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On the spectra of a new duplication based corona of graphs

Renny P. Varghese[1] and D. Susha[2]

[1] Department of Mathematics, Catholicate College
Pathanamthitta, Kerala 689 645, India
e-mail: rennypv1@gmail.com

[2] Department of Mathematics, Catholicate College
Pathanamthitta, Kerala 689 645, India
e-mail: sushad70@gmail.com

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Abstract: In this paper we introduce a new corona-type product of graphs namely *duplication corresponding corona*. Here we mainly determine the adjacency, Laplacian and signless Laplacian spectra of the new graph product. In addition to that we find out the incidence energy, the number of spanning trees, Kirchhoff index and Laplacian-energy-like invariant of the new graph. Also we discuss some new classes of cospectral graphs.

Keywords: Spectrum, Corona, Duplication graph, Incidence energy, Spanning tree, Kirchhoff index, Laplacian-energy-like invariant.
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1 Introduction

The graphs considered in this paper have no loops and parallel edges and are undirected. Let G be an arbitrary graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $A(G)$ be the adjacency matrix of G. It is an $n \times n$ symmetric matrix, $A(G) = (a_{ij})_{n\times n}$, where $a_{ij} = 1$ if v_i and v_j are connected by an edge in G, and 0 elsewhere.

Let the degree of v_i in G be d_i (number of vertices adjacent to v_i) and the diagonal degree matrix of G be $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$. Laplacian matrix and signless Laplacian matrix are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. For more details see [2,4,7,14]. The characteristic polynomial of the graph G is defined as $f_G(A:x) = \det(xI_n - A)$, where I_n is the identity matrix of order n. The *eigenvalues* of G are the roots of $f_G(A:x) = 0$. They are denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and usually called adjacency spectrum or A-*spectrum* of G. Similarly the eigenvalues of $L(G)$ and $Q(G)$ are denoted by $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$. They are called the Laplacian and signless Laplacian spectrum (or L-*spectrum* and Q-*spectrum* respectively) of G. The eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are real numbers since the matrices are real and symmetric. *Cospectral* graphs are those graphs with the same spectrum.

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $U(G) = \{x_1, x_2, \ldots, x_n\}$ be another set. Draw x_i adjacent to all the vertices in $\mathcal{N}(v_i)$, the neighborhood set of v_i, in G for each i and delete the edges of G only. The graph thus obtained is called the *duplication graph* [15] of G and we denote it as $\mathcal{DG}$. Let G_1 and G_2 be two graphs with n_1 and n_2 vertices. The *corona* of G_1 and G_2 is the graph obtained by taking one copy of G_1 and n_1 copies of G_2 and joining i-th vertex of G_1 to every vertices in the i-th copy of G_2. Frucht and Harary in [5] introduced this concept and their spectrum by Cvetković et. al [3]. Gopalapillai in [6] introduced neighborhood corona of graphs and calculated the corresponding spectrum. In [17, 19] Varghese and Susha defined some new join and corona-type graphs based on duplication graph of an arbitrary graph. In [10, 16, 18] new product related to corona of graph are studied. Motivated from these works, here we define a new corona-type graph namely, *duplication corresponding corona* and determine its adjacency, Laplacian and signless Laplacian spectrum.

The paper is organized as follows. In Section 2 we state some necessary preliminaries. In Section 3 we define duplication corresponding corona and find their adjacency, Laplacian and signless Laplacian spectrum. Also we discuss some concepts like the number of spanning trees, the Kirchhoff index and the incidence energy of the new graph. Also we introduce some new classes of cospectral graphs using this new product.

2 Preliminaries

Lemma 2.1 ([2]). *Let* $M = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}$ *be a symmetric block matrix of order* 2×2. *Then the eigenvalues of* M *are those of* $M_1 + M_2$ *together with* $M_1 - M_2$.

Proposition 2.2 ([2]). *Let* P_0, P_1, P_2 *and* P_3 *be matrices of order* $n_1 \times n_1, n_1 \times n_2, n_2 \times n_1, n_2 \times n_2$ *respectively. Then*

$$\det \begin{bmatrix} P_0 & P_1 \\ P_2 & P_3 \end{bmatrix} = \begin{cases} \det(P_0)\ \det(P_3 - P_2 P_0^{-1} P_1), & \text{if } P_0 \text{ is invertible} \\ \det(P_3)\ \det(P_0 - P_1 P_3^{-1} P_2), & \text{if } P_3 \text{ is invertible} \end{cases}$$

Definition 2.3 ([13]). Let A be the adjacency matrix of a graph G with n vertices. The determinant, $det(xI - A) = f_G(A:x) \neq 0$, is invertible being the characteristic matrix of A. The A-*coronal*, $\chi_A(x)$, of G is defined to be the sum of the entries of the matrix $(xI - A)^{-1}$. We denote this as

$$\chi_A(x) = \mathbf{j}_n^T (xI - A)^{-1} \mathbf{j}_n, \tag{1}$$

where $\mathbf{j}_n$ is a $n \times 1$ column vector with all entries equal to 1.

