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Pell–Padovan generalized quaternions

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Abstract: The aim of this article is to introduce Pell–Padovan generalized quaternions. It also derives new properties associated with these and takes into account negative indices. Additionally, it presents generating function, Binet-like formula, Simson formula, matrix representations, and several summation properties.
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1 Introduction

Special numbers play a significant role in mathematics, architecture, engineering, music, art, photography, painting, multiple other areas. There have been a large number of investigations in the literature into special numbers such as Fibonacci, Lucas, Tribonacci, Gibonacci, Jacobsthal, and others, [11, 21–23]. The Pell sequence is the most popular one and is named after the well-known English mathematician J. Pell (see in [1, 6, 12, 15, 16, 22]). Moreover, the Padovan sequence is named after the Italian architect R. Padovan who attributed its discovery to the architect Hans van der Laan in [25] in 1994. The sequence was determined by I. Stewart in [40] in 1996, who designated them Padovan numbers in honour of Richard Padovan. In addition, Stewart discussed these numbers in a section of his book [41] (for more exhaustive information on this topic, see the following: [4, 20, 25, 26, 28, 29, 31, 34–36, 45, 46]).

The Pell–Padovan sequence is determined by the following recurrence relation:

$$T_{n+3} = 2T_{n+1} + T_n, \ \forall n \in \mathbb{N} \tag{1}$$

with initial values $T_0 = T_1 = T_2 = 1$, [29, 31, 32]. Several studies have been conducted into Pell–Padovan sequences, [2, 7, 8, 38, 42].

The well-known mathematician W. R. Hamilton invented quaternions in 1843, [14]. Quaternion algebra is associative and non-commutative 4-dimensional Clifford algebra. Quaternions play an effective and important role in pure mathematics, applied mathematics, differential geometry, physics, and other disciplines. The set of all real quaternions is denoted by H and represented by $H = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ where i, j, k are quaternionic units that satisfy the following requirements:

$$i^2 = j^2 = k^2 = -1, \ ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j, \ ijk = -1.$$

In 1849, J. Cockle obtained split quaternions (coquaternions) with $i^2 = -1, j^2 = k^2 = 1$, $ijk = 1$, [5]. Generalized quaternions have also been studied in [17, 18, 24, 27, 43] and are denoted by the notation $H_{\alpha\beta}$. For $q = a_0 + ia_1 + ja_2 + ka_3 \in H_{\alpha\beta}$, the quaternionic units i, j, k satisfy the following requirements:

$$\begin{gathered} i^2 = -\alpha, \ j^2 = -\beta, \ k^2 = -\alpha\beta, \\ ij = -ji = k, \ jk = -kj = \beta i, \ ki = -ik = \alpha j, \end{gathered} \tag{2}$$

where $a_0, a_1, a_2, a_3, \alpha, \beta \in \mathbb{R}$. Quaternions with Padovan and Pell–Padovan coefficients can be found in several studies, [9, 10, 13, 42].

This paper, examines Pell–Padovan generalized quaternions and consists of 4 sections. In Section 1 and Section 2, several significant definitions and properties relating to generalized quaternions and Pell–Padovan numbers are given. In Section 3, Pell–Padovan generalized quaternions and their special properties are determined. These quaternions have also been classified in relation to quaternionic units, and we identify new and practical recurrence relations not only for Pell–Padovan numbers but also Pell–Padovan generalized quaternions. We then present the Binet-like formula, generating function, Simson formula, matrix representations, and several special summation formulas of these. Section 4, presents overall conclusions.

2 Preliminaries

The Pell–Padovan sequence (A066983 in [33]) is determined by the recurrence relation given in the equation (1). For all $n \geq 1$, it can be written that

$$T_{-n} = T_{-n+3} - 2T_{-n+1}. \tag{3}$$

Therefore, $T_{n+3} = 2T_{n+1} + T_n$ can be written as $\forall n \in \mathbb{Z}$ from equations (1) and (3). In Table 1, several values of Pell–Padovan numbers are presented.

n	...	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	...
T_n	...	25	-15	9	-5	3	-1	1	1	1	3	3	7	9	17	...

Table 1. Pell–Padovan numbers

Also, from Table 1, $\forall n \in \mathbb{Z}$,

$$\begin{cases} T_n = -T_{-n+2} + 2; & \text{if } n \text{ is odd} \quad (4a) \\ T_n = T_{-n+2}; & \text{if } n \text{ is even} \quad (4b) \end{cases}$$

and

$$T_{n+2} = T_{n+1} + T_n - (-1)^n \tag{5}$$

can be obtained. The characteristic equation of Pell–Padovan sequence is given by $x^3-2x-1=0$. Moreover, $\forall n \in \mathbb{Z}$, the Binet-like formula of this sequence is as follows, [42]:

$$T_n = 2\left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right) - 2\left(\frac{r_1^{n} - r_2^{n}}{r_1 - r_2}\right) + r_3^{n+1}, \tag{6}$$

where

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}, \quad r_3 = -1$$

are the roots of equation $x^3 - 2x - 1 = 0$. Here, $r_1 + r_2 + r_3 = 0$, $r_1r_2 + r_1r_3 + r_2r_3 = -2$, and $r_1r_2r_3 = 1$ are satisfied. Also, the Simson formula for the Pell–Padovan numbers can be found in [37]. Furthermore, the *Pell–Padovan matrix* is examined in [7, 38], using the Kalman's matrix formula in [19], such that (for detailed information see [30, 44]):

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The other type of Pell–Padovan matrix can be seen in [7].

On the other hand, if the set of generalized quaternions $H_{\alpha\beta}$ are examined for special cases, real quaternions (for $\alpha = 1, \beta = 1$), split quaternions (for $\alpha = 1, \beta = -1$), semi-quaternions (for $\alpha = 1, \beta = 0$), split semi-quaternions (for $\alpha = -1, \beta = 0$) and $1/4$ quaternions (for $\alpha = 0, \beta = 0$) are obtained. A generalized quaternion q can be expressed through $q = S_q + \overrightarrow{V}_q$, where $S_q = a_0$ and $\overrightarrow{V}_q = ia_1 + ja_2 + ka_3$. In this equation, S_q is called the scalar part and $\overrightarrow{V}_q$ is called the vector part of q. The addition or substraction of two generalized quaternions is given as: $q \pm p = a_0 \pm b_0 + i\,(a_1 \pm b_1) + j\,(a_2 \pm b_2) + k\,(a_3 \pm b_3)$. In this case, $S_{q\pm p} = a_0 \pm b_0 = S_q \pm S_p$ and $\overrightarrow{V}_{q\pm p} = \overrightarrow{V}_q \pm \overrightarrow{V}_p$. In addition, the multiplication of q by a scalar c is as follows: $cq = ca_0 + ica_1 + jca_2 + kca_3$. The multiplication of two generalized quaternions can be identified as:

