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The abundancy index of divisors of odd perfect numbers – Part II

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Abstract: In this note, we show that if $N = q^k n^2$ is an odd perfect number with special prime q, and N is not divisible by 3, then the inequality $q < n$ holds. We then give another unconditional proof for the inequality $q < n$ which is independent of the results of Brown and Starni.
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1 Introduction

If J is a positive integer, then we write $\sigma(J)$ for the sum of the divisors of J. A positive integer L is *perfect* if $\sigma(L) = 2L$. We denote the *abundancy index* I of the positive integer x as $I(x) = \sigma(x)/x$.

An even perfect number M is said to be given in *Euclidean form* if $M = (2^p - 1) \cdot 2^{p-1}$, where p and $2^p - 1$ are primes. We call $M_p = 2^p - 1$ the *Mersenne prime* factor of M. Currently, there are only 51 known Mersenne primes [12], which correspond to 51 even perfect numbers.

An odd perfect number N is said to be given in *Eulerian form* if $N = q^k n^2$, where q is prime with $q \equiv k \equiv 1 \pmod 4$ and $\gcd(q, n) = 1$. We call q^k the *Euler part* of N while n^2 is called the *non-Euler part* of N. (We will call q the *special prime* factor of N.)

It is currently unknown whether there are infinitely many even perfect numbers, or whether any odd perfect numbers exist. It is widely believed that there is an infinite number of even perfect numbers. On the other hand, no examples for an odd perfect number have been found (despite extensive computer searches), nor has a proof for their nonexistence been established.

Ochem and Rao [9] recently proved that $N > 10^{1500}$. Nielsen [8] obtained the lower bound $\omega(N) \geq 10$ for the number of *distinct* prime factors of N, improving on his last result $\omega(N) \geq 9$ (see [7]).

After testing large numbers $X = t^r s^2$ with $\omega(X) = 8$ for perfection, Sorli conjectured in [10] that $r = \nu_t(X) = 1$ always holds. (More recently, Beasley [2] points out that Descartes was the first to conjecture $k = \nu_q(N) = 1$ "in a letter to Marin Mersenne in 1638, with Frenicle's subsequent observation occurring in 1657".) Dris conjectured in [4] and [5] that the divisors q^k and n are related by the inequality $q^k < n$. This conjecture was made on the basis of the result $I(q^k) < \sqrt[3]{2} < I(n)$.

In a recent preprint [3], Brown claims a proof for $q < n$ "(and mildly stronger results) for all odd perfect numbers". Starni proves in [11] and Dris shows in [6] that $q < n$ holds unconditionally. The proof for $3 \nmid n \Rightarrow q < n$ that we present here is independent of Brown's and Starni's. We use the following lemma (first proved in [6]) to prove the implication $3 \nmid n \Rightarrow q < n$ in this paper.

Lemma 1.1. *If $N = q^k n^2$ is an odd perfect number given in Eulerian form, then the biconditionals*

$$q^k < n \iff \sigma(q^k) < \sigma(n) \iff \frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}$$

hold.

2 Preparations

The following result was communicated to the second author (via e-mail, by Pascal Ochem) on April 17, 2013.

Theorem 2.1. *If $N = q^k n^2$ is an odd perfect number given in Eulerian form, then*

$$I(n) > \left(\frac{8}{5}\right)^{\frac{\ln(4/3)}{\ln(13/9)}} \approx 1.44440557.$$

The proof of Theorem 2.1 uses the following lemma.

Lemma 2.2. *Let $n > 1$, $L(n) = \ln(I(n))$, and $x(n) = L(n^2)/L(n)$. If $\gcd(a, b) = 1$, then*

$$\min(x(a), x(b)) < x(ab) < \max(x(a), x(b)).$$

Proof. First, note that $x(a) \neq x(b)$ (since $\gcd(a, b) = 1$). Without loss of generality, we may assume that $x(a) < x(b)$. Thus, since