Let $A = (a_{ij})$ and B be matrices. Then the Kronecker product, $A \otimes B$, of A and B is defined as the partition matrix $(a_{ij}B)$. For details see [2].

- $(A \otimes B)^T = A^T \otimes B^T$, where A and B are of appropriate order;
- $(A + B) \otimes C = A \otimes C + B \otimes C$;
- $(A \otimes B)(C \otimes D) = AC \otimes BD$ whenever the product AC and BD exist;
- $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ for the non-singular matrix A and B;
- $\det(A \otimes B) = (\det A)^p \, (\det B)^n$ where A and B are $n \times n$ and $p \times p$ matrices.

3 Duplication corresponding corona of graphs

The following definition describes a new graph corona-type product based on the duplication graph of a graph.

Definition 3.1. Let G_1 and G_2 be two vertex disjoint graphs with n_1 and n_2 vertices, respectively. Let $V(G_1) \cup U(G_1)$ where, $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $U(G_1) = \{x_1, x_2, \ldots, x_{n_1}\}$ be vertex set of $\mathcal{DG}_1$, the duplication graph of G_1. The vertex x_i is a duplication of v_i for each $i = 1, 2, \ldots n_1$. *Duplication corresponding corona* of G_1 and G_2, denoted by $G_1 \underline{\divideontimes} G_2$ and is the graph obtained from $\mathcal{DG}_1$ and n_1 copies of G_2 by making x_i and v_i adjacent to every vertices in the i-th copy of G_2 for $i = 1, 2, \ldots, n_1$.

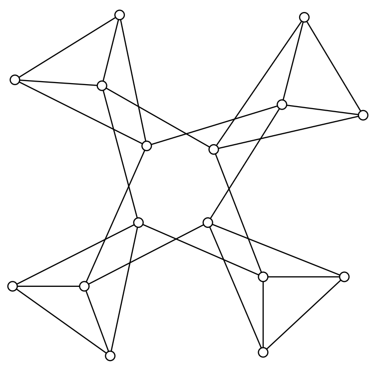

Figure 1. Duplication corresponding corona $C_4 \underline{\divideontimes} K_2$

Theorem 3.2. *Let G_i be two graphs with spectrum $\lambda_{i1} \geq \lambda_{i2} \geq \cdots \geq \lambda_{in_i}$ for $i = 1, 2$. Then the characteristic polynomial of $G_1 \underline{\divideontimes} G_2$ is*

$$f_{G_1 \underline{\divideontimes} G_2}(A : x) = \prod_{j=1}^{n_2} (x - \lambda_{2j})^{n_1} \prod_{i=1}^{n_1} (x + \lambda_{1i}) \prod_{i=1}^{n_1} (x - 2\chi_{A_2}(x) - \lambda_{1i}).$$

Proof. Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, respectively. Let the vertex set of G_1 be $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $U(G_1) = \{x_1, x_2, \ldots, x_{n_1}\}$ be a set of vertices corresponding to $V(G_1)$. The vertex in the i-th copy of G_2 is $\{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$ and let $W_j = \{u_j^1, u_j^2, \ldots, u_j^{n_2}\}$ for $j = 1, 2, \ldots, n_1$. Then $V(G_1) \cup U(G_1) \cup \{W_1 \cup W_2 \cup \ldots \cup W_{n_1}\}$ is a vertex partition of $G_1 \underline{\divideontimes} G_2$ with $n_1(n_2 + 2)$ vertices.

By this vertex partitioning, the block form of the adjacency matrix of $G_1 \underline{\divideontimes} G_2$ is

$$A(G_1 \underline{\divideontimes} G_2) = \begin{bmatrix} 0 & A_1 & \mathbf{j}_{n_2}^T \otimes I_{n_1} \\ A_1 & 0_{n_1 \times n_1} & \mathbf{j}_{n_2}^T \otimes I_{n_1} \\ \mathbf{j}_{n_2} \otimes I_{n_1} & \mathbf{j}_{n_2} \otimes I_{n_1} & A_2 \otimes I_{n_1} \end{bmatrix},$$

where A_1 and A_2 are the adjacency matrices of G_1 and G_2, respectively. $\mathbf{j}_{n_2}$ is a $n_2 \times 1$ column vector with all entries equal to 1 and I_{n_1} is an identity matrix of order n_1.

The characteristic polynomial of $G_1 \underline{\divideontimes} G_2$ is

$$\begin{aligned} f_{G_1 \underline{\divideontimes} G_2}(A : x) &= \begin{vmatrix} xI_{n_1} & -A_1 & -\mathbf{j}_{n_2}^T \otimes I_{n_1} \\ -A_1 & xI_{n_1} & -\mathbf{j}_{n_2}^T \otimes I_{n_1} \\ -\mathbf{j}_{n_2} \otimes I_{n_1} & -\mathbf{j}_{n_2} \otimes I_{n_1} & (xI_{n_2} - A_2) \otimes I_{n_1} \end{vmatrix}, \\ &= \det((xI_{n_2} - A_2) \otimes I_{n_2}) \ \det(S), \\ \text{where, } S &= \begin{bmatrix} xI_{n_1} & -A_1 \\ -A_1 & xI_{n_1} \end{bmatrix} \\ &- \begin{bmatrix} -\mathbf{j}_{n_2}^T \otimes I_{n_1} \\ -\mathbf{j}_{n_2}^T \otimes I_{n_1} \end{bmatrix} ((xI_{n_2} - A_2) \otimes I_{n_1})^{-1} \begin{bmatrix} -\mathbf{j}_{n_2} \otimes I_{n_1} & -\mathbf{j}_{n_2} \otimes I_{n_1} \end{bmatrix}. \end{aligned}$$