$$pq = \mathrm{S}_p\mathrm{S}_q - g(\overrightarrow{V}_p, \overrightarrow{V}_q) + \mathrm{S}_p\overrightarrow{V}_q + \mathrm{S}_q\overrightarrow{V}_p + \overrightarrow{V}_p \times \overrightarrow{V}_q,$$

where $g(\overrightarrow{V}_p, \overrightarrow{V}_q)$ and $\overrightarrow{V}_p \times \overrightarrow{V}_q$ represent generalized inner and vector products, respectively. The conjugate of q is $\overline{q} = a_0 - ia_1 - ja_2 - ka_3$ and it is clear that $\overline{q} = S_q - \overrightarrow{V}_q$. Moreover, the norm of q is $N_q = q\overline{q} = \overline{q}q = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2$, [18].

3 Pell–Padovan generalized quaternions

In this section, we determine the Pell–Padovan generalized quaternions with positive and negative indices, and derive relations pertaining to these. Additionally, we establish several new properties concerning these special quaternions.

Definition 3.0.1. $\forall n \in \mathbb{N}$, *the Pell–Padovan generalized quaternions are described by the following formula:*

$$\widetilde{T}_n = T_n + iT_{n+1} + jT_{n+2} + kT_{n+3}, \tag{7}$$

where T_n is the n-th Pell–Padovan number and i, j, k are quaternionic units that satisfy the properties given in equation (2)*, where $\alpha, \beta \in \mathbb{R}$.*

In addition, $\forall n \in \mathbb{Z}^+$ the Pell–Padovan generalized quaternions with negative indices are defined by:

$$\widetilde{T}_{-n} = T_{-n} + iT_{-n+1} + jT_{-n+2} + kT_{-n+3}, \tag{8}$$

where T_{-n} is the $-n$-th Pell–Padovan number and i, j, k are quaternionic units that satisfy the properties given in equation (2)*, $\alpha, \beta \in \mathbb{R}$.*

Thus, the equation (7) is valid $\forall n \in \mathbb{Z}$.

We denote the set of the Pell–Padovan generalized quaternions by $\widetilde{T}_{\alpha\beta}$ and present the classification of these quaternions in Table 2:

$\alpha = 1, \beta = 1$	Pell–Padovan real quaternions [3, 42]
$\alpha = 1, \beta = -1$	Pell–Padovan split quaternions
$\alpha = 1, \beta = 0$	Pell–Padovan semi-quaternions
$\alpha = -1, \beta = 0$	Pell–Padovan split semi-quaternions
$\alpha = 0, \beta = 0$	Pell–Padovan 1/4 quaternions

Table 2. Classification for $\widetilde{T}_{\alpha\beta}$

Definition 3.0.2. *Let*

$$\widetilde{T}_n = T_n + iT_{n+1} + jT_{n+2} + kT_{n+3}\,, \widetilde{U}_n = U_n + iU_{n+1} + jU_{n+2} + kU_{n+3} \in \widetilde{T}_{\alpha\beta}.$$

Then the following definitions can be given for $\widetilde{T}_n$:

- *Scalar part:* $S_{\widetilde{T}_n} = T_n$,
- *Vector part:* $\overrightarrow{V}_{\widetilde{T}_n} = iT_{n+1} + jT_{n+2} + kT_{n+3}$,
- *Conjugate:*

$$\overline{\widetilde{T}_n} = T_n - iT_{n+1} - jT_{n+2} - kT_{n+3}, \tag{9}$$

- *Addition and substraction:*

$$\widetilde{T}_n \pm \widetilde{U}_n = (T_n \pm U_n) + i(T_{n+1} \pm U_{n+1}) + j(T_{n+2} \pm U_{n+2}) + k(T_{n+3} \pm U_{n+3}), \tag{10}$$

- *Multiplication by a scalar:*

$$c\widetilde{T}_n = cT_n + icT_{n+1} + jcT_{n+2} + kcT_{n+3},\ c \in \mathbb{R},$$

- *Multiplication:*

$$\begin{aligned}\widetilde{T}_n\widetilde{U}_n &= (T_n + iT_{n+1} + jT_{n+2} + kT_{n+3})(U_n + iU_{n+1} + jU_{n+2} + kU_{n+3}) \\ &= \mathrm{S}_{\widetilde{T}_n}\mathrm{S}_{\widetilde{U}_n} - g(\overrightarrow{V}_{\widetilde{T}_n}, \overrightarrow{V}_{\widetilde{U}_n}) + \mathrm{S}_{\widetilde{T}_n}\overrightarrow{V}_{\widetilde{U}_n} + \mathrm{S}_{\widetilde{U}_n}\overrightarrow{V}_{\widetilde{T}_n} + \overrightarrow{V}_{\widetilde{T}_n} \times \overrightarrow{V}_{\widetilde{U}_n},\end{aligned} \tag{11}$$

where

$$g(\overrightarrow{V}_{\widetilde{T}_n}, \overrightarrow{V}_{\widetilde{U}_n}) = \alpha T_{n+1}U_{n+1} + \beta T_{n+2}U_{n+2} + \alpha\beta T_{n+3}U_{n+3},$$

and

$$\overrightarrow{V}_{\widetilde{T}_n} \times \overrightarrow{V}_{\widetilde{U}_n} = \begin{vmatrix} \beta i & \alpha j & k \\ T_{n+1} & T_{n+2} & T_{n+3} \\ U_{n+1} & U_{n+2} & U_{n+3} \end{vmatrix} = \begin{matrix} i(\beta T_{n+2}U_{n+3} - \beta T_{n+3}U_{n+2}) \\ +j(\alpha T_{n+3}U_{n+1} - \alpha T_{n+1}U_{n+3}) \\ +k(T_{n+1}U_{n+2} - T_{n+2}U_{n+1}). \end{matrix}$$

Theorem 3.1. *Let $\widetilde{T}_n, \widetilde{T}_{-n} \in \widetilde{T}_{\alpha\beta}$. The following recurrence relations are then satisfied:*

$$\widetilde{T}_{n+3} = 2\widetilde{T}_{n+1} + \widetilde{T}_n,\ \forall n \in \mathbb{N}, \tag{12}$$

$$\widetilde{T}_{-n} = \widetilde{T}_{-n+3} - 2\widetilde{T}_{-n+1},\ \forall n \in \mathbb{Z}^+. \tag{13}$$

Proof. Using the equations (1), (7), and (10), we obtain:

$$\begin{aligned} 2\widetilde{T}_{n+1} + \widetilde{T}_n &= 2(T_{n+1} + iT_{n+2} + jT_{n+3} + kT_{n+4}) + (T_n + iT_{n+1} + jT_{n+2} + kT_{n+3}) \\ &= 2T_{n+1} + T_n + i(2T_{n+2} + T_{n+1}) + j(2T_{n+3} + T_{n+2}) \\ &\quad +k(2T_{n+4} + T_{n+3}) \\ &= T_{n+3} + iT_{n+4} + jT_{n+5} + kT_{n+6} \\ &= \widetilde{T}_{n+3}. \end{aligned}$$

Equation (13) can be easily depicted using the same method. □

The recurrence relation (12) is also valid $\forall n \in \mathbb{Z}$.

