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A note on $\mathcal{Q}$-matrices and higher order Fibonacci polynomials

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Abstract: The results described in a recent article, relative to a representation formula for the generalized Fibonacci sequences in terms of $\mathcal{Q}$-matrices are extended to the case of Fibonacci, Tribonacci and R-bonacci polynomials.
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1 Introduction

In a recent article the classical equation permitting to compute the powers of the Golden ratio Phi in terms of Fibonacci's numbers, by using the companion matrix associated to the second order recursion, has been recovered. The history of this equation was reported by H .W. Gould [8] and the considered technique was later extended to general recursions by J. Ivie [11].

It is worth to note that links between the Golden ratio and the Fibonacci numbers – usually ascribed to J. Kepler, or to É. Lucas – actually date back to Indian mathematicians, in connection with Sanscrit metrics. In a historical article by P. Singh [22], it is recalled that, long before L. Pisano (about 1220 BC), Virahanka (600–800 BC), Gopala (before 1135 BC) and Hemachandra (about 1150 BC) introduced the Fibonacci numbers and the method of their generation.

In [21] the recalled equation, even in the more general case of Tribonacci and, theoretically, of higher order number sequences has been derived. The method used there is based on the

Cayley–Hamilton theorem, according to the results in [7] and previous papers by the author on this subject [4, 20].

In this article, after recalling the $\mathcal{Q}$-matrix method [17] for Fibonacci polynomials, the same technique proposed in [21] is applied to the Tribonacci and R-bonacci cases [10]. A different equation with respect to that derived in [11] is proposed.

In a recent article K. Yordzhev [24] considered the set of square boolean matrices with the same number of 1's in each row and each column. Associating to each matrix an ordered n-tuple of natural numbers and introducing a suitable equivalence relation between the factor-sets, he was able to prove, for particuar values of the parametes, a connection with the sequence of Fibonacci numbers. Considering the sequence of Tribonacci (or R-bonacci) numbers, we conjecture that it should be possible to extend the procedure considered in [24] to the case of 3-dimensional (or R-dimensional) boolean tensors.

2 Basic definitions

Definition. Given an $r \times r$, matrix $\mathcal{A} = (a_{ij})$, with real or complex entries its *characteristic polynomial* is given by

$$P(\lambda) := \det(\lambda \mathcal{I} - \mathcal{A}) = \lambda^r - u_1 \lambda^{r-1} + u_2 \lambda^{r-2} + \ldots + (-1)^r u_r \,. \tag{1}$$

and the coefficients

$$\begin{cases} u_1 := tr\,\mathcal{A} = a_{11} + a_{22} + \ldots + a_{rr} \\ u_2 := \displaystyle\sum_{i<j}^{1,r} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ u_3 := \displaystyle\sum_{i<j<k}^{1,r} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} \\ \vdots \\ u_r := \det\,\mathcal{A} \end{cases} \tag{2}$$

are called the *invariants* of $\mathcal{A}$.

2.1 Recalling the $F_{k,n}$ functions

It is well known [4, 19, 20] that a basis for the r-dimensional vector space of solutions of the homogeneous linear bilateral (i.e., running for all integer indexes) recurrence relation with constant coefficients u_k $(k = 1, 2, \ldots, r)$, with $u_r \neq 0$,

$$X_n = u_1 X_{n-1} - u_2 X_{n-2} + \cdots + (-1)^{r-1} u_r X_{n-r}\,, \qquad (n \in \mathbf{Z})\,, \tag{3}$$

is given by the functions $F_{k,n} = F_{k,n}(u_1, u_2, \ldots, u_r)$, $(k = 1, 2, \ldots, r\,,\ n \geq -1)$, defined by the initial conditions below:

$$
\begin{array}{llll}
F_{1,-1}=0 & F_{1,0}=0 & \dots & F_{1,r-2}=1, \\
F_{2,-1}=0 & F_{2,0}=1 & \dots & F_{2,r-2}=0, \\
\vdots & \vdots & \ddots & \vdots \\
F_{r,-1}=1 & F_{r,0}=0 & \dots & F_{r,r-2}=0.
\end{array} \tag{4}
$$

Remark 1. The $F_{k,n}$ functions constitute a different basis with respect to the usual one, which uses the roots of the characteristic equation [2]. This basis does not imply the knowledge of roots and does not depend on their multiplicity, so that it is sometimes more convenient.