By using the property of Kronecker product we can substantiate that

$$(\mathbf{j}_{n_2}^T \otimes I_{n_1})((xI_{n_2} - A_2)^{-1} \otimes I_{n_1})(\mathbf{j}_{n_2} \otimes I_{n_1}) = \chi_{A_2}(x) I_{n_1}.$$

Therefore,

$$\begin{aligned} S &= \begin{bmatrix} xI_{n_1} & -A_1 \\ -A_1 & xI_{n_1} \end{bmatrix} - \begin{bmatrix} \chi_{A_2}(x)I_{n_1} & \chi_{A_2}(x)I_{n_1} \\ \chi_{A_2}(x)I_{n_1} & \chi_{A_2}(x)I_{n_1} \end{bmatrix}, \\ &= \begin{bmatrix} xI_{n_1} - \chi_{A_2}(x)I_{n_1} & -A_1 - \chi_{A_2}(x)I_{n_1} \\ -A_1 - \chi_{A_2}(x)I_{n_1} & xI_{n_1} - \chi_{A_2}(x)I_{n_1} \end{bmatrix}. \end{aligned}$$

This implies that, $\det(S) = \det(xI - M)$, where

$$M = \begin{bmatrix} \chi_{A_2}(x)I_{n_1} & A_1 + \chi_{A_2}(x)I_{n_1} \\ A_1 + \chi_{A_2}(x)I_{n_1} & \chi_{A_2}(x)I_{n_1} \end{bmatrix}.$$

By Lemma 2.1 the eigenvalues of M are those of $A_1 + 2\chi_{A_2}(x)I_{n_1}$ and $-A_1$, i.e., the roots of $\det(xI_{n_1} - 2\chi_{A_2}(x)I_{n_1} - A_1) = 0$ and $\det(xI_{n_1} + A_1) = 0$.

Also

$$\begin{aligned} \det((xI_{n_2} - A_2) \otimes I_{n_1}) &= (\det(xI_{n_2} - A_2))^{n_1} (\det(I_{n_1}))^{n_2} \\ &= \prod_{j=1}^{n_2} (x - \lambda_{2j})^{n_1}. \end{aligned}$$

Hence,

$$f_{G_1 \underline{\circledast} G_2}(A:x) = \prod_{j=1}^{n_2}(x-\lambda_{2j})^{n_1}\prod_{i=1}^{n_1}(x+\lambda_{1i})\prod_{i=1}^{n_1}(x-2\chi_{A_2}(x)-\lambda_{1i}). \qquad \square$$

Corollary 3.3. *For $i = 1, 2$, let G_i be two graphs with n_i vertices with spectrum $\lambda_{i1} \geq \lambda_{i2} \geq \cdots \geq \lambda_{in_i}$. If G_2 is r_2-regular, then A-spectrum of $G_1 \underline{\circledast} G_2$, consists of*

1. *λ_{2j}, repeated n_1 times for $j = 2, 3, \ldots, n_2$;*
2. *$-\lambda_{1i}$, for $i = 1, 2, \ldots, n_1$;*
3. *$\frac{(r_2+\lambda_{1i})\pm\sqrt{(r_2-\lambda_{1i})^2+8n_2}}{2}$, for $i = 1, 2, \ldots, n_1$.*

Proof. Since G_2 is r_2-regular, from Equation 1 we get, $\chi_{A_2}(x) = \frac{n_2}{x-r_2}$.

$$\begin{aligned} x - 2\chi_{A_2}(x) - \lambda_{1i} &= x - 2\frac{n_2}{x-r_2} - \lambda_{1i} \\ &= \frac{x^2 - xr_2 - 2n_2 - x\lambda_{1i} + r_2\lambda_{1i}}{x-r_2} \\ &= \frac{x^2 - (\lambda_{1i}+r_2)x + r_2\lambda_{1i} - 2n_2}{x-r_2}. \end{aligned}$$

Hence from Theorem 3.2 and use the fact $\lambda_{21} = r_2$, we get

$$f_{G_1 \underline{\circledast} G_2}(A:x) = \prod_{j=2}^{n_2}(x-\lambda_{2j})^{n_1}\prod_{i=1}^{n_1}(x+\lambda_{1i})\prod_{i=1}^{n_1}(x^2-(\lambda_{1i}+r_2)x+r_2\lambda_{1i}-2n_2).$$