Example 3.1.1. *The following are several values of $\widetilde{T}_n$, $\forall n \in \mathbb{Z}$:*

$$\begin{aligned} \widetilde{T}_0 &= 1 + i + j + k3, & \widetilde{T}_4 &= 3 + i7 + j9 + k17, \\ \widetilde{T}_1 &= 1 + i + j3 + k3, & \widetilde{T}_5 &= 7 + i9 + j17 + k25, \\ \widetilde{T}_2 &= 1 + i3 + j3 + k7, & \widetilde{T}_6 &= 9 + i17 + j25 + k43, \\ \widetilde{T}_3 &= 3 + i3 + j7 + k9, & \widetilde{T}_7 &= 17 + i25 + j43 + k67, \end{aligned}$$

and

$$\begin{aligned} \widetilde{T}_{-1} &= -1 + i + j + k, & \widetilde{T}_{-5} &= -15 + i9 - j5 + k3, \\ \widetilde{T}_{-2} &= 3 - i + j + k, & \widetilde{T}_{-6} &= 25 - i15 + j9 - k5, \\ \widetilde{T}_{-3} &= -5 + i3 - j + k, & \widetilde{T}_{-7} &= -41 + i25 - j15 + k9, \\ \widetilde{T}_{-4} &= 9 - i5 + j3 - k, & \widetilde{T}_{-8} &= 67 - i41 + j25 - k15. \end{aligned}$$

Theorem 3.2. *Let T_n and $\widetilde{T}_n$ be the n-th Pell–Padovan number and the n-th Pell–Padovan generalized quaternion, respectively. $\forall n \in \mathbb{Z}$, then the following properties are satisfied:*

$$\begin{cases} \widetilde{T}_n = -T_{-n+2} + 2 + iT_{-n+1} + j\left(-T_{-n} + 2\right) + kT_{-n-1}; & \text{if } n \text{ is odd}, \quad (14a) \\ \widetilde{T}_n = T_{-n+2} + i\left(-T_{-n+1} + 2\right) + jT_{-n} + k\left(-T_{-n-1} + 2\right); & \text{if } n \text{ is even}. \quad (14b) \end{cases}$$

Proof. If we assume that n is odd, using equations (7), (4a), and (4b), we obtain:

$$\begin{aligned} \widetilde{T}_n &= T_n + iT_{n+1} + jT_{n+2} + kT_{n+3} \\ &= -T_{-n+2} + 2 + iT_{-n+1} + j\left(-T_{-n} + 2\right) + kT_{-n-1}. \end{aligned}$$

Similarly, equation (14b) is obvious. □

Example 3.2.1. *Let T_n and $\widetilde{T}_n$ be the n-th Pell–Padovan number and the n-th Pell–Padovan generalized quaternion, respectively.*

- If we take $n = 19$ in the property (14a), then we obtain:

$$\begin{aligned} &-T_{-17} + 2 + iT_{-18} + j[(-T_{-19}) + 2] + kT_{-20} \\ &= -(-5167) + 2 + i8361 + j[-(-13529) + 2] + k21891 \\ &= 5169 + i8361 + j13531 + k21891 \\ &= \widetilde{T}_{19}. \end{aligned}$$

- If we take $n = 30$ in the property (14b), then we obtain:

$$\begin{aligned} &T_{-28} + i[(-T_{-29}) + 2] + jT_{-30} + k[(-T_{-31}) + 2] \\ &= 1028457 + i[-(-1664079) + 2] + j(2692537) + k[-(-4356617) + 2] \\ &= 1028457 + i1664081 + j2692537 + k4356619 \\ &= \widetilde{T}_{30}. \end{aligned}$$

Theorem 3.3. *If $\widetilde{T}_n \in \widetilde{T}_{\alpha\beta}$ and $\forall n \in \mathbb{Z}$, the following property is satisfied:*

$$\widetilde{T}_{n+2} = \widetilde{T}_{n+1} + \widetilde{T}_n - (-1)^n(1 - i + j - k). \tag{15}$$

Proof. The proof can be shown using equations (5) and (7). □

Theorem 3.4. *Let T_n be the n-th Pell–Padovan number, $\widetilde{T}_n \in \widetilde{T}_{\alpha\beta}$ and $\overline{\widetilde{T}_n}$ be the conjugate of $\widetilde{T}_n$. $\forall n \in \mathbb{Z}$, the following properties are obtained:*

(i) $$\widetilde{T}_n - i\widetilde{T}_{n+1} - j\widetilde{T}_{n+2} - k\widetilde{T}_{n+3} = (1 + \alpha\beta)T_n + (\beta + 4\alpha\beta)T_{n+1} + (\alpha + 2\beta + 4\alpha\beta)T_{n+2}, \tag{16}$$

(ii) $$\widetilde{T}_n\overline{\widetilde{T}_n} = T_n^2 + \alpha T_{n+1}^2 + \beta T_{n+2}^2 + \alpha\beta T_{n+3}^2, \tag{17}$$

(iii) $$\widetilde{T}_n + \overline{\widetilde{T}_n} = 2T_n, \tag{18}$$

(iv) $$\widetilde{T}_n^2 = 2\widetilde{T}_n T_n - \widetilde{T}_n\overline{\widetilde{T}_n}, \tag{19}$$

where $\alpha, \beta \in \mathbb{R}$.