$$x(n) = \frac{L(n^2)}{L(n)},$$

we have

$$\frac{L(a^2)}{L(a)} < \frac{L(b^2)}{L(b)}.$$

This implies that

$$L(a^2)L(b) < L(b^2)L(a).$$

Adding $L(a^2)L(a)$ to both sides of the last inequality, we get

$$L(a^2)\Big(L(b) + L(a)\Big) < L(a)\Big(L(b^2) + L(a^2)\Big).$$

Using the identity $L(X) + L(Y) = L(XY)$, we can rewrite the last inequality as

$$L(a^2)L(ab) < L(a)L((ab)^2)$$

since $I(x)$ is a multiplicative function of x and $\gcd(a, b) = 1$. It follows that

$$\min(x(a), x(b)) = x(a) = \frac{L(a^2)}{L(a)} < \frac{L((ab)^2)}{L(ab)} = x(ab).$$

Under the same assumption $x(a) < x(b)$, we can show that

$$x(ab) < x(b) = \max(x(a), x(b))$$

by adding $L(b^2)L(b)$ (instead of $L(a^2)L(a)$) to both sides of the inequality

$$L(a^2)L(b) < L(b^2)L(a).$$

This finishes the proof. □

Remark 2.3. *From Lemma 2.2, we note that* $1 < x(n) < 2$ *follows from* $I(n) < I(n^2) < (I(n))^2$ *and* $I(n^2) = (I(n))^{x(n)}$.

The trivial lower bound for $I(n)$ *is*

$$\left(\frac{8}{5}\right)^{1/2} < I(n).$$

This bound is more general than the stronger bound as given in Theorem 2.1 (which is also unconditional). Note that decreasing the denominator in the exponent gives an increase in the lower bound for $I(n)$.

Remark 2.4. *We sketch a proof for Theorem 2.1 here, as communicated to the second author by Pascal Ochem.*

Suppose that $N = q^k n^2$ *is an odd perfect number given in Eulerian form.*

We want to obtain a lower bound on $I(n)$. *We know that*

$$I(n^2) = 2/I(q^k) > 2/(5/4) = 8/5.$$

(Please see the paper by Dris [5] for a proof of the upper bound $I(q^k) < 5/4$.*)*

We need to improve the trivial bound $I(n^2) < (I(n))^2$.

Let $x(n)$ *be such that*

$$I(n^2) = \Big(I(n)\Big)^{x(n)}.$$

That is, $x(n) = L(n^2)/L(n)$. *We want an upper bound on* $x(n)$ *for* n *odd. By Lemma 2.2, we consider the component* r^s *with* r *prime that maximizes* $x(r^s)$.

We have

$$I(r^s) = \frac{r^{s+1}-1}{r^s(r-1)} = 1 + \frac{1}{r-1} - \frac{1}{r^s(r-1)}.$$

Also,

$$I(r^{2s}) = \frac{r^{2s+1}-1}{r^{2s}(r-1)} = I(r^s)\Bigg(1 + \left(\frac{1-r^{-s}}{r^{s+1}-1}\right)\Bigg).$$

So,

$$x(r^s) = \frac{L(r^{2s})}{L(r^s)} = \frac{L(r^s) + \ln(1 + \left(\frac{1-r^{-s}}{r^{s+1}-1}\right))}{L(r^s)},$$

from which it follows that

$$x(r^s) = 1 + \frac{\ln(1 + \left(\frac{1-r^{-s}}{r^{s+1}-1}\right))}{\ln(1 + \frac{1}{r-1} - \frac{1}{r^s(r-1)})}.$$

We can check that

$$x(r^s) > x(r^t)$$

if $s < t$ and $r \geq 3$. Therefore, $x(r^s)$ is maximized for $s = 1$.

Now,

$$x(r) = 1 + \frac{\ln(1 + (1/(r(r+1))))}{\ln(1 + (1/r))} = \frac{\ln(1 + (1/r) + (1/r)^2)}{\ln(1 + (1/r))} = L(r^2)/L(r),$$

which is maximized for $r = 3$. So,

$$x(3) = L(3^2)/L(3) = \ln(13/9)/\ln(4/3) \approx 1.27823.$$

The claim in Theorem 2.1 then follows, and the proof is complete.

The argument in Remark 2.4 can be improved to account for the divisibility of n by primes r other than 3, whereby $x(r) \leq x(5) < x(3)$. We outline an attempt on such an improvement in the next section.