This completes the proof of the corollary. $\square$

Corollary 3.4. *Let G_1 be an arbitrary graph with n_1 vertices and spectrum $\lambda_{11} \geq \lambda_{12} \geq \cdots \geq \lambda_{1n_1}$. If $G_2 = \overline{K_{n_2}}$ (totally disconnected graph on n_2 vertices), then A-spectrum of $G_1 \underline{\circledast} G_2$ consists of,*

1. *0, repeated $n_1(n_2-1)$ times;*
2. *$-\lambda_{1i}$, for $i = 1, 2, \ldots, n_1$;*
3. *$\frac{\lambda_{1i}\pm\sqrt{\lambda_{1i}^2+8n_2}}{2}$, for $i = 1, 2, \ldots, n_1$.*

Proof. Since G_2 is $\overline{K_{n_2}}$, $\chi_{A_2}(x) = \frac{n_2}{x}$ and the $\lambda_{2j} = 0$ for $j = 1, 2, \ldots, n_2$.
Using Theorem 3.2 and Corollary 3.3 we get the characteristic polynomial as,

$$f_{G_1 \underline{\circledast} G_2}(\mathcal{A}:x) = x^{n_1(n_2-1)}\prod_{i=1}^{n_1}(x+\lambda_{1i})\prod_{i=1}^{n_1}(x^2-\lambda_{1i}x-2n_2).$$

This completes the proof of the corollary. $\square$

Theorem 3.5. *Let G_1 be an r_1-regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices with Laplacian spectrum $0 = \mu_{j1} \leq \mu_{j2} \leq \cdots \leq \mu_{jn_j}$ for $j = 1, 2$. Then L - spectrum of $G_1 \underline{\circledast} G_2$ consists of,*
(i) 0 *;*
(ii) $n_2 + 2$ *;*
(iii) $2 + \mu_{2i}$*, repeated* n_1 *times for* $i = 2, 3, \ldots, n_2$ *;*
(iii) $n_2 + 2r_1 - \mu_{1i}$*, for* $i = 1, 2, \ldots, n_1$*;*
(iv) Two roots of the equation,

$$x^2 - (n_2 + \mu_{1i} + 2)x + 2\mu_{1i} = 0, \; i = 2, 3, \ldots, n_1.$$

Proof. Let G_1 be an r_1-regular graph with n_1 vertices and G_2 be an arbitrary graph with n_2 vertices. Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ be the vertex set of G_1 and $U(G_1) = \{x_1, x_2, \ldots, x_{n_1}\}$ be the set of vertices corresponding to $V(G_1)$. The vertex in the i-th copy of G_2 is $\{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$. The degree of the vertices of $G_1 \underline{\circledast} G_2$ are
$d_{G_1 \underline{\circledast} G_2}(v_i) = d_{G_1 \underline{\circledast} G_2}(x_i) = n_2 + r_1$, $i = 1, 2, \ldots, n_1$ and
$d_{G_1 \underline{\circledast} G_2}(u_j^i) = d_{G_2}(u_j) + 2$, for $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$.
The diagonal degree matrix of $G_1 \underline{\circledast} G_2$ is,

$$D(G_1 \underline{\circledast} G_2) = \begin{bmatrix} (r_1 + n_2)I_{n_1} & 0 & 0 \\ 0 & (r_1 + n_2)I_{n_1} & 0 \\ 0 & 0 & (D(G_2) + 2I_{n_2}) \otimes I_{n_1} \end{bmatrix},$$

where $D(G_2)$ is the diagonal degree matrix of the graph G_2.

$$\begin{aligned} (D(G_2) + 2I_{n_2}) \otimes I_{n_1} - A_2 \otimes I_{n_1} &= (D(G_2) + 2I_{n_2} - A_2) \otimes I_{n_1} \\ &= (L_2 + 2I_{n_2}) \otimes I_{n_1}. \end{aligned}$$

Also from the proof of Theorem 3.2, the adjacency matrix of $G_1 \underline{\circledast} G_2$ is,

$$A = \begin{bmatrix} 0 & A_1 & \mathbf{j}_{n_2}^T \otimes I_{n_1} \\ A_1 & 0_{n_1 \times n_1} & \mathbf{j}_{n_2}^T \otimes I_{n_1} \\ \mathbf{j}_{n_2} \otimes I_{n_1} & \mathbf{j}_{n_2}^T \otimes I_{n_1} & A_2 \otimes I_{n_1} \end{bmatrix},$$

where A_1 and A_2 are the adjacency matrices of G_1 and G_2, respectively.
The Laplacian matrix of $G_1 \underline{\circledast} G_2$ is,

$$\begin{aligned} L &= D - A \\ &= \begin{bmatrix} (r_1 + n_2)I_{n_1} & -A_1 & -\mathbf{j}_{n_2}^T \otimes I_{n_1} \\ -A_1 & (r_1 + n_2)I_{n_1} & -\mathbf{j}_{n_2}^T \otimes I_{n_1} \\ -\mathbf{j}_{n_2} \otimes I_{n_1} & -\mathbf{j}_{n_2} \otimes I_{n_1} & (L_2 + 2I_{n_2}) \otimes I_{n_1} \end{bmatrix}, \end{aligned}$$

where L_2 is the Laplacian matrix of the graph G_2.
Here we use the properties of Kronecker product and can easily proved that

$$(\mathbf{j}_{n_2}^T \otimes I_{n_1})(((x - 2)I_{n_2} - L_2)^{-1} \otimes I_{n_1})(\mathbf{j}_{n_2} \otimes I_{n_1}) = \chi_{L_2}(x - 2)I_{n_1}.$$