Proof. (i) From equations (2), (7), we obtain:

$$\begin{aligned}\widetilde{T}_n - i\widetilde{T}_{n+1} - j\widetilde{T}_{n+2} - k\widetilde{T}_{n+3} = &\ T_n + iT_{n+1} + jT_{n+2} + kT_{n+3} \\ &-i(T_{n+1} + iT_{n+2} + jT_{n+3} + kT_{n+4}) \\ &-j(T_{n+2} + iT_{n+3} + jT_{n+4} + kT_{n+5}) \\ &-k(T_{n+3} + iT_{n+4} + jT_{n+5} + kT_{n+6}) \\ = &\ T_n + \alpha T_{n+2} + \beta T_{n+4} + \alpha\beta T_{n+6}.\end{aligned}$$

Through equation (1), we obtain the following:

$$\begin{aligned}\widetilde{T}_n - i\widetilde{T}_{n+1} - j\widetilde{T}_{n+2} - k\widetilde{T}_{n+3} = &\ T_n + \alpha T_{n+2} + \beta(2T_{n+2} + T_{n+1}) \\ &+\alpha\beta(4T_{n+2} + 4T_{n+1} + T_n) \\ = &\ (1+\alpha\beta)T_n + (\beta + 4\alpha\beta)T_{n+1} \\ &+(\alpha + 2\beta + 4\alpha\beta)T_{n+2}.\end{aligned}$$

(ii) From equations (2), (7), (9), and (11), we easily obtain:

$$\begin{aligned}\widetilde{T}_n\overline{\widetilde{T}_n} = &\ (T_n + iT_{n+1} + jT_{n+2} + kT_{n+3})\,(T_n - iT_{n+1} - jT_{n+2} - kT_{n+3}) \\ = &\ T_n^2 + \alpha T_{n+1}^2 + \beta T_{n+2}^2 + \alpha\beta T_{n+3}^2.\end{aligned}$$

(iii) Using equations (7) and (10), we obtain:

$$\begin{aligned}\widetilde{T}_n + \overline{\widetilde{T}_n} = &\ (T_n + iT_{n+1} + jT_{n+2} + kT_{n+3}) + (T_n - iT_{n+1} - jT_{n+2} - kT_{n+3}) \\ = &\ 2T_n.\end{aligned}$$

(iv) Utilizing the property calculated in part (iii), we obtain:

$$\widetilde{T}_n^2 = \widetilde{T}_n\widetilde{T}_n = \widetilde{T}_n(2T_n - \overline{\widetilde{T}_n}) = 2\widetilde{T}_nT_n - \widetilde{T}_n\overline{\widetilde{T}_n}.$$

Ultimately, as shown in Table 3, we derive special cases with respect to the classification of Pell–Padovan generalized quaternions regarding properties (16) and (17).

For	$\widetilde{T}_n - i\widetilde{T}_{n+1} - j\widetilde{T}_{n+2} - k\widetilde{T}_{n+3}$	$\widetilde{T}_n\overline{\widetilde{T}_n}$
$\alpha = 1, \beta = 1$	$2T_n + 5T_{n+1} + 7T_{n+2}$	$T_n^2 + T_{n+1}^2 + T_{n+2}^2 + T_{n+3}^2$
$\alpha = 1, \beta = -1$	$-5T_{n+1} - 5T_{n+2}$	$T_n^2 + T_{n+1}^2 - T_{n+2}^2 - T_{n+3}^2$
$\alpha = 1, \beta = 0$	$T_n + T_{n+2}$	$T_n^2 + T_{n+1}^2$
$\alpha = -1, \beta = 0$	$T_n - T_{n+2}$	$T_n^2 - T_{n+1}^2$
$\alpha = 0, \beta = 0$	T_n	T_n^2

Table 3. Special cases

This completes the proof. □

Theorem 3.5. *Let $\widetilde{T}_n, \widetilde{U}_n \in \widetilde{T}_{\alpha\beta}$. $\forall n \in \mathbb{Z}$, the following properties are satisfied:*

(i) $$\widetilde{T}_n\overline{\widetilde{U}_n} - \overline{\widetilde{T}_n}\widetilde{U}_n = 2i(T_{n+1}U_n - T_nU_{n+1}) + 2j(T_{n+2}U_n - T_nU_{n+2}) + 2k(T_{n+3}U_n - T_nU_{n+3}),$$

(ii) $$\begin{aligned}\widetilde{T}_n\overline{\widetilde{U}_n} + \overline{\widetilde{T}_n}\widetilde{U}_n = 2[&T_nU_n + \alpha T_{n+1}U_{n+1} + \beta T_{n+2}U_{n+2} + \alpha\beta T_{n+3}U_{n+3}\\ &+ \beta i(T_{n+3}U_{n+2} - T_{n+2}U_{n+3})\\ &+ \alpha j(T_{n+1}U_{n+3} - T_{n+3}U_{n+1})\\ &+ k(T_{n+2}U_{n+1} - T_{n+1}U_{n+2})],\end{aligned}$$

(iii) $$\widetilde{T}_n\widetilde{U}_n - \overline{\widetilde{T}_n}\,\overline{\widetilde{U}_n} = 2i(T_nU_{n+1} + T_{n+1}U_n) + 2j(T_nU_{n+2} + T_{n+2}U_n) + 2k(T_nU_{n+3} + T_{n+3}U_n),$$

(iv) $$\begin{aligned}\widetilde{T}_n\widetilde{U}_n + \overline{\widetilde{T}_n}\,\overline{\widetilde{U}_n} = 2[&T_nU_n - \alpha T_{n+1}U_{n+1} - \beta T_{n+2}U_{n+2} - \alpha\beta T_{n+3}U_{n+3}\\ &+ \beta i(T_{n+2}U_{n+3} - T_{n+3}U_{n+2})\\ &+ \alpha j(T_{n+3}U_{n+1} - T_{n+1}U_{n+3})\\ &+ k(T_{n+1}U_{n+2} - T_{n+2}U_{n+1})].\end{aligned}$$

Proof. (i) Using the equations (2), (7), (9), and (11), we obtain the following:

$$\begin{aligned}\widetilde{T}_n\overline{\widetilde{U}}_n - \overline{\widetilde{T}}_n\widetilde{U}_n =\ & (T_n + iT_{n+1} + jT_{n+2} + kT_{n+3})(U_n - iU_{n+1} - jU_{n+2} - kU_{n+3})\\ & -(T_n - iT_{n+1} - jT_{n+2} - kT_{n+3})(U_n + iU_{n+1} + jU_{n+2} + kU_{n+3})\\ =\ & 2i(T_{n+1}U_n - T_nU_{n+1}) + 2j(T_{n+2}U_n - T_nU_{n+2})\\ & +2k(T_{n+3}U_n - T_nU_{n+3}).\end{aligned}$$

