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Bi-unitary multiperfect numbers, IV(b)

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Dedicated to the memory of Prof. D. Suryanarayana

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Abstract: A divisor d of a positive integer n is called a unitary divisor if $\gcd(d, n/d) = 1$; and d is called a bi-unitary divisor of n if the greatest common unitary divisor of d and n/d is unity. The concept of a bi-unitary divisor is due to D. Surynarayana (1972). Let $\sigma^{**}(n)$ denote the sum of the bi-unitary divisors of n. A positive integer n is called a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$ for some $k \geq 3$. For $k = 3$ we obtain the bi-unitary triperfect numbers.

Peter Hagis (1987) proved that there are no odd bi-unitary multiperfect numbers. The present paper is part IV(b) in a series of papers on even bi-unitary multiperfect numbers. In parts I, II and III we considered bi-unitary triperfect numbers of the form $n = 2^a u$, where $1 \leq a \leq 6$ and u is odd. In part IV(a) we solved partly the case $a = 7$. We proved that if n is a bi-unitary triperfect number of the form $n = 2^7.5^b.17^c.v$, where $(v, 2.5.17) = 1$, then $b \geq 2$. We then solved completely the case $b = 2$. In the present paper we give some partial results concerning the case $b \geq 3$ under the assumption $3 \nmid n$.

Keywords: Perfect numbers, Triperfect numbers, Multiperfect numbers, Bi-unitary analogues.
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1 Introduction

Throughout this paper, all lower case letters denote positive integers; p and q denote primes. The letters u, v and w are reserved for odd numbers.

*Prof. Varanasi Sitaramaiah passed away on 2 October 2020.

A divisor d of n is called a unitary divisor if $\gcd(d, n/d) = 1$. If d is a unitary divisor of n, we write $d \| n$. A divisor d of n is called a *bi-unitary* divisor if $(d, n/d)^{**} = 1$, where the symbol $(a, b)^{**}$ denotes the greatest common unitary divisor of a and b. The concept of a bi-unitary divisor is due to D. Suryanarayana (cf. [7]). Let $\sigma^{**}(n)$ denote the sum of bi-unitary divisors of n. The function $\sigma^{**}(n)$ is multiplicative, that is, $\sigma^{**}(1) = 1$ and $\sigma^{**}(mn) = \sigma^{**}(m)\sigma^{**}(n)$ whenever $(m, n) = 1$. If p^α is a prime power and α is odd, then every divisor of p^α is a bi-unitary divisor; if α is even, each divisor of p^α is a bi-unitary divisor except for $p^{\alpha/2}$. Hence

$$\sigma^{**}(p^\alpha) = \begin{cases} \sigma(p^\alpha) = \frac{p^{\alpha+1}-1}{p-1} & \text{if } \alpha \text{ is odd,} \\ \sigma(p^\alpha) - p^{\alpha/2} & \text{if } \alpha \text{ is even.} \end{cases} \tag{1.3}$$

If α is even, say $\alpha = 2k$, then $\sigma^{**}(p^\alpha)$ can be simplified to

$$\sigma^{**}(p^\alpha) = \left(\frac{p^k - 1}{p - 1}\right).(p^{k+1} + 1). \tag{1.4}$$

From (1.3), it is not difficult to observe that $\sigma^{**}(n)$ is odd only when $n = 1$ or $n = 2^\alpha$.

The concept of a bi-unitary perfect number was introduced by C. R. Wall [8]; a positive integer n is called a bi-unitary perfect number if $\sigma^{**}(n) = 2n$. C. R. Wall [8] proved that there are only three bi-unitary perfect numbers, namely $6, 60$ and 90. A positive integer n is called a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$ for some $k \geq 3$. For $k = 3$ we obtain the bi-unitary triperfect numbers.

Peter Hagis [1] proved that there are no odd bi-unitary multiperfect numbers. Our present paper is part VI(b) in a series of papers on even bi-unitary multiperfect numbers. In parts I, II and III (see [2, 3, 4]) we considered bi-unitary triperfect numbers of the form $n = 2^a u$, where $1 \leq a \leq 6$ and u is odd. In part IV(a) we solved partly the case $a = 7$. We proved that if n is a bi-unitary triperfect number of the form $n = 2^7.5^b.17^c.v$, where $(v, 2.5.17) = 1$, then $b \geq 2$. We then solved completely the case $b = 2$. We proved that in this case c has to equal 1 and further showed that $n = 2^7.3^2.5^2.7.13.17 = 44553600$ is the only bi-unitary triperfect number of the form considered there. In this paper we examine the case $b \geq 3$ with the restriction that $3 \nmid n$. We present some necessary conditions for triperfect numbers.

For a general account on various perfect-type numbers, we refer to [6].

2 Preliminaries

We assume that the reader has parts I, II, III, IV(a) (see [2, 3, 4, 5]) available. We, however, recall Lemmas 2.1 to 2.6 from part IV(a), because they are so important also here.

Lemma 2.1. (I) *If α is odd, then for any prime p,*

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} > \frac{\sigma^{**}(p^{\alpha+1})}{p^{\alpha+1}}.$$

(II) For any $\alpha \geq 2\ell - 1$ and any prime p,

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} \geq \left(\frac{1}{p-1}\right)\left(p - \frac{1}{p^{2\ell}}\right) - \frac{1}{p^\ell} = \frac{1}{p^{2\ell}}\left(\frac{p^{2\ell+1} - 1}{p - 1} - p^\ell\right).$$

(III) If p is any prime and α is a positive integer, then

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} < \frac{p}{p-1}.$$

Remark 2.1. (I) and (III) of Lemma 2.1 are mentioned in C. R. Wall [8]; (II) of Lemma 2.1 has been used by him [8] without explicitly stating it.