The Laplacian characteristic polynomial of $G_1 \underline{\circledast} G_2$ is,

$$f_{G_1\underline{\circledast}G_2}(L:x) = \begin{vmatrix} (x-r_1-n_2)I_{n_1} & A_1 & \mathbf{j}_{n_2}^T \otimes I_{n_1} \\ A_1 & (x-r_1-n_2)I_{n_1} & \mathbf{j}_{n_2}^T \otimes I_{n_1} \\ \mathbf{j}_{n_2} \otimes I_{n_1} & \mathbf{j}_{n_2} \otimes I_{n_1} & ((x-2)I_{n_2}-L_2)\otimes I_{n_1} \end{vmatrix}$$
$$= \det(((x-2)I_{n_2} - L_2) \otimes I_{n_1})\ \det(S),$$

$$\text{where, } S = \begin{bmatrix} (x-r_1-n_2)I_{n_1} & A_1 \\ A_1 & (x-r_1-n_2)I_{n_1} \end{bmatrix}$$
$$- \begin{bmatrix} \mathbf{j}_{n_2}^T \otimes I_{n_1} \\ \mathbf{j}_{n_2}^T \otimes I_{n_1} \end{bmatrix} (((x-2)I_{n_2} - L_2) \otimes I_{n_1})^{-1} \begin{bmatrix} \mathbf{j}_{n_2} \otimes I_{n_1} & \mathbf{j}_{n_2} \otimes I_{n_1} \end{bmatrix}$$
$$= \begin{bmatrix} (x-r_1-n_2)I_{n_1} & A_1 \\ A_1 & (x-r_1-n_2)I_{n_1} \end{bmatrix}$$
$$- \begin{bmatrix} \chi_{L_2}(x-2)I_{n_1} & \chi_{L_2}(x-2)I_{n_1} \\ \chi_{L_2}(x-2)I_{n_1} & \chi_{L_2}(x-2)I_{n_1} \end{bmatrix}$$
$$= \begin{bmatrix} (x-r_1-n_2-\chi_{L_2}(x-2))I_{n_1} & A_1 - \chi_{L_2}(x-2)I_{n_1} \\ A_1 - \chi_{L_2}(x-2)I_{n_1} & (x-r_1-n_2-\chi_{L_2}(x-2))I_{n_1} \end{bmatrix}.$$

Hence, $\det(S) = \det(xI - M)$, where,

$$M = \begin{bmatrix} (r_1+n_2+\chi_{L_2}(x-2))I_{n_1} & -A_1 + \chi_{L_2}(x-2)I_{n_1} \\ -A_1 + \chi_{L_2}(x-2)I_{n_1} & (r_1+n_2+\chi_{L_2}(x-2))I_{n_1} \end{bmatrix}.$$

By Lemma 2.1 the eigenvalues of M are those of $(r_1+n_2+\chi_{L_2}(x-2))I_{n_1} - A_1 + \chi_{L_2}(x-2)I_{n_1}$ together with $(r_1+n_2+\chi_{L_2}(x-2))I_{n_1} + A_1 - \chi_{L_2}(x-2)I_{n_1}$, i.e., those of $(r_1+n_2+2\chi_{L_2}(x-2))I_{n_1} - A_1$ together with $(n_2+r_1)I_{n_1} + A_1$.

We use the fact that $L = D - A$, so, for a r_1-regular graph we have $L = r_1 I_{n_1} - A_1$ and $\mu_{1i} = r_1 - \lambda_{1i}$ for $i = 1, 2, \ldots, n$.

Therefore the eigenvalues of M are those of $(n_2 + 2\chi_{L_2}(x-2))I_{n_1} + L_1$ together with $(n_2 + 2r_1)I_{n_1} - L_1$.

$$\det(((x-2)I_{n_2} - L_2) \otimes I_{n_1}) = \det((x-2)I_{n_2} - L_2)^{n_1} (\det(I_{n_1}))^{n_2}$$
$$= \prod_{j=1}^{n_2} (x - 2 - \mu_{2j})^{n_1}.$$

Hence,

$$f_{G_1\underline{\circledast}G_2}(L:x) = \prod_{j=1}^{n_2}(x-2-\mu_{2j})^{n_1} \prod_{i=1}^{n_1}(x-2r_1-n_2+\mu_{1i}) \prod_{i=1}^{n_1}(x-n_2-2\chi_{L_2}(x-2)-\mu_{1i}).$$