(ii) In the same manner as the proof of part (i), we obtain:

$$\begin{aligned}\widetilde{T}_n\overline{\widetilde{U}}_n + \overline{\widetilde{T}}_n\widetilde{U}_n =\ & (T_n + iT_{n+1} + jT_{n+2} + kT_{n+3})(U_n - iU_{n+1} - jU_{n+2} - kU_{n+3})\\ & +(T_n - iT_{n+1} - jT_{n+2} - kT_{n+3})(U_n + iU_{n+1} + jU_{n+2} + kU_{n+3})\\ =\ & 2[T_nU_n + \alpha T_{n+1}U_{n+1} + \beta T_{n+2}U_{n+2} + \alpha\beta T_{n+3}U_{n+3}\\ & +\beta i(T_{n+3}U_{n+2} - T_{n+2}U_{n+3})\\ & +\alpha j(T_{n+1}U_{n+3} - T_{n+3}U_{n+1})\\ & +k(T_{n+2}U_{n+1} - T_{n+1}U_{n+2})].\end{aligned}$$

(iii) Utilizing the same method as the proof of part (i), we obtain:

$$\begin{aligned}\widetilde{T}_n\widetilde{U}_n - \overline{\widetilde{T}}_n\overline{\widetilde{U}}_n =\ & (T_n + iT_{n+1} + jT_{n+2} + kT_{n+3})(U_n + iU_{n+1} + jU_{n+2} + kU_{n+3})\\ & -(T_n - iT_{n+1} - jT_{n+2} - kT_{n+3})(U_n - iU_{n+1} - jU_{n+2} - kU_{n+3})\\ =\ & 2i(T_nU_{n+1} + T_{n+1}U_n) + 2j(T_nU_{n+2} + T_{n+2}U_n)\\ & +2k(T_nU_{n+3} + T_{n+3}U_n).\end{aligned}$$

(iv) Using the same method as the proof of part (i), we obtain:

$$\begin{aligned}\widetilde{T}_n\widetilde{U}_n + \overline{\widetilde{T}}_n\overline{\widetilde{U}}_n =\ & (T_n + iT_{n+1} + jT_{n+2} + kT_{n+3})(U_n + iU_{n+1} + jU_{n+2} + kU_{n+3})\\ & +(T_n - iT_{n+1} - jT_{n+2} - kT_{n+3})(U_n - iU_{n+1} - jU_{n+2} - kU_{n+3})\\ =\ & 2[T_nU_n - \alpha T_{n+1}U_{n+1} - \beta T_{n+2}U_{n+2} - \alpha\beta T_{n+3}U_{n+3}\\ & +\beta i(T_{n+2}U_{n+3} - T_{n+3}U_{n+2})\\ & +\alpha j(T_{n+3}U_{n+1} - T_{n+1}U_{n+3})\\ & +k(T_{n+1}U_{n+2} - T_{n+2}U_{n+1})].\end{aligned}$$

Hence, we have also completed the proofs. □

In addition, we can classify the properties in part (ii) and (iv) of the Theorem 3.5 as follows:

Special Cases:

For part (ii), we have the following special cases:

$$\widetilde{T}_n\overline{\widetilde{U}_n}+\overline{\widetilde{T}_n}\widetilde{U}_n=\begin{cases}2[T_nU_n+T_{n+1}U_{n+1}+T_{n+2}U_{n+2}+T_{n+3}U_{n+3} & \alpha=1,\beta=1,\\ +i(T_{n+3}U_{n+2}-T_{n+2}U_{n+3}) & \\ +j(T_{n+1}U_{n+3}-T_{n+3}U_{n+1}) & \\ +k(T_{n+2}U_{n+1}-T_{n+1}U_{n+2})]; & \\ & \\ 2[T_nU_n+T_{n+1}U_{n+1}-T_{n+2}U_{n+2}-T_{n+3}U_{n+3} & \alpha=1,\beta=-1,\\ -i(T_{n+3}U_{n+2}-T_{n+2}U_{n+3}) & \\ +j(T_{n+1}U_{n+3}-T_{n+3}U_{n+1}) & \\ +k(T_{n+2}U_{n+1}-T_{n+1}U_{n+2})]; & \\ & \\ 2[T_nU_n+T_{n+1}U_{n+1} & \alpha=1,\beta=0,\\ +j(T_{n+1}U_{n+3}-T_{n+3}U_{n+1}) & \\ +k(T_{n+2}U_{n+1}-T_{n+1}U_{n+2})]; & \\ & \\ 2[T_nU_n-T_{n+1}U_{n+1} & \alpha=-1,\beta=0,\\ -j(T_{n+1}U_{n+3}-T_{n+3}U_{n+1}) & \\ +k(T_{n+2}U_{n+1}-T_{n+1}U_{n+2})]; & \\ & \\ 2[T_nU_n+k(T_{n+2}U_{n+1}-T_{n+1}U_{n+2})]; & \alpha=0,\beta=0.\end{cases}$$

For part (iv), a similar characterization can be given. The other parts are independent of the values of α and β.

Inspired by the study [4], we now give the several recurrence relations related to Pell–Padovan numbers and Pell–Padovan generalized quaternions in the following two theorems, respectively:

Theorem 3.6. *If T_n is the n-th Pell–Padovan number, then*

$$T_n=\rho_aT_{n-a}+\sigma_aT_{n-2a}+T_{n-3a}$$

is obtained with (ρ_a,σ_a) such that $\rho_a,\sigma_a\in\mathbb{Z}$; $1\leq a\leq 10$; $a\in\mathbb{N}$; $n\in\mathbb{Z}$ (see Table 4).

Proof. We now use mathematical induction on n for $a=1$. For $n=0,1,2,3$, the equation is true from Table 1. We assume that the equation is valid for $n\in\mathbb{Z}$: $T_n=2T_{n-2}+T_{n-3}$. We now have to show that the equation is true for $n+1\in\mathbb{Z}$. Using the equation (1), we obtain:

$$2T_{n-1}+T_{n-2}\quad=(T_{n+1}-T_{n-2})+T_{n-2}=T_{n+1}.$$

For $2\leq a\leq 10$, the other equalities can easily be demonstrated. □

a	(ρ_a, σ_a)	$T_n = \rho_a T_{n-a} + \sigma_a T_{n-2a} + T_{n-3a}$
1	$(0, 2)$	$T_n = 2T_{n-2} + T_{n-3}$
2	$(4, -4)$	$T_n = 4T_{n-2} - 4T_{n-4} + T_{n-6}$
3	$(3, 5)$	$T_n = 3T_{n-3} + 5T_{n-6} + T_{n-9}$
4	$(8, -8)$	$T_n = 8T_{n-4} - 8T_{n-8} + T_{n-12}$
5	$(10, 12)$	$T_n = 10T_{n-5} + 12T_{n-10} + T_{n-15}$
6	$(19, -19)$	$T_n = 19T_{n-6} - 19T_{n-12} + T_{n-18}$
7	$(28, 30)$	$T_n = 28T_{n-7} + 30T_{n-14} + T_{n-21}$
8	$(48, -48)$	$T_n = 48T_{n-8} - 48T_{n-16} + T_{n-24}$
9	$(75, 77)$	$T_n = 75T_{n-9} + 77T_{n-18} + T_{n-27}$
10	$(124, -124)$	$T_n = 124T_{n-10} - 124T_{n-20} + T_{n-30}$