It has been shown by É. Lucas [16, 20] that all $\{F_{k,n}\}_{n\in\mathbf{Z}}$ functions are expressed through the only bilateral sequence $\{F_{1,n}\}_{n\in\mathbf{Z}}$ the bilateral sequence $\{F_{1,n}\}_{n\in\mathbf{Z}}$, corresponding to the initial conditions in (4). More precisely, the following equations hold

$$
\begin{cases}
F_{1,n} = u_1 F_{1,n-1} + F_{2,n-1} \\
F_{2,n} = -u_2 F_{1,n-1} + F_{3,n-1} \\
\vdots \\
F_{r-1,n} = (-1)^{r-2} u_{r-1} F_{1,n-1} + F_{r,n-1} \\
F_{r,n} = (-1)^{r-1} u_r F_{1,n-1}
\end{cases} \tag{5}
$$

Therefore, the bilateral sequence $\{F_{1,n}\}_{n\in\mathbf{Z}}$ called the *fundamental solution* of (3) (*"fonction fondamentale"* by É. Lucas [16]).

The functions $F_{1,n}(u_1,\dots,u_r)$ are called in literature [19] *generalized Lucas polynomials of the second kind*, and are related to the multivariate Chebyshev polynomials (see e.g. R. Lidl and C. Wells [15], R. Lidl [14], T. Koornwinder [12, 13], M. Bruschi and P. E. Ricci [3], K. B. Dunn and R. Lidl [6], R. J. Beerends [1]).

2.2 Matrix powers representation

In preceding articles [4], [20], the following result has been proved:

Theorem 2.1. *Given an $r\times r$ matrix $\mathcal{A}$, with real or complex entries, and denoting by equation* (1) *its characteristic polynomial, the matrix powers $\mathcal{A}^n$, with integer exponent n, are given by the equation:*

$$
\mathcal{A}^n = F_{1,n-1}(u_1,\dots,u_r)\mathcal{A}^{r-1} + F_{2,n-1}(u_1,\dots,u_r)\mathcal{A}^{r-2} + \cdots + F_{r,n-1}(u_1,\dots,u_r)\mathcal{I}\,, \tag{6}
$$

where the functions $F_{k,n}(u_1,\dots,u_r)$ are defined in Section 2.1.

For more information, see e.g. [20, 23].

2.3 The general companion matrix

In what follows, we consider the companion matrix

$$\mathcal{Q}_{r\times r} = \mathcal{Q} := \begin{pmatrix} u_1 & 1 & 0 & \cdots & 0 & 0 \\ u_2 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{r-2} & 0 & 0 & \cdots & 1 & 0 \\ u_{r-1} & 0 & 0 & \cdots & 0 & 1 \\ u_r & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{7}$$

associated with the linear recurrence relation (3).

The invariants of the matrix (7) are given by

$$u_1, \quad -u_2, \quad u_3, \quad \ldots, \quad (-1)^r u_r\,,$$

so that its characteristic polynomial $P(\lambda)$ is given by equation (1), and its highest eigenvalue (in modulus) is a solution of the equation

$$\lambda^r = u_1\lambda^{r-1} - u_2\lambda^{r-2} + \ldots + (-1)^{r-1}u_{r-1}\lambda + (-1)^r u_r\,. \tag{8}$$

Recalling Cauchy's bounds for the roots of polynomials [9, 18], it immediately follows that, in case of the equation (8), the highest (in modulus) eigenvalue is bounded by

$$1 + \max\left\{\frac{|u_{r-1}|}{|u_r|}, \frac{|u_{r-2}|}{|u_r|}, \cdots, \frac{1}{|u_r|}\right\}. \tag{9}$$

3 Extending a Lucas' formula

The well known equation

$$\Phi^n = f_n\Phi + f_{n-1}\,, \tag{10}$$

relating the powers of the Golden ratio with the classical sequence of Fibonacci numbers $\{f_n\}$, ($f_{n+1} = f_n + f_{n-1}$, with $f_0 = 0,\ f_1 = 1$) can be derived by using the $\mathcal{Q}$-matrix

$$\mathcal{Q} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{11}$$

in the form:

$$\mathcal{Q}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}. \tag{12}$$

In [21] the equation (12) has been recovered by using equation (6), with $r = 2$.

