alpha = 1$.

Similarly, if $\beta \geq 2$, then $17^2|13^{k+1}+1$; but this is equivalent to $k+1 = 34u$. Consequently, $13^{34}+1|13^{k+1}+1$ but

$$1021|\frac{13^{34}+1}{2}|\frac{13^{k+1}+1}{2} = 5.17^{\beta},$$

which is impossible. Hence $\beta = 1$. Thus $\frac{13^{k+1}+1}{2} = 5.17$ so that $k = 1$. But $k \geq 3$, a contradiction.

It follows that $\frac{13^{k+1}+1}{2}$ is divisible by an odd prime $q' \notin \{5, 17\}$ and so $q \notin [3, 293]$. Thus $q > 293$ and $q|\frac{13^{k+1}+1}{2}$. From (3.7*b*), it is clear that $q|w'$. Thus (II) holds.

Now, p and q are distinct factors of w' and $p, q > 293$. By (3.7*c*), $w' = p^g.q^h$. From (3.7*a*), we have $n = 2^8.3.5^2.11^2.13^e.61^f.p^g.q^h$. Also, we may assume that $p \geq 307$ and $q \geq 311$. Hence

$$3 = \frac{\sigma^{**}(n)}{n} \leq \frac{495}{256}.\frac{4}{3}.\frac{26}{25}.\frac{122}{121}.\frac{13}{12}.\frac{61}{60}.\frac{307}{306}.\frac{311}{310} = 2.99687138 < 3,$$

a contradiction.

The proof of Lemma 3.2 is complete. □

Note 3.1. Let $n = 2^8.3^2.5^c.11^d.v$, where $(v, 2.3.5.11) = 1$, be a bi-unitary triperfect number. Taking $b = 2$ in (3.1*b*), we obtain after simplification

$$2^7.3.5^{c-2}.11^{d-1}.v = \sigma^{**}(5^c).\sigma^{**}(11^d).\sigma^{**}(v). \tag{3.10}$$

It is clear from (3.10) that $c \geq 2$.

Note 3.2. Since $\sigma^{**}(5^2) = 26 = 2.13$, taking $c = 2$ in (3.10), we obtain

$$2^6.3.11^{d-1}.v = 13.\sigma^{**}(11^d).\sigma^{**}(v). \tag{3.10'}$$

From (3.10'), $13|v$. Let $v = 13^e.w$, where $(w, 2.3.5.11.13) = 1$. Hence we have

$$n = 2^8.3^2.5^2.11^d.13^e.w, \tag{3.10a}$$

and from (3.10'), we obtain

$$2^6.3.11^{d-1}.13^{e-1}.w = \sigma^{**}(11^d).\sigma^{**}(13^e).\sigma^{**}(w), \tag{3.10b}$$

where

w has at most four odd prime factors and is prime to 2.3.5.11.13. (3.10*c*)

Lemma 3.3. *Let* $n = 2^8.3^2.5^2.11.13^e.w$*, where* $(w, 2.3.5.11.13) = 1$*, be a bi-unitary triperfect number. Then* $e = 1$ *and* $w = 7$ *so that* $n = 2^8.3^2.5^2.11.13.7 = 57657600$.

Proof. Taking $d = 1$ in (3.10*a*) and (3.10*b*), we get

$$n = 2^8.3^2.5^2.11.13^e.w \tag{3.10d}$$

and

$$2^4.13^{e-1}.w = \sigma^{**}(13^e).\sigma^{**}(w); \tag{3.10e}$$

w has not more than three odd prime factors.

We distinguish the following cases:

Case 1. Let $e = 1$. Taking $e = 1$ in (3.10*e*), we get

$$2^3.w = 7.\sigma^{**}(w). \tag{3.10f}$$

From (3.10*f*), $7|w$. Let $w = 7^f.w'$, where $(w', 2.3.5.11.13.7) = 1$. Then from (3.10*d*) and (3.10*f*), we get

$$n = 2^8.3^2.5^2.11.13.7^f.w' \tag{3.10g}$$

and

$$2^3.7^{f-1}.w' = \sigma^{**}(7^f).\sigma^{**}(w'). \tag{3.10h}$$

Let $f = 1$. From (3.10*h*), we get $w' = \sigma^{**}(w')$ after simplification and so $w' = 1$. Hence $n = 2^8.3^2.5^2.11.13.7 = 57657600$ is a bi-unitary triperfect number.

Let $f \geq 2$. If $f = 2$, since $\sigma^{**}(7^2) = 50 = 2.5^2$, from (3.10*h*) ($f = 2$), we see that 5 divides its right hand side but 5 is not a factor of its left hand side. Hence $f = 2$ is not admissible.

We may assume that $f \geq 3$. Then by Lemma 2.1, $\frac{\sigma^{**}(7^f)}{7^f} \geq \frac{2752}{2401}$. From (3.10*g*), we obtain

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{12}{11}.\frac{14}{13}.\frac{2752}{2401} = 3.008746356 > 3,$$

a contradiction.

Case 2. Let $e = 2$. Since $\sigma^{**}(13^2) = 170 = 2.5.17$, taking $e = 2$ in (3.10*e*), we see that 5 is a factor of its right hand side but it is not so with respect to its left hand side. Hence $e = 2$ is not admissible.