Using equation (1) we have,

$$\chi_{L_2}(x-2) = \frac{n_2}{x-2}.$$

$$x - n_2 - 2\chi_{L_2}(x-2) - \mu_{1i} = x - n_2 - \frac{2n_2}{x-2} - \mu_{1i}$$
$$= \frac{x^2 - (n_2 + 2 + \mu_{1i})x + 2\mu_{1i}}{x-2}$$
$$\prod_{i=1}^{n_1}(x - n_2 - 2\chi_{L_2}(x-2) - \mu_{1i}) = \frac{x(x-n_2-2)}{(x-2)^{n_1}}\prod_{i=2}^{n_1}(x^2 - (n_2+2+\mu_{1i})x + 2\mu_{1i})$$

Also, since $\mu_{11} = \mu_{21} = 0$, the characteristic polynomial becomes

$$f_{G_1 \underline{\divideontimes} G_2}(L:x) = x(x-n_2-2)\prod_{j=2}^{n_2}(x-2-\mu_{2j})^{n_1}\prod_{i=1}^{n_1}(x-n_2-2r_1+\mu_{1i})$$
$$\prod_{i=2}^{n_1}(x^2 - (n_2+\mu_{1i}+2)x + 2\mu_{1i}).$$

Hence the theorem is proved. □

Remark 3.6. Spanning tree of a graph is a subgraph of it which is also a tree. The number of spanning trees of a graph G is denoted by $t(G)$. If G is a connected graph with n vertices and the Laplacian spectrum $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$, then [11] the number of spanning trees

$$t(G) = \frac{\mu_2(G)\mu_3(G)\cdots\mu_n(G)}{n}. \tag{2}$$

Corollary 3.7. *Let G_1 be an r_1-regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices. Then the number of spanning trees of $G_1 \underline{\divideontimes} G_2$ is,*

$$t(G_1 \underline{\divideontimes} G_2) = 2^{n_1-1}\, t(G_1)\prod_{j=2}^{n_2}(2+\mu_{2j})^{n_1}\prod_{i=1}^{n_1}(n_2+2r_1-\mu_{1i}).$$

Proof. The proof follows from Theorem 3.5 and equation (2). □

Remark 3.8. Klein and Randić in [9] introduced a new notion named *resistance distance* based on electric resistance in a network corresponding to a graph, in which the resistance distance between any two adjacent vertices is 1 ohm. The electric resistance is calculated by means of the Kirchhoff laws called *Kirchhoff index*.

For a graph G with n vertices, where $n \geq 2$, and the Laplacian spectrum $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$, then the Kirchhoff index, $Kf(G)$, is defined as

$$Kf(G) = n\sum_{i=2}^{n}\frac{1}{\mu_i}. \tag{3}$$

Corollary 3.9. *Let G_1 be an r_1-regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices. Then Kirchhoff index of $G_1 \underline{\divideontimes} G_2$ is,*

$$Kf(G_1 \underline{\divideontimes} G_2) = \frac{2n_1(n_2+r_1+1)}{n_2+2r_1} + \frac{n_1(n_1-1)(n_2+2)}{2} + \frac{(n_2+2)^2}{2}Kf(G_1)$$
$$+ n_1(n_2+2)\left[\sum_{j=2}^{n_2}\frac{n_1}{2+\mu_{2j}} + \sum_{i=2}^{n_1}\frac{1}{n_2+2r_1-\mu_{1i}}\right].$$

Proof. Let y_{i1} and y_{i2} be the roots of the equation $x^2-(n_2+\mu_{1i}+2)x+2\mu_{1i}=0,\ i=2,3,\ldots,n_1$.

$$\begin{aligned}\frac{1}{y_{i1}}+\frac{1}{y_{i2}} &= \frac{y_{i1}+y_{i2}}{y_{i1}y_{i2}}\\ &= \frac{n_2+2+\mu_{1i}}{2\mu_{1i}}\\ &= \frac{n_2+2}{2\mu_{1i}}+\frac{1}{2}\\ \sum_{i=2}^{n_1}\left(\frac{1}{y_{i1}}+\frac{1}{y_{i2}}\right) &= \frac{n_2+2}{2n_1}Kf(G_1)+\frac{n_1-1}{2}.\end{aligned}$$

The remaining proof follows from Theorem 3.5 and equation (3). □

J. Liu and B. Liu defined a new Laplacian graph invariant in [12], Laplacian-energy-like invariant (LEL).

Let G be a graph with n vertices and the Laplacian spectrum $0=\mu_1(G)\le\mu_2(G)\cdots\le\mu_n(G)$. LEL of G is defined as,

$$LEL(G)=\sum_{i=2}^{n}\sqrt{\mu_i}. \tag{4}$$

Corollary 3.10. *Let G_1 be an r_1-regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices with Laplacian spectrum $\{0=\mu_1,\mu_2,\ldots,\mu_n\}$. Then,*

$$\begin{aligned}LEL(G_1\underline{\circledast}G_2) = \sqrt{n_2+2}\ + n_1\sum_{j=2}^{n_2}\sqrt{2+\mu_{2j}}\ +\sum_{i=1}^{n_1}\sqrt{n_2+2r_1-\mu_{1i}}\\ +\sum_{i=2}^{n_1}\left[(n_2+\mu_{1i}+2)+2\sqrt{2\mu_{1i}}\right]^{1/2}.\end{aligned}$$