Remark 2. Note that the definition of the Fibonacci sequence is sometimes started form the initial conditions $f_0 = 1,\ f_1 = 1$, which simply implies a shift of the index.

3.1 The Fibonacci polynomial case

Now, it is well known that the companion matrix

$$\mathcal{Q}(x) := \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \tag{13}$$

is associated to the recursion:

$$f_{n+1}(x) = x\, f_n(x) + f_{n-1}(x)\,, \tag{14}$$

and, assuming the initial conditions: $f_0(x) = 0, \quad f_1(x) = 1,$ the sequence of Fibonacci polynomials $\{f_n(x)\}$ follows.

Here, and in what follows, the variable x is assumed to be real or complex, as computations remain valid in both cases.

In the polynomial case, it is possible to use the same technique described in [21]. In fact, starting from equation (6), with $r = 2$, since

$$F_{1,n-1}(x,-1) = f_n(x) \qquad F_{2,n-1}(x,-1) = f_{n-1}(x)\,, \tag{15}$$

we find

$$\mathcal{Q}^n(x) = f_n(x)\, \mathcal{Q}(x) + f_{n-1}(x)\, \mathcal{I}\,, \tag{16}$$

and then we recover the well known equation:

$$\mathcal{Q}^n(x) = \begin{pmatrix} f_{n+1}(x) & f_n(x) \\ f_n(x) & f_{n-1}(x) \end{pmatrix}. \tag{17}$$

3.2 The Tribonacci polynomial case

We consider now the Tribonacci polynomials, defined by the recursion:

$$\begin{cases} t_{n+2}(x) = x^2\, t_{n+1}(x) + x\, t_n(x) + t_{n-1}(x), \\ t_0(x) = 0,\ t_1(x) = 1,\ t_2(x) = x^2\,, \end{cases} \tag{18}$$

so that, by using the above notation, we have:

$$u_1 = x^2,\ u_2 = -x,\ u_3 = 1\,.$$

The first few Tribonacci polynomials are:

$$t_1(x) = 1\,, \quad t_2(x) = x^2\,, \quad t_3(x) = x^4 + x\,, \quad t_4(x) = x^6 + 2x^3 + 1\,, \quad t_5(x) = x^8 + 3x^5 + 3x^2\,.$$

According to the results in [21], it is suitable to consider even the associated Tribonacci polynomials $\{t^*_n(x)\}$, satisfying the same recursion, but with different initial conditions:

$$\begin{cases} t^*_{n+2}(x) = x^2\, t^*_{n+1}(x) + x\, t^*_n(x) + t^*_{n-1}(x), \\ t^*_0(x) = 1,\ t^*_1(x) = 0,\ t^*_2(x) = x\,, \end{cases} \tag{19}$$

The first few associated Tribonacci polynomials are:

$$t^*_0(x) = 1\,, \quad t^*_1(x) = 0\,, \quad t^*_2(x) = x\,, \quad t^*_3(x) = x^3 + 1\,, \quad t^*_4(x) = x^5 + 2x^2\,, \quad t^*_5(x) = x^7 + 3x^4 + 2x\,.$$

Then, we can prove the following result.