Case 3. Let $e \geq 3$. We now prove that $7 \nmid n$. On the contrary, let $7|n$ so that $7|w$. Let $w = 7^f.w'$, where w' is relatively prime to $2.3.5.11.13.7$. From (3.10*d*) and (3.10*e*), we get

$$n = 2^8.3^2.5^2.11.13^e.7^f.w' \tag{3.11a}$$

and

$$2^4.13^{e-1}.7^f.w' = \sigma^{**}(13^e).\sigma^{**}(7^f)\sigma^{**}(w'), \tag{3.11b}$$

where

$$w' \text{ has at most two odd prime factors and is prime to } 2.3.5.11.13.7. \tag{3.11c}$$

Since $e \geq 3$, we have $\frac{\sigma^{**}(13^e)}{13^e} \geq \frac{30772}{28561}$. If $f \geq 3$, from (3.11*a*), we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{12}{11}.\frac{30772}{28561}.\frac{2752}{2401} = 3.010115825 > 3,$$

a contradiction. Hence $f = 1$ or $f = 2$.

If $f=1$, again from (3.11a) ($f=1$), we have

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{12}{11}.\frac{30772}{28561}.\frac{8}{7}=3.001365498>3,$$

a contradiction.

Let $f=2$. Since $\sigma^{**}(7^2)=50=2.5^2$, taking $f=2$ in (3.11b), we see that 5 divides its right hand side whereas its left hand side is not divisible by 5, a contradiction.

Hence $7\nmid n$. We now prove that $s\nmid n$, where $s\in\{17,19,23,29\}$. On the contrary, we assume that $s|n$ so that $s|w$. Let $w=s^f.w'$. From (3.10d) and (3.10e), we obtain

$$n=2^8.3^2.5^2.11.13^e.s^f.w',\quad (e\geq 3) \tag{3.12a}$$

and

$$2^4.13^{e-1}.s^f.w'=\sigma^{**}(13^e).\sigma^{**}(s^f).\sigma^{**}(w'), \tag{3.12b}$$

where

$$w' \text{ has at most two odd prime factors and is prime to } 2.3.5.11.13.s.7. \tag{3.12c}$$

We will obtain a contradiction by examining the factors of $\sigma^{**}(13^e)$.

If e is odd or $4|e$, we have $7|\sigma^{**}(13^e)$. In these cases, from (3.12b), it follows that $7|n$. But we proved that $7\nmid n$. Hence we may assume that $e=2k$ and k is odd; also, since $e\geq 3$, clearly, $k\geq 3$. We have

$$\sigma^{**}(13^e)=\left(\frac{13^k-1}{12}\right)(13^{k+1}+1)\quad (k\geq 3, k \text{ odd}).$$

We now prove that

(I) $\frac{13^k-1}{12}$ is divisible by an odd prime $p>29$ and $p|w'$,

(II) $\frac{13^{k+1}+1}{2}$ is divisible by an odd prime $q>29$ and $q|w'$,

(III) p and q are distinct primes.

By replacing the interval $[3,293]$ by the interval $[3,29]$ in Lemma 2.5 of Part IV(a), see [5], we arrive at the following:

Result 3.1. Given that k is odd and ≥ 3. Let $p\neq 13$. Then we have:

(a) If $p\in[3,29]-\{3\}$, $r=ord_p13$ is odd and $p|13^k-1$, then we can find an odd prime $p'>29$.

(b) If $q\in[3,29]-\{5,17\}$, $s=\frac{1}{2}ord_q13$ is even and $q|13^{k+1}+1$, then we can find an odd prime $q'|\frac{13^{k+1}+1}{2}$ and $q'>29$.

Proof of (I). Let

$$S_{13}=\{p|13^k-1 : p\in[3,29]-\{3\} \text{ and } ord_p13 \text{ is odd}\}.$$

If S_{13} is non-empty, the statement in (I) follows from Result 3.1(a) stated above. We may assume that S_{13} is empty. Since $p\nmid 13^k-1$ if ord_p13 is odd, it follows that 13^k-1 is not divisible by any prime $p\in[3,29]$, except for possibly 3. This is true with respect to $\frac{13^k-1}{12}$. Also, $3|13^k-1$ but $9\nmid 13^k-1$, since 3 is not a factor of the left hand side of (3.12b). Hence $\frac{13^k-1}{12}$ is odd, >1 and not divisible by any prime in $[3,29]$. If $p|\frac{13^k-1}{12}$, $p>29$. Also, from (3.12b), $p|w'$. This proves (I).

Proof of (II). Let

$$T_{13} = \{q | 13^{k+1} + 1 : q \in [3, 29] - \{5, 17\} \text{ and } s = \frac{1}{2} ord_q 13 \text{ is even}\}.$$

If T_{13} is non-empty, (II) follows immediately from Result 3.1(b). We may assume that T_{13} is empty. Since $q \nmid 13^{k+1} + 1$ when $s = \frac{1}{2} ord_q 13$ is odd, it follows that $13^{k+1} + 1$ is not divisible by any prime in $[3, 29] - \{5, 17\}$. We may note that $5 | 13^{k+1} + 1 \iff k + 1 = 2u \iff 17 | 13^{k+1} + 1$. We may note that 5 is not a factor of the left hand side of (3.12*b*). Hence $5 \nmid 13^{k+1} + 1$ and so $17 \nmid 13^{k+1} + 1$. It follows that $\frac{13^{k+1}+1}{2}$ is odd, > 1 and not divisible by any prime in $[3, 29]$. If $q | \frac{13^{k+1}+1}{2}$, then $q > 29$ and $q | w'$ from (3.12*b*). This proves (II).