Proof. The proof follows from Theorem 3.5, equation (4) and the identity

$$(\sqrt{x}+\sqrt{y})^2=(x+y)+2\sqrt{xy}.$$

□

Corollary 3.11. *For two L-cospectral regular graphs G_1 and G_2 and an arbitrary graph H, the graphs $G_1\underline{\circledast}H$ and $G_2\underline{\circledast}H$ are L-cospectral. Also if G is a regular graph and H_1 and H_2 are L-cospectral graphs, then $G\underline{\circledast}H_1$ and $G\underline{\circledast}H_2$ are L-cospectral.*

Theorem 3.12. *Let G_1 be an r_1-regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices with signless Laplacian spectrum $\nu_{i1}\ge\nu_{i2}\ge\cdots\ge\nu_{in_i}$ for $i=1,2$, respectively. Then the signless Laplacian characteristic polynomial of $G_1\underline{\circledast}G_2$ is,*

$$f_{G_1\underline{\circledast}G_2}(Q:x)=\prod_{j=1}^{n_2}(x-2-\nu_{2j})^{n_1}\prod_{i=1}^{n_1}(x-n_2-2r_1+\nu_{1i})\prod_{i=1}^{n_1}(x-n_2-2\chi_{Q_2}(x-2)-\nu_{1i}).$$

Proof. As like the notations defined in Theorem 3.5, we can define the signless Laplacian matrix of $G_1\underline{\circledast}G_2$ is:

$$Q=\begin{bmatrix}(r_1+n_2)I_{n_1} & A_1 & \mathbf{j}_{n_2}^T\otimes I_{n_1}\\ A_1 & (r_1+n_2)I_{n_1} & \mathbf{j}_{n_2}^T\otimes I_{n_1}\\ \mathbf{j}_{n_2}\otimes I_{n_1} & \mathbf{j}_{n_2}\otimes I_{n_1} & (2I_{n_2}+Q_2)\otimes I_{n_1}\end{bmatrix}.$$

$$\begin{aligned}f_{G_1\underline{\circledast}G_2}(Q:x)&=\begin{vmatrix}(x-r_1-n_2)I_{n_1} & -A_1 & -\mathbf{j}_{n_2}^T\otimes I_{n_1}\\ -A_1 & (x-r_1-n_2)I_{n_1} & -\mathbf{j}_{n_2}^T\otimes I_{n_1}\\ -\mathbf{j}_{n_2}\otimes I_{n_1} & -\mathbf{j}_{n_2}\otimes I_{n_1} & ((x-2)I_{n_2}-Q_2)\otimes I_{n_1}\end{vmatrix},\\ &=\det((x-2)I_{n_2}-Q_2)\otimes I_{n_1}\ \det(S),\end{aligned}$$

where Q_2 is the signless Laplacian matrix of G_2.

As like in Theorem 3.5 proved above,

$$S=\begin{bmatrix}(x-r_1-n_2-\chi_{Q_2}(x-2))I_{n_1} & -A_1-\chi_{Q_2}(x-2)I_{n_1}\\ -A_1-\chi_{Q_2}(x-2)I_{n_1} & (x-r_1-n_2-\chi_{Q_2}(x-2))I_{n_1}\end{bmatrix}.$$

Hence $\det S=\det(xI-M)$, where,

$$M=\begin{bmatrix}(r_1+n_2+\chi_{Q_2}(x-2))I_{n_1} & A_1+\chi_{Q_2}(x-2)I_{n_1}\\ A_1+\chi_{Q_2}(x-2)I_{n_1} & (r_1+n_2+\chi_{Q_2}(x-1))I_{n_1}\end{bmatrix}.$$

By Lemma 2.1 the eigenvalues of M are those of $(r_1+n_2+2\chi_{Q_2}(x-2))I_{n_1}+A_1$ together with $(r_1+n_2)I_{n_1}-A_1$.

We use the fact that $Q=D+A$, since G_1 is an r_1-regular graph we have $Q=r_1I+A_1$ and then, $\nu_{1i}=r_1+\lambda_{1i}$ for $i=1,2,\dots,n$.

Therefore, the eigenvalues of M are those of $(n_2+2\chi_{Q_2}(x-2))I_{n_1}+Q_1$ together with $(n_2+2r_1)I_{n_1}-Q_1$.