Table 4. Some recurrence relations for Pell–Padovan numbers

Theorem 3.7. *If $\widetilde{T}_n \in \widetilde{T}_{\alpha\beta}$, then*

$$\widetilde{T}_n = \rho_a \widetilde{T}_{n-a} + \sigma_a \widetilde{T}_{n-2a} + \widetilde{T}_{n-3a}$$

is obtained with (ρ_a, σ_a) such that $\rho_a, \sigma_a \in \mathbb{Z}$; $1 \leq a \leq 10$; $a \in \mathbb{N}$; $n \in \mathbb{Z}$.

a	(ρ_a, σ_a)	$\widetilde{T}_n = \rho_a \widetilde{T}_{n-a} + \sigma_a \widetilde{T}_{n-2a} + \widetilde{T}_{n-3a}$
1	$(0, 2)$	$\widetilde{T}_n = 2\widetilde{T}_{n-2} + \widetilde{T}_{n-3}$
2	$(4, -4)$	$\widetilde{T}_n = 4\widetilde{T}_{n-2} - 4\widetilde{T}_{n-4} + \widetilde{T}_{n-6}$
3	$(3, 5)$	$\widetilde{T}_n = 3\widetilde{T}_{n-3} + 5\widetilde{T}_{n-6} + \widetilde{T}_{n-9}$
4	$(8, -8)$	$\widetilde{T}_n = 8\widetilde{T}_{n-4} - 8\widetilde{T}_{n-8} + \widetilde{T}_{n-12}$
5	$(10, 12)$	$\widetilde{T}_n = 10\widetilde{T}_{n-5} + 12\widetilde{T}_{n-10} + \widetilde{T}_{n-15}$
6	$(19, -19)$	$\widetilde{T}_n = 19\widetilde{T}_{n-6} - 19\widetilde{T}_{n-12} + \widetilde{T}_{n-18}$
7	$(28, 30)$	$\widetilde{T}_n = 28\widetilde{T}_{n-7} + 30\widetilde{T}_{n-14} + \widetilde{T}_{n-21}$
8	$(48, -48)$	$\widetilde{T}_n = 48\widetilde{T}_{n-8} - 48\widetilde{T}_{n-16} + \widetilde{T}_{n-24}$
9	$(75, 77)$	$\widetilde{T}_n = 75\widetilde{T}_{n-9} + 77\widetilde{T}_{n-18} + \widetilde{T}_{n-27}$
10	$(124, -124)$	$\widetilde{T}_n = 124\widetilde{T}_{n-10} - 124\widetilde{T}_{n-20} + \widetilde{T}_{n-30}$

Table 5. Some recurrence relations for Pell–Padovan generalized quaternions

Proof. For $a = 1$, using the equations given in Table 4, (7), (10), and (12), we obtain:

$$\begin{aligned} 2\widetilde{T}_{n-2} + \widetilde{T}_{n-3} &= 2(T_{n-2} + iT_{n-1} + jT_n + kT_{n+1}) + T_{n-3} + iT_{n-2} + jT_{n-1} + kT_n \\ &= 2T_{n-2} + T_{n-3} + i(2T_{n-1} + T_{n-2}) + j(2T_n + T_{n-1}) \\ &\quad + k(2T_{n+1} + T_n) \\ &= T_n + iT_{n+1} + jT_{n+2} + kT_{n+3} \\ &= \widetilde{T}_n. \end{aligned}$$

For $2 \leq a \leq 10$, the other equalities can easily be shown using the same method. □

Theorem 3.8 (Generating functions). *If $\widetilde{T}_n \in \widetilde{T}_{\alpha\beta}$, then the generating functions for $\widetilde{T}_n$ with positive and negative indices are obtained as follows, respectively:*

$$H(x) = \frac{\widetilde{T}_0 + \widetilde{T}_1 x + (\widetilde{T}_2 - 2\widetilde{T}_0)x^2}{1 - 2x^2 - x^3}, \tag{20}$$

$$G(x) = \frac{\widetilde{T}_0 + (2\widetilde{T}_0 + \widetilde{T}_{-1})x + (2\widetilde{T}_{-1} + \widetilde{T}_{-2})x^2}{1 + 2x - x^3}. \tag{21}$$

Proof. We assume the function

$$H(x) = \sum_{n=0}^{\infty} \widetilde{T}_n\, x^n = \widetilde{T}_0 + \widetilde{T}_1 x + \widetilde{T}_2 x^2 + \cdots + \widetilde{T}_n x^n + \cdots$$

is generating function of the Pell–Padovan generalized quaternions. Additionally, by multiplying both sides of the equality with the term $2x^2$, we obtain:

$$2x^2 H(x) = 2\widetilde{T}_0 x^2 + 2\widetilde{T}_1 x^3 + 2\widetilde{T}_2 x^4 + \cdots + 2\widetilde{T}_{n+1} x^{n+3} + \cdots$$

Furthermore, with the term x^3, we obtain:

$$x^3 H(x) = \widetilde{T}_0 x^3 + \widetilde{T}_1 x^4 + \widetilde{T}_2 x^5 + \cdots + \widetilde{T}_n x^{n+3} + \cdots$$

By performing necessary calculations, we obtain the following:

$$\begin{aligned}(1 - 2x^2 - x^3)H(x) = {} & \widetilde{T}_0 + \widetilde{T}_1 x + (\widetilde{T}_2 - 2\widetilde{T}_0)x^2 + (\widetilde{T}_3 - 2\widetilde{T}_1 - \widetilde{T}_0)x^3 \\ & + \cdots + (\widetilde{T}_{n+3} - 2\widetilde{T}_{n+1} - \widetilde{T}_n)x^{n+3} + \cdots\end{aligned}$$

Eventually, utilizing (12), we obtain (20), and the equality (21) can also be arrived at easily. □

Theorem 3.9 (Binet-like formula). *Let $\widetilde{T}_n \in \widetilde{T}_{\alpha\beta}$. $\forall n \in \mathbb{Z}$, the Binet-like formula for $\widetilde{T}_n$ is given as:*

$$\widetilde{T}_n = 2\left(\frac{\underline{a} r_1^{n+1} - \underline{b} r_2^{n+1}}{r_1 - r_2}\right) - 2\left(\frac{\underline{a} r_1^{n} - \underline{b} r_2^{n}}{r_1 - r_2}\right) + \underline{c} r_3^{n+1}, \tag{22}$$

where $\underline{a} = 1 + ir_1 + jr_1^2 + kr_1^3$, $\underline{b} = 1 + ir_2 + jr_2^2 + kr_2^3$, $\underline{c} = 1 + ir_3 + jr_3^2 + kr_3^3$ and r_1, r_2, r_3 are the roots of the equation $x^3 - 2x - 1 = 0$.