Proof of (III). It is easy to see that $\frac{13^k-1}{12}$ and $\frac{13^{k+1}+1}{2}$ are relatively prime. Hence p and q in (I) and (II) are distinct odd primes. This proves (II).

From (3.12*a*), (3.12*c*), (I) and (II), we have $n = 2^8.3^2.5^2.11.13^e.s^f.p^g.q^h$, where we can assume that $p \geq 31$ and $q \geq 37$. Also, $s \geq 17$. Hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{12}{11}.\frac{17}{16}.\frac{31}{30}.\frac{37}{36} = 2.979719148 < 3,$$

a contradiction.

This proves that n is not divisible by 17 or 19 or 23 or 29.

Consider now the factor $\sigma^{**}(13^e)$ in the equation (3.10*e*). Since $7 \nmid n$, e can neither be odd nor $4|e$. We can assume that $e = 2k$, where k is odd and $k \geq 3$. Using Result 3.1, it is not difficult to show that $\frac{13^k-1}{12}$ and $\frac{13^{k+1}+1}{2}$ are respectively divisible by two distinct odd primes p and q respectively and $p, q > 29$ and both these primes are factors of w in (3.10*e*). We may assume that $p \geq 31$ and $q \geq 37$. In (3.10*e*), w has not more than three odd prime factors. Assuming that w has three odd prime factors, since n is not divisible by 17 or 19 or 23 or 29, we may assume that the possible third prime factor of w, say $r \geq 41$. From (3.10*d*), we have $n = 2^8.3^2.5^2.11.13^e.p^f.q^g.r^h$, so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{12}{11}.\frac{13}{26}.\frac{31}{30}.\frac{37}{36}.\frac{41}{40} = 2.87455259 < 3,$$

a contradiction.

The proof of Lemma 3.3 is complete. □

Lemma 3.4. *Let* $n = 2^8.3^2.5^2.11^d.13^e.w$, *where* $(w, 2.3.5.11.13) = 1$, *be as in (3.10a), satisfying (3.10b) and (3.10c) with* $d \geq 2$. *Then* n *cannot be a bi-unitary triperfect number.*

Proof. We first show that $7 \nmid n$. On the contrary suppose that $7|n$. Hence $7|w$ and let $w = 7^f.w'$. From (3.10*a*), (3.10*b*) and (3.10*c*), we get

$$n = 2^8.3^2.5^2.11^d.13^e.7^f.w' \quad (d \geq 2) \tag{3.13a}$$

and

$$2^6.3.11^{d-1}.13^{e-1}.7^f.w' = \sigma^{**}(11^d).\sigma^{**}(13^e).\sigma^{**}(7^f).\sigma^{**}(w'), \tag{3.13b}$$

where

$$w' \text{ has at most three odd prime factors and } (w', 2.3.5.11.13.7) = 1. \tag{3.13c}$$

As 5 cannot be a factor of the left hand side of (3.13*b*), we can assume that $e \neq 2$ and $f \neq 2$.

By Lemma 2.1, for $d \geq 3$, $\frac{\sigma^{**}(11^d)}{11^d} \geq \frac{15984}{14641}$; for $e \geq 3$, $\frac{\sigma^{**}(13^e)}{13^e} \geq \frac{30772}{28561}$; and for $f \geq 3$, $\frac{\sigma^{**}(7^f)}{7^f} \geq \frac{2752}{2401}$. From (3.13$a$), if $d \geq 3$, $e \geq 3$, and $f \geq 3$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{15984}{14641}.\frac{30772}{28561}.\frac{2752}{2401} = 3.01237738 > 3,$$

a contradiction. Thus $d \geq 3$, $e \geq 3$, and $f \geq 3$ cannot hold simultaneously. Recalling that $d \geq 2$, $e \neq 2$ and $f \neq 2$, the following cases arise:

(i) $d = 2;\ e \geq 3;\ f \geq 3$ (ii) $d \geq 3;\ e = 1;\ f \geq 3$ (iii) $d \geq 3;\ e \geq 3;\ f = 1$

(iv) $d = 2;\ e = 1;\ f \geq 3$ (v) $d = 2;\ e \geq 3;\ f = 1$ (vi) $d \geq 3;\ e = 1;\ f = 1$

(vii) $d = 2;\ e = 1;\ f = 1$.

In each of the above seven cases we obtain a contradiction. First we dispose off the cases (ii), (iii), (v), (vi) and (vii).

(ii) Let $d \geq 3$, $e = 1$ and $f \geq 3$. From (3.13a) ($e = 1$), we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{15984}{14641}.\frac{14}{13}.\frac{2752}{2401} = 3.011006871 > 3,$$

a contradiction.

(iii) Let $d \geq 3$, $e \geq 3$ and $f = 1$. From (3.13a) ($f = 1$), we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{15984}{14641}.\frac{30772}{28561}.\frac{8}{7} = 3.003620469 > 3,$$

a contradiction.

(v), (vii) We can bring (v) and (vii) under the case $d = 2$, $f = 1$. Taking $d = 2$ and $f = 1$ in (3.13a), we get $n = 2^8.3^2.5^2.11^2.13^e.7.w'$. Since $\sigma^{**}(7) = 8$, taking $f = 1$ in (3.13b), we see that $w' = 1$ or $w' = p^\alpha$, where p is an odd prime ≥ 17. Hence we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{122}{121}.\frac{13}{12}.\frac{8}{7}.\frac{17}{16} = 2.963558577 < 3,$$

a contradiction.