Also,

$$\begin{aligned}\det(((x-2)I_{n_2}-Q_2)\otimes I_{n_1})&=\det((x-2)I_{n_2}-Q_2)^{n_1}(\det(I_{n_1}))^{n_2}\\ &=\prod_{j=1}^{n_2}(x-2-\nu_{2j})^{n_1}.\end{aligned}$$

Hence,

$$f_{G_1\underline{\circledast}G_2}(Q:x)=\prod_{j=1}^{n_2}(x-2-\nu_{2j})^{n_1}\prod_{i=1}^{n_1}(x-n_2-2r_1+\nu_{1i})\prod_{i=1}^{n_1}(x-n_2-2\chi_{Q_2}(x-2)-\nu_{1i}).$$

This completes the proof. □

Corollary 3.13. *Let G_i be an r_i-regular graph on n_i vertices $i=1,2$ with signless Laplacian spectrum $2r_i=\nu_{i1}\geq\nu_{i2}\geq\cdots\geq\nu_{in_i}$ for $i=1,2$, respectively. Then the Q-spectrum of $G_1\underline{\circledast}G_2$ consists of:*

(i) $\nu_{2i}+2$*, repeats* n_1 *times for* $i=2,3,\dots,n_2$ *;*

(ii) $n_2+2r_1-\nu_{1i}$ *for* $i=1,2,\dots,n_1$ *;*

(iii) two roots of the equation

$$x^2-(n_2+2r_2+\nu_{1i}+2)x+2(n_2r_2+\nu_{1i}+r_2\nu_{1i})=0 \textit{ for } i=2,3,\dots,n_2.$$

Proof. Since G_2 is r_2-regular, $\nu_{21}=2r_2$. Using equation (1) we have,

$$\chi_{Q_2}(x-2)=\frac{n_2}{x-2-2r_2}.$$

Therefore,

$$f_{G_1\underline{\circledast}G_2}(Q:x)=\prod_{j=2}^{n_2}(x-2-\nu_{2j})^{n_1}\prod_{i=1}^{n_1}(x-2r_1-n_2+\nu_{1i})$$
$$\prod_{i=1}^{n_1}(x^2-(2+2r_2+n_2+\nu_{1i})x-2(n_2r_2+\nu_{1i}+r_2\nu_{1i})). \qquad \square$$

Corollary 3.14. *For two Q-cospectral regular graphs G_1 and G_2 and an arbitrary graph H, the graphs $G_1\underline{\circledast}H$ and $G_2\underline{\circledast}H$ are Q-cospectral. Also if G is a regular graph and H_1 and H_2 are identical Q-cospectral graphs ($\chi_{Q(H_1)}(x)=\chi_{Q(H_2)}(x)$), then $G\underline{\circledast}H_1$ and $G\underline{\circledast}H_2$ are Q-cospectral.*

Let A be any real $n\times m$ matrix with transpose A^T. The square roots of the eigenvalues of AA^T are called the singular values of the matrix A. The *incidence matrix* [1] of G is the $0-1$ matrix $R=(r_{ij})$ with rows indexed by vertices and columns by edges where $r_{ij}=1$ when the vertex v_i is an end point of the edge e_j and 0 otherwise.

Definition 3.15. [8] Let $\rho_1,\rho_2,\dots,\rho_n$ be the singular values of the incidence matrix R of a graph G. The *incidence energy* of G is defined as $\mathcal{IE}(G)=\sum_{i=1}^n\rho_i$.

It is clear that $\mathcal{IE}(G)\geq 0$ and the equality holds if and only if G is totally disconnected.
We have, $RR^T=Q$ [1] and let $\nu_1\geq\nu_2\geq\cdots\geq\nu_n$ be the signless Laplace spectrum of G. In [8] Jooyandeh et. al. defined the incidence energy as

$$\mathcal{IE}(G)=\sum_{i=1}^n\sqrt{\nu_i}. \tag{5}$$

Corollary 3.16. *For $i=1,2$ let G_i be an r_i-regular graph on n_i vertices with signless Laplacian spectrum $\{2r_i=\nu_{i1},\nu_{i2},\dots,\nu_{in_i}\}$. Then the incidence energy of $G_1\underline{\circledast}G_2$ is,*

$$\mathcal{IE}(G_1\underline{\circledast}G_2)=n_1\sum_{j=2}^{n_2}\sqrt{2+\nu_{2j}}\;+\sum_{i=1}^{n_1}\sqrt{n_2+2r_1-\nu_{1i}}$$
$$+\sum_{i=1}^{n_1}\left[(n_2+2r_2+\mu_{1i}+2)+2\sqrt{2(n_2r_2+\nu_{1i}+r_2\nu_{1i})}\right]^{1/2}.$$

Proof. The proof follows the identity $(\sqrt{x}+\sqrt{y})^2=(x+y)+2\sqrt{xy}$ and using Corollary 3.13 and Equation (5).

Let x_{i1} and x_{i2} be the roots of the equation $x^2-(n_2+2r_2+\nu_{1i}+2)x+2(n_2r_2+\nu_{1i}+r_2\nu_{1i})=0$.

$$\begin{aligned}(\sqrt{x_{i1}}+\sqrt{x_{i2}})^2 &= (x_{i1}+x_{i2})+2\sqrt{x_{i1}x_{i2}}\\ &= (n_2+2r_2+\mu_{1i}+2)+2\sqrt{2(n_2r_2+\nu_{1i}+r_2\nu_{1i})}\\ \sqrt{x_{i1}}+\sqrt{x_{i2}} &= \left[(n_2+2r_2+\mu_{1i}+2)+2\sqrt{2(n_2r_2+\nu_{1i}+r_2\nu_{1i})}\right]^{1/2}.\end{aligned}$$

Then the corollary follows. □

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