Theorem 3.1. *Introducing the $\mathcal{Q}_{3\times 3}(x)$-matrix*

$$\mathcal{Q}_{3\times 3}(x) = \mathcal{Q}(x) = \begin{pmatrix} x^2 & 1 & 0 \\ x & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{20}$$

the positive integer powers of $\mathcal{Q}(x)$ (for $n \geq 2$) are represented by

$$\mathcal{Q}^n(x) = \begin{pmatrix} t_{n+1}(x) & t_n(x) & t_{n-1}(x) \\ t^*_{n+1}(x) & t^*_n(x) & t^*_{n-1}(x) \\ t_n(x) & t_{n-1}(x) & t_{n-2}(x) \end{pmatrix}. \tag{21}$$

Proof. Note that, putting $r = 3$ and taking into account that $u_1 = x^2, u_2 = -x, u_3 = 1$, the recursion (3) for the $F_{1,n-1}$ functions is given by

$$F_{1,n-1}(x^2, -x, 1) = x^2\, F_{1,n-2}(x^2, -x, 1) + x\, F_{2,n-3}(x^2, -x, 1) + F_{3,n-4}(x^2, -x, 1)\,, \tag{22}$$

with initial conditions $F_{1,-1} = F_{1,0} = 0,\ F_{1,1} = 1$, so that $F_{1,2} = x^2$. Consequently we find: $F_{1,2} = t_2(x)$, and in general, $F_{1,n-1}(x^2, -x, 1) = t_{n-1}(x)$.

A similar result holds for the $F_{3,n-1}$ functions, since we can put the shifted initial conditions $F_{3,0} = 0,\ F_{3,1} = 0,\ F_{3,2} = 1$, so that $F_{3,3} = x^2$. Consequently we find: $F_{3,3} = t_2(x)$, and in general, $F_{3,n-1}(x^2, -x, 1) = t_{n-2}(x)$.

For the $F_{2,n}$ functions we must recall equation (3), which gives:

$$F_{2,n-1}(x^2, -x, 1) = x^2\, F_{2,n-2}(x^2, -x, 1) + x\, F_{2,n-3}(x^2, -x, 1) + F_{2,n-4}(x^2, -x, 1)\,,$$

with initial conditions $F_{2,0} = 1,\ F_{2,1} = 0,\ F_{2,2} = x$, so that $F_{2,3} = x^3 + 1$. Consequently we find: $F_{2,3} = t^*_3(x)$, and in general, $F_{2,n-1}(x^2, -x, 1) = t^*_{n-1}(x)$.
Then equation (6)

$$\mathcal{Q}^n(x) = F_{1,n-1}(x^2, -x, 1)\, \mathcal{Q}^2(x) + F_{2,n-1}(x^2, -x, 1)\, \mathcal{Q} + F_{3,n-1}(x^2, -x, 1)\, \mathcal{I}$$

becomes:

$$\mathcal{Q}^n(x) = t_{n-1}(x)\, \mathcal{Q}^2(x) + t^*_{n-1}(x)\, \mathcal{Q} + t_{n-2}(x)\, \mathcal{I}\,, \tag{23}$$

which is equivalent to (21). □

3.3 Checking the first few powers

In order to check equation (21), consider the first matrix powers:

$$\mathcal{Q}^2(x) = \begin{pmatrix} x^4 + x & x^2 & 1 \\ x^3 + 1 & x & 1 \\ x^2 & 1 & 0 \end{pmatrix}, \qquad \mathcal{Q}^3(x) = \begin{pmatrix} x^6 + 2x^3 + 1 & x^4 + x & x^2 \\ x^5 + 2x^2 & x^3 + 1 & x \\ x^4 + x & x^2 & 1 \end{pmatrix},$$

$$\mathcal{Q}^4(x) = \begin{pmatrix} x^8 + 3x^5 + 3x^2 & x^6 + 2x^3 + 1 & x^4 + x \\ x^7 + 3x^4 + 2x & x^5 + 2x^2 & x^3 + 1 \\ x^6 + 2x^3 + 1 & x^4 + x & x^2 \end{pmatrix}.$$

Remark 3. Note that in the computation of $\mathcal{Q}^{n+1}(x)$ it is important to find only its first column, since, at each step, the last column in $\mathcal{Q}^n(x)$ disappears and the first two columns in $\mathcal{Q}^n(x)$ are shifted to the last two ones in $\mathcal{Q}^{n+1}(x)$. The general expression of companion matrix powers is analyzed, by the combinatorial point of view, in [5].

Assuming $n = 3$ and $n = 4$, and recalling the first few Tribonacci polynomials and their associated ones, it is immediately seen that the corresponding rows coincide. Then equation (21) holds for every n, because the matrix powers satisfy the same recursion of the R-Bonacci polynomials as well as their associated ones.