(vi) Let $d \geq 3$, $e = 1$ and $f = 1$. Hence from (3.13a) ($e = 1, f = 1$), we obtain

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{15984}{14641}.\frac{14}{13}.\frac{8}{7} = 3.002253944 > 3,$$

a contradiction.

(i), (iv) We cover the cases (i) and (iv) under the case $d = 2$ and $f \geq 3$. Let $d = 2$ and $f \geq 3$. Since $\sigma^{**}(11^2) = 122 = 2.61$, taking $d = 2$ in (3.13b), we obtain

$$2^5.3.11.13^{e-1}.7^f.w' = 61.\sigma^{**}(13^e).\sigma^{**}(7^f).\sigma^{**}(w'). \qquad (3.13d)$$

Hence $61|w'$ and let $w' = 61^g.w''$. Hence from (3.13a) and (3.13d), we get

$$n = 2^8.3^2.5^2.11^2.13^e.7^f.61^g.w'' \quad (f \geq 3) \qquad (3.14a)$$

and

$$2^5.3.11.13^{e-1}.7^f.61^{g-1}.w'' = \sigma^{**}(13^e).\sigma^{**}(7^f).\sigma^{**}(61^g).\sigma^{**}(w''), \qquad (3.14b)$$

where

$$w'' \text{ has at most two odd prime factors and } (w'', 2.3.5.11.13.7.61) = 1. \qquad (3.14c)$$

By examining the factors of $\sigma^{**}(7^f)$ we will obtain a contradiction.

If f is odd or $4|f$, then $8|\sigma^{**}(7^f)$. From (3.14b), it follows that $w'' = 1$. Hence $n = 2^8.3^2.5^2.11^2.13^e.7^f.61^g$, and so

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{122}{121}.\frac{13}{12}.\frac{7}{6}.\frac{61}{60} = 2.89479627 < 3,$$

a contradiction.

We may assume that $f = 2k$ and k is odd. Since $f \geq 3$, we have $k \geq 3$. We claim that (when k is odd and ≥ 3)

(I) $\frac{7^k-1}{6}$ is divisible by a prime $p' > 71$ and $p'|w''$,

(II) $7^{k+1}+1$ is divisible by a prime $q' > 71$ and $q'|w''$,

(III) the primes p' and q' are distinct.

By replacing the intervals $[3, 2520]$ and $[3, 1193]$ in Lemma 2.4 (a) and (b) of Part IV(a) (see [5]) by the interval $[3, 71]$, we arrive at the following.

Result 3.2. Given that k is odd and ≥ 3. Let $p \neq 7$. Then we have:

(a) If $p \in [3, 71] - \{3, 19, 37\}$, $ord_p 7$ is odd and $p|7^k - 1$, then we can find an odd prime $p'|7^k - 1$ and $p' > 71$.

(b) If $q \in [3, 71] - \{5, 13\}$, $\frac{1}{2} ord_p 7$ is even and $q|7^{k+1}+1$, then we can find an odd prime $q'|7^{k+1}+1$ and $q' > 71$.

Proof of (I). Let

$$S_7 = \{p|7^k - 1 : \ p \in [3, 71] - \{3, 19, 37\} \ \text{ and } \ ord_p 7 \ \text{ is odd}\}.$$

If S_7 is non-empty, by Result 3.2(a), the statement in (I) follows immediately. We may assume that S_7 is empty. Since $p \nmid 7^k - 1$ when $ord_p 7$ is even, it follows that $p \nmid 7^k - 1$ for any $p \in [3, 71] - \{3, 19, 37\}$. We shall examine the divisibility of $7^k - 1$ by $p \in \{3, 19, 37\}$.

First we dispose of the case when $p = 37$. We have $37|7^k - 1 \iff 9|k$. Hence $37|7^k - 1$ implies that $7^9 - 1|7^k - 1$. Also, $7^9 - 1 = 2.3^3.19.37.1063$. Hence $\frac{7^k-1}{6}|\sigma^{**}(7^f)$ is divisible by 19, 37 and 1063. From (3.14b), it follows that w'' is divisible by these three prime factors. This contradicts (3.14c). Thus $37 \nmid 7^k - 1$.

Clearly, $3|7^k - 1$. We show that $27 \nmid 7^k - 1$. If $27|7^k - 1$, then $9|\frac{7^k-1}{6}|\sigma^{**}(7^f)$. From (3.14$b$), it follows that $3|w''$. But this is not the case. Hence $27 \nmid 7^k - 1$. Further $9|7^k - 1 \iff 3|k \iff 19|7^k - 1$. Thus if $9 \nmid 7^k - 1$, then $19 \nmid 7^k - 1$ and $3\|7^k - 1$.

Thus if $9 \nmid 7^k - 1$, then it follows that $\frac{7^k-1}{6}$ is divisible by none of the primes in $[3, 71]$. If $p'|\frac{7^k-1}{6}$, then $p' > 71$ and from (3.14b), $p'|w''$. This proves (I) in this case.

We may assume that $9|7^k - 1$ and so $9\|7^k - 1$. Then $19|7^k - 1$. Consider $\frac{7^k-1}{18}$. This is not divisible by any prime in $[3, 71]$ except for 19. We show that $\frac{7^k-1}{18}$ is not divisible by 19 alone. On the other hand, let $\frac{7^k-1}{18} = 19^\alpha$ for some positive integer α. If $\alpha \geq 2$, then $19^2|7^k - 1$; but this is equivalent to $57|k$ and this implies that $7^{57} - 1|7^k - 1$. But $419|\frac{7^{57}-1}{18}\frac{7^k-1}{18} = 19^\alpha$. This is impossible. Hence $\alpha = 1$ and so $\frac{7^k-1}{18} = 19$ or $k = 3$. Hence $f = 2k = 6$. We show that $f = 6$ is not admissible.