The following claim follows from Acquaah and Konyagin's estimate for the prime factors of an odd perfect number [1].

Lemma 2.5. *If $N = q^k n^2$ is an odd perfect number given in Eulerian form, then $q < n\sqrt{3}$.*

Proof. First, we observe that we have the estimate $q < n$ when $k > 1$ (see [5]).

Now assume that $k = 1$. By Acquaah and Konyagin's estimate for the Euler prime q:

$$q < (3N)^{1/3} \Longrightarrow q^3 < 3N = 3q^k n^2 \Longrightarrow q^2 = q^{3-k} < 3n^2.$$

It follows that $q < n\sqrt{3}$. □

2.1 Proof of Lemma 1.1

Lastly, we have the following result.

Lemma 2.6. *If $N = q^k n^2$ is an odd perfect number given in Eulerian form with $\sigma(q^k) \neq \sigma(n)$, then the following biconditionals are true:*

$$q^k < n \Longleftrightarrow \sigma(q^k) < \sigma(n) \Longleftrightarrow \frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}.$$

Proof. Note that we have the bounds

$$\frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < 2 \cdot \left(\frac{q^k}{n} + \frac{n}{q^k}\right),$$

so that the sum

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

is bounded if and only if the sum

$$\frac{q^k}{n} + \frac{n}{q^k}$$

is bounded. In general, since the sum $\frac{q^k}{n} + \frac{n}{q^k}$ is not bounded, we do not expect the sum $\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$ to be bounded.

Next, we can show that:

If $N = q^k n^2$ is an odd perfect number given in Eulerian form, then

$$I(q^k) + I(n) < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

holds if and only if $q^k < n \iff \sigma(q^k) < \sigma(n)$.

Similarly, we can prove that:

If $N = q^k n^2$ is an odd perfect number given in Eulerian form, then

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < I(q^k) + I(n)$$

holds if and only if $q^k < n \iff \sigma(n) < \sigma(q^k)$.

Since

$$I(q^k) + I(n) = \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

is true if and only if $\sigma(q^k) = \sigma(n)$ holds, and since assuming

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < I(q^k) + I(n)$$

implies that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < I(q^k) + I(n) < I(q^k) + I(n^2) < 3,$$

which contradicts the remarks above, if $\sigma(q^k) \neq \sigma(n)$ then the following inequality must be true

$$I(q^k) + I(n) < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}.$$

We then obtain

$$q^k < n \iff \sigma(q^k) < \sigma(n),$$

which further implies that the biconditionals

$$q^k < n \iff \sigma(q^k) < \sigma(n) \iff \frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}$$

hold. □

3 Main results

First, observe that if we have $n < q$, then $k = 1$ (by [5]) and therefore that $q > 10^{500}$ (by [9]). Second, we prove a preliminary result.

Lemma 3.1. *If $N = q^k n^2$ is an odd perfect number given in Eulerian form (with smallest prime factor u satisfying $u \geq 5$), then $q + 1 \neq \sigma(n)$.*

Proof. Let $N = q^k n^2$ be an odd perfect number given in Eulerian form, with smallest prime factor u satisfying $u \geq 5$. Suppose to the contrary that $\sigma(q) = q + 1 = \sigma(n)$. (Observe that this assumption implies that $n < q$, from which follows that $k = 1$ (see [5]); N then takes the form $N = qn^2$.)

By Remark 2.4 and Lemma 2.5, we obtain

$$2.799 \approx 1 + 2^{\frac{\log(6/5)}{\log(31/25)}} \leftarrow \frac{q+1}{q} + \left(\frac{2q}{q+1}\right)^{\frac{\log(I(5))}{\log(I(5^2))}} \leq I(q) + \left(I(n^2)\right)^{\frac{\log(I(u))}{\log(I(u^2))}}$$

$$< \frac{\sigma(q)}{q} + \frac{\sigma(n)}{n} = \frac{\sigma(q)}{n} + \frac{\sigma(n)}{q} = \frac{\sigma(q)}{n} + \frac{\sigma(q)}{q} < \sqrt{3} \cdot \left(1 + 10^{-500}\right) + \left(1 + 10^{-500}\right) \approx 2.732,$$

which is a contradiction. □

We are now ready to prove our main result in this note.