Proof. Using the equations (6) and (7), we obtain:

$$\begin{aligned}\widetilde{T}_n = {} & T_n + iT_{n+1} + jT_{n+2} + kT_{n+3} \\ = {} & \frac{2}{r_1 - r_2} r_1^{n+1}\left(1 + ir_1 + jr_1^2 + kr_1^3\right) - \frac{2}{r_1 - r_2} r_2^{n+1}\left(1 + ir_2 + jr_2^2 + kr_2^3\right) \\ & - \frac{2}{r_1 - r_2} r_1^{n}\left(1 + ir_1 + jr_1^2 + kr_1^3\right) - \frac{2}{r_1 - r_2}\left(-r_2^{n}\right)\left(1 + ir_2 + jr_2^2 + kr_2^3\right) \\ & + r_3^{n+1}\left(1 + ir_3 + jr_3^2 + kr_3^3\right) \\ = {} & \frac{2}{r_1 - r_2}\underline{a} r_1^{n+1} - \frac{2}{r_1 - r_2}\underline{b} r_2^{n+1} - \frac{2}{r_1 - r_2}\underline{a} r_1^{n} + \frac{2}{r_1 - r_2}\underline{b} r_2^{n} + \underline{c} r_3^{n+1}.\end{aligned}$$

Ultimately, we obtain the Binet-like formula for $\widetilde{T}_n$. □

Theorem 3.10. *If $\forall n \in \mathbb{R}$, then the Simson formula for Pell–Padovan generalized quaternions are given by:*

$$\widetilde{S}_n = \begin{vmatrix} \widetilde{T}_{n+2} & \widetilde{T}_{n+1} & \widetilde{T}_n \\ \widetilde{T}_{n+1} & \widetilde{T}_n & \widetilde{T}_{n-1} \\ \widetilde{T}_n & \widetilde{T}_{n-1} & \widetilde{T}_{n-2} \end{vmatrix}.$$

Additionally, it is calculated as follows:

$$\widetilde{S}_n \ = [(1+2\alpha-4\beta+2\alpha\beta)+i\,(-\alpha+4\beta-4\alpha\beta)+j\,(4-\beta+2\alpha\beta)+k\,(4-\alpha\beta)]\,S_n,$$

where S_n is the Simson formula for Pell–Padovan numbers[1].

Proof. Using equation (12) and properties of determinant, the proof is clear. □

We now give the matrix representations of $\widetilde{T}_n$ and $\widetilde{T}_{-n}$.

Theorem 3.11. *Let $\widetilde{T}_n$, $\widetilde{T}_{-n} \in \widetilde{T}_{\alpha\beta}$. $\forall n \in \mathbb{Z}^+$, the following properties are obtained:*

(i) $$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \widetilde{T}_2 \\ \widetilde{T}_1 \\ \widetilde{T}_0 \end{pmatrix} = \begin{pmatrix} \widetilde{T}_{n+2} \\ \widetilde{T}_{n+1} \\ \widetilde{T}_n \end{pmatrix},$$

(ii) $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix}^n \begin{pmatrix} \widetilde{T}_0 \\ \widetilde{T}_{-1} \\ \widetilde{T}_{-2} \end{pmatrix} = \begin{pmatrix} \widetilde{T}_{-n} \\ \widetilde{T}_{-n-1} \\ \widetilde{T}_{-n-2} \end{pmatrix},$$

(iii) $$\begin{pmatrix} \widetilde{T}_0 & \widetilde{T}_1 & \widetilde{T}_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \widetilde{T}_n & \widetilde{T}_{n+1} & \widetilde{T}_{n+2} \end{pmatrix},$$

(iv) $$\begin{pmatrix} \widetilde{T}_{-2} & \widetilde{T}_{-1} & \widetilde{T}_0 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^n = \begin{pmatrix} \widetilde{T}_{-n-2} & \widetilde{T}_{-n-1} & \widetilde{T}_{-n} \end{pmatrix},$$

(v) $$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \widetilde{T}_2 & \widetilde{T}_1 & \widetilde{T}_0 \\ \widetilde{T}_1 & \widetilde{T}_0 & \widetilde{T}_{-1} \\ \widetilde{T}_0 & \widetilde{T}_{-1} & \widetilde{T}_{-2} \end{pmatrix} = \begin{pmatrix} \widetilde{T}_{n+2} & \widetilde{T}_{n+1} & \widetilde{T}_n \\ \widetilde{T}_{n+1} & \widetilde{T}_n & \widetilde{T}_{n-1} \\ \widetilde{T}_n & \widetilde{T}_{n-1} & \widetilde{T}_{n-2} \end{pmatrix},$$

(vi) $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix}^n \begin{pmatrix} \widetilde{T}_2 & \widetilde{T}_1 & \widetilde{T}_0 \\ \widetilde{T}_1 & \widetilde{T}_0 & \widetilde{T}_{-1} \\ \widetilde{T}_0 & \widetilde{T}_{-1} & \widetilde{T}_{-2} \end{pmatrix} = \begin{pmatrix} \widetilde{T}_{-n+2} & \widetilde{T}_{-n+1} & \widetilde{T}_{-n} \\ \widetilde{T}_{-n+1} & \widetilde{T}_{-n} & \widetilde{T}_{-n-1} \\ \widetilde{T}_{-n} & \widetilde{T}_{-n-1} & \widetilde{T}_{-n-2} \end{pmatrix}.$$

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[1]Simson formula for Pell–Padovan numbers is given by (see in [37]): $S_n = \begin{vmatrix} T_{n+2} & T_{n+1} & T_n \\ T_{n+1} & T_n & T_{n-1} \\ T_n & T_{n-1} & T_{n-2} \end{vmatrix}$.