Let $f = 6$. We have $\sigma^{**}(7^6) = 2.3.19.1201$. Taking $f = 6$ in (3.14b), we get

$$2^4.11.13^{e-1}.7^6.61^{g-1}.w'' = 19.1201.\sigma^{**}(13^e).\sigma^{**}(61^g).\sigma^{**}(w''). \qquad (3.14d)$$

From (3.14b), w'' is divisible by 19 and 1201. By (3.14c), we have $w'' = 19^h.(1201)^i$. Hence from (3.14a) ($f = 6$) and (3.14d), we get

$$n = 2^8.3^2.5^2.11^2.13^e.7^6.61^g.19^h.(1201)^i \qquad (3.15a)$$

and

$$2^4.11.13^{e-1}.7^6.61^{g-1}.19^{h-1}.(1201)^{i-1} = \sigma^{**}(13^e).\sigma^{**}(61^g).\sigma^{**}(19^h).\sigma^{**}((1201)^i). \qquad (3.15b)$$

We obtain a contradiction by examining the factors of $\sigma^{**}(19^h)$.

If h is odd or $4|h$, then $5|\sigma^{**}(19^h)$. From (3.15b), it follows that 5 should divide its left hand side and this is not possible. We may assume that $h = 2\ell$, where ℓ is odd. We have

$$\sigma^{**}(19^h) = \left(\frac{19^\ell - 1}{18}\right).(19^{\ell+1} + 1).$$

If $\ell = 1$, then $h = 2$ and $\sigma^{**}(19^2) = 362 = 2.181$. Taking $h = 2$ in (3.15b), we see that 181 is a factor of its left hand side. But this is not so. Hence $\ell \geq 3$.

We prove that $\frac{19^\ell-1}{18}|\sigma^{**}(19^h)$ is not divisible by any of the primes $7, 11, 13, 61$ and 1201. This leads to a contradiction in view of (3.15b).

We note that

(a) $11|19^t - 1 \iff 10|t$; (b) $13|19^t - 1 \iff 12|t$; (c) $7|19^t - 1 \iff 6|t$;

(d) $61|19^t - 1 \iff 30|t$; and (e) $1201|19^t - 1 \iff 200|t$.

Thus in order that $19^t - 1$ is divisible by any one of the primes $7, 11, 13, 61$ and 1201, t must be even. Since ℓ is odd, $19^\ell - 1$ is divisible by none of the primes $7, 11, 13, 61$ and 1201; trivially $19^\ell - 1$ is not divisible by 19. Also, $2\|19^\ell - 1$, since ℓ is odd; $27|19^\ell - 1 \iff 3|\ell$; this implies that $19^3 - 1|19^\ell - 1$. But $19^3 - 1 = 2.3^3.127$. Hence $3|\frac{19^3-1}{8}|\frac{19^\ell-1}{18}|\sigma^{**}(19^h)$. From (3.15$b$), it follows that 3 is a factor of its left hand side. But this is not the case. Hence $27 \nmid 19^\ell - 1$. As $9|19^\ell - 1$, it follows that $9\|19^\ell - 1$.

Thus $\frac{19^\ell-1}{18} > 1$, odd and not divisible by $7, 11, 13, 19, 61$ and 1201. Since $\frac{19^\ell-1}{18}$ is a factor of $\sigma^{**}(19^h)$, from (3.15b), this should not happen.

Hence $f = 6$ is not admissible. Thus $\frac{7^k-1}{18}$ is divisible by an odd prime, say $p' \neq 19$ and $p' \notin [3, 71]$. Clearly from (3.14b), $p'|w''$. Thus $p'|\frac{7^k-1}{18}|\frac{7^k-1}{6}$, $p'|w''$ and $p' > 71$. This proves (I).

Proof of (II). Let

$$T_7 = \{q|7^{k+1} + 1 : \; q \in [3, 71] - \{5, 13\} \text{ and } s = \frac{1}{2}ord_q 7 \text{ is even}\}.$$

By the statement in Result 3.2(b), if T_7 is non-empty then (II) holds. We may assume that T_7 is empty. Since $q \nmid 7^{k+1} + 1$ if $s = \frac{1}{2}ord_q 7$ is even, it follows that $7^{k+1} + 1$ is not divisible by any prime in $[3, 71] - \{5, 13\}$.

We now examine the divisibility of $7^{k+1} + 1$ by 5 and 13.

Since $7^{k+1} + 1|\sigma^{**}(7^f)$ and 5 is not a factor of the left hand side of (3.14b), it follows that $5 \nmid 7^{k+1} + 1$. Also, $13|7^{k+1} + 1 \iff k + 1 = 6u$. Hence if $13|7^{k+1} + 1$, then $5|7^6 + 1|7^{k+1} + 1$. We just proved that $5 \nmid 7^{k+1} + 1$. Hence $13 \nmid 7^{k+1} + 1$.

Thus $7^{k+1} + 1$ is not divisible by any prime in $[3, 71]$. Since $\frac{7^{k+1}+1}{2}$ is odd, > 1 and not divisible by any prime in $[3, 71]$, we have if $q'|\frac{7^{k+1}+1}{2}$, then $q' > 71$ and $q'|w''$. This proves (II).