4 The R-bonacci polynomial case

We consider now the R-bonacci polynomials defined by the recursion:

$$\begin{cases} R_{n+r}(x) = x^{r-1}\, R_{n+r-1}(x) + x^{r-2}\, R_{n+r-2}(x) + \cdots + R_n(x), \\ R_{-k}(x) = 0,\ (k = 0, 1, \ldots, r-3),\ R_1(x) = 1,\ R_2(x) = x^{r-1}. \end{cases} \tag{24}$$

In what follows, we put by definition: $\varphi_n^{(1)}(x) := R_n(x)$. This in order to distinguish the R-bonacci polynomials defined by (23) from the associated R-bonacci-type polynomials verifying the same recurrence relation in (23), but different initial conditions. More precisely, we list below the initial condition for each considered R-bonacci-type sequence:

$$\begin{gathered}
\varphi^{(1)}_{-r+3}(x) = \cdots = \varphi^{(1)}_{-1}(x) = \varphi^{(1)}_{0}(x) = 0,\ \varphi^{(1)}_{1}(x) = 1,\ \varphi^{(1)}_{2}(x) = x^{r-1}\,, \\
\varphi^{(2)}_{-r+3}(x) = \cdots = \varphi^{(2)}_{-1}(x) = 0,\ \varphi^{(2)}_{0}(x) = 1,\ \varphi^{(2)}_{1}(x) = 0,\ \varphi^{(2)}_{2}(x) = x^{r-2}\,, \\
\vdots \\
\varphi^{(r-2)}_{-r+3}(x) = 0,\ \varphi^{(r-2)}_{-r+3}(x) = 1,\ \varphi^{(r-2)}_{-r+4}(x) = \cdots = \varphi^{(r-2)}_{1}(x) = 0,\ \varphi^{(r-2)}_{2}(x) = x^2\,, \\
\varphi^{(r-1)}_{-r+3}(x) = 1,\ \varphi^{(r-1)}_{-r+3}(x) = \varphi^{(r-1)}_{-r+4}(x) = \cdots = \varphi^{(r-1)}_{1}(x) = 0,\ \varphi^{(r-1)}_{2}(x) = x\,.
\end{gathered}$$

Note that the above initial conditions are borrowed from the first $r-1$ rows of $\mathcal{Q}(x)$, read in reverse order:

$$\mathcal{Q}_{r\times r}(x) = \mathcal{Q}(x) := \begin{pmatrix} x^{r-1} & 1 & 0 & \cdots & 0 & 0 \\ x^{r-2} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^2 & 0 & 0 & \cdots & 1 & 0 \\ x & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{25}$$

Therefore, the preceding results can be generalized as follows:

Theorem 4.1. *Putting for shortness:* $(\mathbf{u}) := (x^{r-1}, x^{r-2}, \cdots, x, 1)$*, that is the first column of matrix* (25)*, and starting from equation* (6):

$$\mathcal{Q}^n(x) = F_{1,n-1}(\mathbf{u})\, \mathcal{Q}^{r-1}(x) + F_{2,n-1}(\mathbf{u})\, \mathcal{Q}^{r-2}(x) + \cdots + F_{r,n-1}(\mathbf{u})\, \mathcal{I}\,, \tag{26}$$

it is possible to represent the powers of the $\mathcal{Q}$ *matrix* (25) *in terms of the sequences* $\{\varphi_n^k(x)\}$ $(k = 1, 2, \cdots, r-1)$. *More precisely, the following equation holds:*

$$\mathcal{Q}^n(x) = \begin{pmatrix} \varphi^{(1)}_{n+r-2}(x) & \varphi^{(1)}_{n+r-3}(x) & \dots & \varphi^{(1)}_{n-1}(x) \\ \varphi^{(2)}_{n+r-2}(x) & \varphi^{(2)}_{n+r-3}(x) & \dots & \varphi^{(2)}_{n-1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^{(r-1)}_{n+r-2}(x) & \varphi^{(r-1)}_{n+r-3}(x) & \dots & \varphi^{(r-1)}_{n-1}(x) \\ \varphi^{(1)}_{n+r-3}(x) & \varphi^{(1)}_{n+r-4}(x) & \cdots & \varphi^{(1)}_{n-2}(x) \end{pmatrix}. \tag{27}$$