Proof. (i) We now use the mathematical induction on n. For $n = 1$, it is true using the equation (12):

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{T}_2 \\ \widetilde{T}_1 \\ \widetilde{T}_0 \end{pmatrix} = \begin{pmatrix} 2\widetilde{T}_1 + \widetilde{T}_0 \\ \widetilde{T}_2 \\ \widetilde{T}_1 \end{pmatrix} = \begin{pmatrix} \widetilde{T}_3 \\ \widetilde{T}_2 \\ \widetilde{T}_1 \end{pmatrix} = \begin{pmatrix} \widetilde{T}_{1+2} \\ \widetilde{T}_{1+1} \\ \widetilde{T}_1 \end{pmatrix}.$$

If we now assume that it is true for $n = k \in \mathbb{Z}^+$, then:

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widetilde{T}_2 \\ \widetilde{T}_1 \\ \widetilde{T}_0 \end{pmatrix} = \begin{pmatrix} \widetilde{T}_{k+2} \\ \widetilde{T}_{k+1} \\ \widetilde{T}_k \end{pmatrix}.$$

We then have to demonstrate that it is true for, $n = k + 1 \in \mathbb{Z}^+$ using the power property of matrices and (12):

$$\begin{aligned} \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widetilde{T}_2 \\ \widetilde{T}_1 \\ \widetilde{T}_0 \end{pmatrix} &= \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widetilde{T}_2 \\ \widetilde{T}_1 \\ \widetilde{T}_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{T}_{k+2} \\ \widetilde{T}_{k+1} \\ \widetilde{T}_k \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{T}_{(k+1)+2} \\ \widetilde{T}_{(k+1)+1} \\ \widetilde{T}_{(k+1)} \end{pmatrix}. \end{aligned}$$

Consequently, it is true for all $n \geq 1$ and the other parts can be similarly proved. □

We can obtain the following summation formulas for our special quaternions by utilizing the [39].

Theorem 3.12. *Let $\widetilde{T}_n \in \widetilde{T}_{\alpha\beta}$. $\forall n, m \in \mathbb{N}$, the following are obtained:*

(i) $\sum\limits_{n=0}^{m} \widetilde{T}_n = \frac{1}{2}\Big(\widetilde{T}_{m+2} + \widetilde{T}_{m+1} + \widetilde{T}_m + \widetilde{T}_0 - \widetilde{T}_1 - \widetilde{T}_2\Big)$,

(ii) $\sum\limits_{n=0}^{m} \widetilde{T}_{2n} = \widetilde{T}_{2m+1} + m(\widetilde{T}_2 - \widetilde{T}_1 - \widetilde{T}_0) + \widetilde{T}_0 - \widetilde{T}_1$,

(iii) $\sum\limits_{n=0}^{m} \widetilde{T}_{2n+1} = \frac{1}{2}\Big(\widetilde{T}_{2m+3} + \widetilde{T}_{2m+2} - \widetilde{T}_{2m+1} + 2m(-\widetilde{T}_2 + \widetilde{T}_1 + \widetilde{T}_0) - \widetilde{T}_2 + \widetilde{T}_1 - \widetilde{T}_0\Big)$.

Proof. (i) We now use mathematical induction. For $m = 0, 1, 2$, the equality is obvious. We then assume equality supplied for $m \in \mathbb{N}$. This is as follows:

$$\sum_{n=0}^{m} \widetilde{T}_n = \frac{1}{2}\Big(\widetilde{T}_{m+2} + \widetilde{T}_{m+1} + \widetilde{T}_m + \widetilde{T}_0 - \widetilde{T}_1 - \widetilde{T}_2\Big).$$

We now need to demonstrate that the equality is provided for $m + 1$. Using (12), we obtained the following:

$$\begin{aligned}
\sum_{n=0}^{m+1}\widetilde{T}_n &= \sum_{n=0}^{m}\widetilde{T}_n+\widetilde{T}_{m+1}\\
&= \frac{1}{2}\Big(\widetilde{T}_{m+2}+\widetilde{T}_{m+1}+\widetilde{T}_{m}+\widetilde{T}_{0}-\widetilde{T}_{1}-\widetilde{T}_{2}\Big)+\widetilde{T}_{m+1}\\
&= \frac{1}{2}\Big(\widetilde{T}_{m+3}+\widetilde{T}_{m+2}+\widetilde{T}_{m+1}+\widetilde{T}_{0}-\widetilde{T}_{1}-\widetilde{T}_{2}\Big)\\
&= \frac{1}{2}\Big(\widetilde{T}_{(m+1)+2}+\widetilde{T}_{(m+1)+1}+\widetilde{T}_{m+1}+\widetilde{T}_{0}-\widetilde{T}_{1}-\widetilde{T}_{2}\Big).
\end{aligned}$$

The other parts are now obvious. □

Theorem 3.13. *Let $\widetilde{T}_{-n}\in\widetilde{T}_{\alpha\beta}$. $\forall n,m\in\mathbb{Z}^+$, the following are obtained:*

(i) $\sum_{n=1}^{m}\widetilde{T}_{-n}=\frac{1}{2}\Big(-3\widetilde{T}_{-m-1}-3\widetilde{T}_{-m-2}-\widetilde{T}_{-m-3}+\widetilde{T}_{2}+\widetilde{T}_{1}-\widetilde{T}_{0}\Big)$

(ii) $\sum_{n=1}^{m}\widetilde{T}_{-2n}=-\widetilde{T}_{-2m+1}+\widetilde{T}_{-2m}+m(\widetilde{T}_{2}-\widetilde{T}_{1}-\widetilde{T}_{0})+\widetilde{T}_{1}-\widetilde{T}_{0},$

(iii) $\sum_{n=1}^{m}\widetilde{T}_{-2n+1}=\frac{1}{2}\Big(\widetilde{T}_{-2m+1}-3\widetilde{T}_{-2m}-\widetilde{T}_{-2m-1}+2m(-\widetilde{T}_{2}+\widetilde{T}_{1}+\widetilde{T}_{0})+\widetilde{T}_{2}-\widetilde{T}_{1}+\widetilde{T}_{0}\Big).$

Proof. Using mathematical induction, we can complete the proofs. □

4 Conclusions

In this paper, we have derived Pell–Padovan generalized quaternions and established several new properties relating to these. We have determined the relationship between Pell–Padovan generalized quaternions and Pell–Padovan generalized quaternions with negative indices. In Table 4 and Table 5, we have identified several new useful recurrence relations with respect to Pell–Padovan numbers and Pell–Padovan generalized quaternions, respectively. In addition to these, the Binet-like formula, generating function, and Simson formula were obtained and the matrix representations and summation properties of these quaternions were examined.

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