Proof of (III). Since $\frac{7^k-1}{6}$ and $7^{k+1}+1$ are relatively prime, p' and q' are distinct. Thus (III) holds.

We can assume that $p' \geq 73$ and $q' \geq 79$ in (I) and (II). From (3.14c), $w'' = (p')^h.(q')^i$. Hence from (3.14a),

$$n = 2^8.3^2.5^2.11^2.13^e.7^f.61^g.(p')^h.(q')^i.$$

Hence we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{122}{121}.\frac{13}{12}.\frac{7}{6}.\frac{61}{60}.\frac{73}{72}.\frac{79}{78} = 2.972630002 < 3,$$

a contradiction.

Thus the case $f \geq 3$ and $d = 2$ is not possible.

The proof that $7 \nmid n$ is complete.

To complete the proof of Lemma 3.4, we require the following modification of Lemma 2.5 of Part IV(a) [5], which can be proved easily proceeding in the same way as in [5] :

Result 3.3. Let k be odd and $k \geq 3$. Let $p \neq 13$.

(a) If $p \in [3, 443] - \{3, 61\}$, $r = ord_p 13$ is odd and $p|13^k - 1$, then we can find a prime p' (depending on p) such that $p'|\frac{13^k-1}{12}$ and $p' > 443$.

(b) If $q \in [3, 443] - \{5, 17\}$, $s = \frac{1}{2} ord_q 13$ is even and $q|13^{k+1} + 1$, then we can find a prime q' (depending on q) such that $q'|\frac{13^{k+1}+1}{2}$ and $q' > 443$.

We continue proving Lemma 3.4. We claim that if w is given as in (3.10a),

(A) $\frac{13^k-1}{12}$ is divisible by an odd prime $p' > 443$ and $p'|w$,

(B) $\frac{13^{k+1}+1}{2}$ is divisible by an odd prime $q' > 443$ and $q'|w$,

and p' and q' are distinct.

Proof of (A). Let

$$S'_{13} = \{p|13^k - 1 : p \in [3, 443] - \{3, 61\} \text{ and } r = ord_p 13 \text{ is odd}\}.$$

If S'_{13} is non-empty, then (A) holds by (a) of Result 3.3. We may assume that S'_{13} is empty. Since $p \nmid 13^k - 1$ if $ord_p 13$ is even, it follows that $p \nmid 13^k - 1$ if $p \in [3, 443]$, except for possibly $p \in \{3, 61\}$.

Clearly, $3|13^k - 1$. We note that $9|13^k - 1 \iff 3|k \iff 61|13^k - 1$. Suppose that $9 \nmid 13^k - 1$ so that $61 \nmid 13^k - 1$. Also, in this case $3\|13^k - 1$. Hence $\frac{13^k-1}{12}$ is not divisible by any prime in $[3, 443]$. Also, $\frac{13^k-1}{12}$ is odd and > 1. Let $p'|\frac{13^k-1}{12}$. Then $p' > 443$ and from (3.10b), $p'|w$. This proves (A) in this case.

Suppose that $9|13^k - 1$ and so $61|13^k - 1$. Also, $27 \nmid 13^k - 1$; if this is not so, then $9|\frac{13^k-1}{12}|\sigma^{**}(13^e)$. Hence $3|w$ from (3.10b). This is not possible. Thus $3\|\frac{13^k-1}{12}$ and as a consequence $\frac{13^k-1}{36}$ is odd, > 1 and not divisible by 3 but divisible by 61.

We wish to show that $\frac{13^k-1}{36}$ must be divisible by an odd prime $p' \neq 61$. On the contrary, let $\frac{13^k-1}{36} = 61^\alpha$, for some positive integer α. If $\alpha \geq 2$, then $61^2|13^k - 1$; this holds if and only if $183|k$. Hence $61|183|k$ and so $13^{61} - 1|13^k - 1$. But $4027|\frac{13^{61}-1}{36}|\frac{13^k-1}{36} = 61^\alpha$, which is impossible. Hence $\alpha = 1$ and $\frac{13^k-1}{36} = 16$ or $k = 3$. So $e = 6$.

We now prove that $e = 6$ is not admissible in (3.10b).

Let $e = 6$. We have $\sigma^{**}(13^6) = 2.3.61.14281$. Taking $e = 6$ in (3.10*b*), we get

$$2^5.11^{d-1}.13^5.w = 61.14281.\sigma^{**}(11^d).\sigma^{**}(w). \qquad (3.15c)$$

From (3.15*c*), w is divisible by 61 and 14281. Let $w = 61^f.(14281)^g.w'$. From (3.10*a*) and (3.15*c*), we obtain (when $e = 6$),

$$n = 2^8.3^2.5^2.11^d.13^6.61^f.(14281)^g.w' \qquad (3.16a)$$

and

$$2^5.11^{d-1}.13^5.61^{f-1}.(14281)^{g-1}.w' = \sigma^{**}(11^d).\sigma^{**}(61^f).\sigma^{**}((14281)^g).\sigma^{**}(w'). \qquad (3.16b)$$

where

$$w' \text{ has at most two odd prime factors and } (w', 2.3.5.7.11.13.61.14281) = 1; \qquad (3.16c)$$

note that w' is prime to 7 since we proved that $7 \nmid n$ when n is given by (3.10*a*).

We examine $\sigma^{**}(11^d)$ in (3.16*b*) to obtain a contradiction to $e = 6$.