Theorem 3.2. *If $N = q^k n^2$ is an odd perfect number given in Eulerian form (with smallest prime factor u satisfying $u \geq 5$), then the inequality $q < n$ is true.*

Proof. Let $N = q^k n^2$ be an odd perfect number given in Eulerian form (with smallest prime factor at least 5), and suppose to the contrary that $n < q$. This implies that

$$k = 1.$$

By Lemma 3.1, $q + 1 \neq \sigma(n)$, so that Lemma 2.6 applies. Since $n < q$ and $k = 1$, we have either of the inequalities

$$n < \sigma(n) \leq q < \sigma(q), \quad \text{or}$$
$$n < q < \sigma(n) < \sigma(q).$$

Since $\sigma(q) = q + 1$, the second case cannot occur. This then gives $\frac{\sigma(n)}{q} \leq 1$ under the first case. We now get an upper bound for $\sigma(q)/n$. Using Theorem 2.1 and Lemma 2.5, we obtain

$$\frac{\sigma(q)}{n} = \frac{q}{n} \cdot I(q) < \sqrt{3} \cdot \left(1 + 10^{-500}\right) \approx \sqrt{3},$$

Consequently,

$$\frac{\sigma(q)}{n} + \frac{\sigma(n)}{q} < 1 + \sqrt{3} \approx 2.732.$$

Again, by Lemma 2.6, we have the lower bound

$$I(q) + I(n) < \frac{\sigma(q)}{n} + \frac{\sigma(n)}{q}.$$

Under the assumption that the smallest prime factor u of $N = qn^2$ satisfies $u \geq 5$, from Remark 2.4, we get

$$\frac{q+1}{q} + \left(\frac{2q}{q+1}\right)^{\frac{\log(I(5))}{\log(I(5^2))}} \leq I(q) + I(n^2)^{\frac{\log(I(u))}{\log(I(u^2))}} \leq I(q) + I(n)$$

so that

$$\lim_{q\to\infty}\left(\frac{q+1}{q} + \left(\frac{2q}{q+1}\right)^{\frac{\log(6/5)}{\log(31/25)}}\right) = 1 + 2^{\frac{\log(6/5)}{\log(31/25)}} \approx 2.799.$$

This is a contradiction. □

Remark 3.3. *We can, in fact, prove $q^k < 2n$ by considering all the possible cases arising from Lemma 2.6 and using the ideas in Theorem 3.2. The details for this result will appear in a sequel.*

4 Concluding remarks

Although we already know that $q < n$ holds unconditionally by Brown [3], Dris [6] and Starni [11], the proof for the implication $3 \nmid n \Rightarrow q < n$ that we present here is essentially an academic exercise. This is to show an alternative proof for the same fact using a different approach/methodology. It is implicit in Dris's approach that the use of the observation

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} \text{ is bounded if and only if } \frac{q^k}{n} + \frac{n}{q^k} \text{ is bounded.}$$

(so that we do not expect the sum

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

to be bounded) is unavoidable. Besides, notice that in the proof of Theorem 3.2, if we had the lower bound $u \geq 3$ for the smallest prime factor u of an odd perfect number $N = q^k n^2$, then under the assumption $n < q$, we have $k = 1$, and as $q \to \infty$:

$$2.7199 \approx 1 + 2^{\frac{\log(4/3)}{\log(13/9)}} \leftarrow \frac{q+1}{q} + \left(\frac{2q}{q+1}\right)^{\frac{\log(I(3))}{\log(I(3^2))}}$$

$$\leq I(q) + I(n^2)^{\frac{\log(I(u))}{\log(I(u^2))}} \leq I(q) + I(n) \leq \frac{\sigma(q)}{n} + \frac{\sigma(n)}{q} < 1 + \sqrt{3} \approx 2.732,$$

which implies that the sum

$$\frac{\sigma(q)}{n} + \frac{\sigma(n)}{q} = \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

is bounded, contradicting the remarks in the proof of Lemma 2.6. Thus, we conclude by the results of this paper and Dris [6] that indeed the inequality $q < n$ holds unconditionally.

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