Proof. The proof follows from the same technique used in Section 3.2. It is sufficient to check that the rows coincide for the first values of the power n. Then equation holds for every n, because the powers of $\mathcal{Q}(x)$ and the R-bonacci polynomials as well as their associated ones satisfy the same recursion.
The particular case of the Quadranacci polynomials is shown in the last section. □

4.1 Example: the Quadranacci polynomial case

Let $r = 4$, and consider the $\mathcal{Q}_{4\times4}(x)$-matrix

$$\mathcal{Q}_{4\times4}(x) = \mathcal{Q}(x) = \begin{pmatrix} x^3 & 1 & 0 & 0 \\ x^2 & 0 & 1 & 0 \\ x & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Note that, using the above notation, the first few Quadranacci and their associated are:

$$\varphi_2^{(1)} = x^3\,, \quad \varphi_3^{(1)} = x^6 + x^2\,, \quad \varphi_4^{(1)} = x^9 + 2x^5 + x\,, \quad \varphi_5^{(1)} = x^{12} + 3x^8 + 3x^4 + 1\,, \ \dots$$
$$\varphi_2^{(2)} = 0\,, \quad \varphi_3^{(2)} = x^2\,, \quad \varphi_4^{(2)} = x^5 + x\,, \quad \varphi_5^{(2)} = x^8 + 2x^4\,, \varphi_6^{(2)} = x^{11} + 3x^7 + 2x^3\,, \ \dots$$
$$\varphi_2^{(3)} = 0\,, \ \varphi_3^{(3)} = 0\,, \ \varphi_4^{(3)} = x\,, \ \varphi_5^{(3)} = x^4 + 1\,, \ \varphi_6^{(3)} = x^7 + 2x^3\,, \ \varphi_7^{(3)} = x^{10} + 3x^6 + 2x^2\,, \ \dots$$

We have:

$$\mathcal{Q}^2(x) = \begin{pmatrix} x^6 + x^2 & x^3 & 1 & 0 \\ x^5 + x & x^2 & 0 & 1 \\ x^4 + 1 & x & 0 & 0 \\ x^3 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{Q}^3(x) = \begin{pmatrix} x^9 + 2x^5 + x & x^6 + x^2 & x^3 & 1 \\ x^8 + 2x^4 + 1 & x^5 + x & x^2 & 0 \\ x^7 + 2x^3 & x^4 + 1 & x & 0 \\ x^6 + x^2 & x^3 & 1 & 0 \end{pmatrix},$$

$$\mathcal{Q}^4(x) = \begin{pmatrix} x^{12} + 3x^8 + 3x^4 + 1 & x^9 + 2x^5 + x & x^6 + x^2 & x^3 \\ x^{11} + 3x^7 + 2x^3 & x^8 + 2x^4 & x^5 + x & x^2 \\ x^{10} + 3x^6 + 2x^2 & x^7 + 2x^3 & x^4 + 1 & x \\ x^9 + 2x^5 + x & x^6 + x^2 & x^3 & 1 \end{pmatrix},$$

so that the initial conditions coincide and consequently equation (27) holds, by the same argument recalled in general.

Remark 4. Obviously, the present results, putting $x = 1$, recover the equations proved in [21] in the case of Fibonacci, Tribonacci and R-bonacci numbers.

5 Conclusion

By using a classical result about a representation formula for matrix powers [7], and the basic solution of a linear recurrence relation, it has been shown that the classical equation for powers of the $\mathcal{Q}(x)$-matrix, in terms of Fibonacci polynomials, can be easily recovered. Furthermore, the used technique have been extended to the case of the Tribonacci polynomial sequence, and to the general case of the higher order R-bonacci polynomial sequences. A representation of these powers, but different from that proposed, in an old article, by J. Ivie [11] in the case of numerical sequences is presented.

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