If d is odd or $4|d$, then $\sigma^{**}(11^d)$ is divisible by 3. It follows from (3.16*b*) that this is not possible as $3 \nmid w'$.

We may assume that $d = 2\ell$, where ℓ is odd.

Let $\ell = 1$ so that $d = 2$. From (3.16*a*) ($d = 2$), we have $n = 2^8.3^2.5^2.11^2.13^6.61^f.(14281)^g.w'$ and w' cannot have more than two odd prime factors. We may assume that $w' = p_1^h.p_2^i$, where $p_1 \geq 17$ and $p_2 \geq 19$. Hence $n = 2^8.3^2.5^2.11^2.13^6.61^f.(14281)^g.p_1^h.p_2^i$ and so we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{122}{121}.\frac{13}{12}.\frac{61}{60}.\frac{14281}{14280}.\frac{17}{16}.\frac{19}{18} = 2.782990097 < 3,$$

a contradiction.

Hence $\ell \geq 3$, since ℓ is odd. We have

$$\sigma^{**}(11^d) = \left(\frac{11^\ell - 1}{10}\right).(11^{\ell+1} + 1) \quad (\ell \geq 3 \text{ and odd}).$$

We prove that

(C) $\frac{11^\ell - 1}{10}$ is divisible by a prime $p' > 23$ and $p'|w'$,

(D) $11^{\ell+1} + 1$ is divisible by a prime $q' > 23$ and $q'|w'$,

and $p' \neq q'$.

Proof of (C). We have

(1) $2 \| 11^\ell - 1$ and $3 \nmid 11^\ell - 1$, since ℓ is odd.

(2) Since 5 is not a factor of the left hand side of (3.16*b*), it follows that $5 \nmid \frac{11^\ell - 1}{10} | \sigma^{**}(11^d)$.

(3) From (1) and (2), $\frac{11^\ell - 1}{10}$ is odd, > 1 (since $\ell \geq 3$) and not divisible by 3 and 5. The left hand side of (3.16*b*) is not divisible by 7. Hence $7 \nmid \frac{11^\ell - 1}{10}$. Also, $7 | 11^\ell - 1 \iff 3|\ell \iff 19 | 11^\ell - 1$. So, $19 \nmid \frac{11^\ell - 1}{10}$.

(4) For any positive integer t, we have (i) $13 | 11^t - 1 \iff 12|t$; (ii) $17 | 11^t - 1 \iff 16|t$ and (iii) $23 | 11^t - 1 \iff 22|t$. In order that $11^t - 1$ is divisible by 13 or 17 or 23, the number t must be even. Since ℓ is odd, we conclude that $11^\ell - 1$ is not divisible by 13 or 17 or 23. Trivially, $11 \nmid 11^\ell - 1$.

From (3) and (4), it follows that $\frac{11^\ell-1}{10} > 1$, is odd and not divisible by any prime in $[3,23]$. Hence every prime factor of $\frac{11^\ell-1}{10}$ is greater than 23. Also, $61|11^\ell - 1 \iff 4|\ell$. But ℓ is odd; hence $61 \nmid 11^\ell - 1$.

Further, $14281|11^\ell - 1 \iff 1785|\ell$. Since $105|1785$, we can conclude that if $14281|11^\ell - 1$, then $11^{105} - 1|11^\ell - 1$. But $7^2|11^{105} - 1$. It follows that $7|w'$ which is not possible. Hence $14281 \nmid 11^\ell - 1$.

From (3.16*b*), it now follows that if $p'|\frac{11^\ell-1}{10}$, then $p' > 23$ and $p'|w'$. This proves (C).

Proof of (D). We note that

(5) $\frac{11^{\ell+1}+1}{2}$ is odd, > 1 and not divisible by 3, 5 and 7, since these are not factors of the left hand side of (3.16*b*).

(6) For any positive integer t, $11^t + 1$ is not divisible by 19. The same is true with respect to $11^{\ell+1} + 1$.

(7) $13|11^{\ell+1} + 1 \iff \ell + 1 = 6u$; hence $13|11^{\ell+1} + 1$ implies that $11^6 + 1|11^{\ell+1} + 1$. But $11^6 + 1 = 2.13.61.1117$. From (3.16*b*), it follows that $1117|w'$. Since w' is divisible by not more than two odd primes, we can assume that $w' = (1117)^h.s^i$, where s is prime ≥ 17. From (3.16*a*), we have $n = 2^8.3^2.5^2.11^d.13^6.61^f.(14281)^g.(1117)^h.s^i$ and hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{11}{10}.\frac{13}{12}.\frac{61}{60}.\frac{14281}{14280}.\frac{1117}{1116}.\frac{17}{16} = 2.87897417 < 3,$$

a contradiction. Hence $13 \nmid 11^{\ell+1} + 1$.

(8) $17|11^{\ell+1} + 1 \iff \ell + 1 = 8u$. Hence $17|11^{\ell+1} + 1$ implies that $11^8 + 1|11^{\ell+1} + 1$. Also, $11^8 + 1 = 2.17.6304673$. Hence from (3.16*b*), 17 and 6304673 are factors of w'. From (3.16*c*), $w' = 17^h.(6304673)^i$. From(3.16*a*), $n = 2^8.3^2.5^2.11^d.13^6.61^f.(14281)^g.17^h.(6304673)^i$ and so we have (using $6304673 > 1117$),

$$\frac{\sigma^{**}(n)}{n} < \frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{11}{10}.\frac{13}{12}.\frac{61}{60}.\frac{14281}{14280}.\frac{17}{16}.\frac{1117}{1116} = 2.87897417 < 3,$$

a contradiction. Hence $17 \nmid 11^{\ell+1} + 1$.

(9) $23|11^{\ell+1} + 1 \iff \ell + 1 = 11u$. Since $\ell + 1$ is even, it follows that $23 \nmid 11^{\ell+1} + 1$.

(10) $14281 \nmid 11^t + 1$ for any positive integer t. In particular, $14281 \nmid 11^{\ell+1} + 1$.

(11) If $61 \nmid \frac{11^{\ell+1}+1}{2}$, then $\frac{11^{\ell+1}+1}{2}$ is not divisible by any prime in $[3,23] \cup \{61, 14281\}$. Hence every prime $q'|\frac{11^{\ell+1}+1}{2}$ divides w' and $q' > 23$. Thus (D) is true in this case.

(12) Suppose that $61|11^{\ell+1} + 1$. We claim that $\frac{11^{\ell+1}+1}{2}$ must be divisible by an odd prime $q' \neq 61$. On the contrary, let $\frac{11^{\ell+1}+1}{2} = 61^\alpha$ for some positive integer α. If $\alpha \geq 2$, then $61^2|11^{\ell+1} + 1$. But this is equivalent to $\ell + 1 = 122u$; hence $733|\frac{11^{122}+1}{2}|\frac{11^{\ell+1}+1}{2} = 61^\alpha$, which is impossible. Hence $\alpha = 1$ and $\frac{11^{\ell+1}+1}{2} = 61$ or $\ell = 1$. But $\ell \geq 3$. This contradiction proves that $\frac{11^{\ell+1}+1}{2}$ is divisible by an odd prime $q' \neq 61$. It follows that $q' \notin [3,23] \cup \{61, 14281\}$ and therefore $q' > 23$ and $q'|w'$. This proves (D) completely.

Also, $p' \neq q'$, since $\frac{11^\ell-1}{10}$ and $11^{\ell+1} + 1$ are relatively prime. Without loss of generality we can assume that $p' \geq 29$ and $q' \geq 31$.

We continue the case $e = 6$ to end up with a contradiction. From (3.16*c*), since p' and q' are odd prime factors of w', we must have $w' = (p')^h.(q')^i$. Hence from (3.16*a*), $n = 2^8.3^2.5^2.11^d.13^6.61^f.(14281)^g.(p')^h.(q')^i$, and we have

$$3=\frac{\sigma^{**}(n)}{n}<\frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{11}{10}.\frac{13}{12}.\frac{61}{60}.\frac{14281}{14280}.\frac{29}{28}.\frac{31}{30}=2.897345302<3,$$

a contradiction. Hence $e=6$ is not possible.

We continue the proof of (A) after Result 3.3. It now follows that $\frac{13^k-1}{36}$ is divisible by an odd prime $p'\neq 61$. Hence $p'\notin[3,443]$ and so $p'>443$. Also, from (3.10*b*), $p'|w$. This proves (A) completely.

Proof of (B). Let

$$T'_{13}=\{q|13^{k+1}+1:\ q\in[3,443]-\{5,17\}\ \text{ and }\ s=\frac{1}{2}ord_q 13\ \text{ is even}\}.$$

If T'_{13} is non-empty, then (B) holds by Result 3.3(b). We may assume that T'_{13} is empty. Since $q\nmid 13^{k+1}+1$ if $s=\frac{1}{2}ord_q 13$ is odd, it follows that $13^{k+1}+1$ is not divisible by any prime $q\in[3,443]$, except for possibly $q\in\{5,17\}$.

We note that $5|13^{k+1}+1\iff k+1=2u\iff 17|13^{k+1}+1$. Since 5 is not a factor of the left hand side of (3.10*b*), it follows that $5\nmid 13^{k+1}+1$. Hence $17\nmid 13^{k+1}+1$.

Thus $13^{k+1}+1$ is not divisible by any prime in $[3,443]$. The same is true with respect to $\frac{13^{k+1}+1}{2}$ which is odd and >1. If $q'|\frac{13^{k+1}+1}{2}$, then $q'>443$ and $q'|w$ from (3.10*b*). This proves (B).

Also, since $\frac{13^k-1}{12}$ is relatively prime to $13^{k+1}+1$, we have $p'\neq q'$.

Completion of proof of Lemma 3.4. We may assume that $p'\geq 449$ and $q'\geq 457$. From (3.10*c*), w has not more than four odd prime factors. Possibly w may have two more odd prime factors apart from p' and q'. If p_1 and p_2 denote these two possible odd prime factors (of w), since w is prime to 2.3.5.11.13 and we already proved that $7\nmid n$, we can assume that $p_1\geq 17$ and $p_2\geq 19$. Thus $n=2^8.3^2.5^2.11^d.13^e.(p')^f.(q')^g.p_1^h.p_2^i$, and hence we have

$$3=\frac{\sigma^{**}(n)}{n}<\frac{495}{256}.\frac{10}{9}.\frac{26}{25}.\frac{11}{10}.\frac{13}{12}.\frac{449}{448}.\frac{457}{456}.\frac{17}{16}.\frac{19}{18}=2.999442728<3,$$

a contradiction.

The proof of Lemma 3.4 is complete. □

Completion of proof of Theorem 3.1. Follows from Lemmas 3.1 to 3.4. □

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