Lemma 2.2. *Let $a > 1$ be an integer not divisible by an odd prime p and let α be a positive integer. Let r denote the least positive integer such that $a^r \equiv 1 \pmod{p^\alpha}$; then r is usually denoted by $ord_{p^\alpha}\, a$. We have the following properties:*

(i) If r is even, then $s = r/2$ is the least positive integer such that $a^s \equiv -1 \pmod{p^\alpha}$. Also, $a^t \equiv -1 \pmod{p^\alpha}$ for a positive integer t if and only if $t = su$, where u is odd.

(ii) If r is odd, then $p^\alpha \nmid a^t + 1$ for any positive integer t.

Remark 2.2. Let a, p, r and $s = r/2$ be as in Lemma 2.2 ($\alpha = 1$). Then $p|a^t - 1$ if and only if $r|t$. If t is odd and r is even, then $r \nmid t$. Hence $p \nmid a^t - 1$. Also, $p|a^t + 1$ if and only if $t = su$, where u is odd. In particular if t is even and s is odd, then $p \nmid a^t + 1$. In order to check the divisibility of $a^t - 1$ (when t is odd) by an odd prime p, we can confine to those p for which $ord_p\, a$ is odd. Similarly, for examining the divisibility of $a^t + 1$ by p when t is even we need to consider primes p with $s = ord_p\, a/2$ even.

Lemma 2.3. *Let k be odd and $k \geq 3$. Let $p \neq 5$.*

(a) If $p \in [3, 2520] - \{11, 19, 31, 71, 181, 829, 1741\}$, $ord_p 5$ is odd and $p|5^k - 1$, then we can find a prime p' (depending on p) such that $p'|\frac{5^k-1}{4}$ and $p' \geq 2521$.

(b) If $q \in [3, 2520] - \{13, 313, 601\}$, $s = \frac{1}{2} ord_q 5$ is even and $q|5^{k+1} + 1$, then we can find a prime q' (depending on q) such that $q'|\frac{5^{k+1}+1}{2}$ and $q' \geq 2521$.

Lemma 2.4. *Let k be odd and $k \geq 3$. Let $p \neq 7$.*

(a) If $p \in [3, 2520] - \{3, 19, 37, 1063\}$, $r = ord_p 7$ is odd and $p|7^k - 1$, then we can find a prime p' (depending on p) such that $p'|\dfrac{7^k - 1}{6}$ and $p' \geq 2521$.

(b) If $q \in [3, 1193] - \{5, 13, 181, 193, 409\}$, $s = \frac{1}{2} ord_q 7$ is even and $q|7^{k+1} + 1$, then we can find a prime q' (depending on q) such that $q'|\frac{7^{k+1}+1}{2}$ and $q' > 1193$.

Lemma 2.5. *Let k be odd and $k \geq 3$. Let $p \neq 13$.*

(a) If $p \in [3, 293] - \{3, 61\}$, $r = ord_p 13$ is odd and $p|13^k - 1$, then we can find a prime p' (depending on p) such that $p'|\frac{13^k-1}{12}$ and $p' \geq 293$.

(b) If $q \in [3, 293] - \{5, 17\}$, $s = \frac{1}{2} ord_q 13$ is even and $q|13^{k+1} + 1$, then we can find a prime q' (depending on q) such that $q'|\frac{13^{k+1}+1}{2}$ and $q' > 293$.

Lemma 2.6. *Let k be odd and $k \geq 3$. Let $p \neq 17$.*

(a) If $p \in [3, 519] - \{307\}$, $r = ord_p 17$ is odd and $p|17^k - 1$, then we can find a prime p' (depending on p) such that $p'|\frac{17^k-1}{16}$ and $p' > 519$.

(b) If $q \in [3, 519] - \{5, 29\}$, $s = \frac{1}{2} ord_q 17$ is even and $q|17^{k+1} + 1$, then we can find a prime q' (depending on q) such that $q'|\frac{17^{k+1}+1}{2}$ and $q' > 519$.

Corollary 2.1. *(I) If k is odd and $k \geq 5$, then $\frac{17^k-1}{16}$ is divisible by an odd prime $p' > 519$.*

(II) If k is odd and $k \geq 3$, then $\frac{17^{k+1}+1}{2}$ is divisible by an odd prime $q' > 519$.

Proof. (I) Let $S_{17} = \{p|17^k - 1 \ : \ p \in [3, 519] - \{307\}$ and $r = ord_p 17$ is odd$\}$. By Lemma 2.6 (a), if S_{17} is non-empty, the statement in (I) above holds.

Let S_{17} be empty. Since $p \nmid 17^k - 1$ if $ord_p 17$ is even, it follows that $17^k - 1$ is not divisible by any prime in $[3, 519]$ except for possibly by 307. As $32|17^t - 1$ if and only if t is even, $32 \nmid 17^k - 1$ since k is odd. Hence $16\|17^k - 1$. Thus $\frac{17^k-1}{16}$ is odd; also it is > 1, since $k \geq 5$.

If $307 \nmid 17^k - 1$, then $\frac{17^k-1}{16}$ is divisible by none of the primes in $[3, 519]$. If $p'|\frac{17^k-1}{16}$, then $p' > 519$. This proves (I) in this case.

Assume that $307|17^k - 1$. We claim that $17^k - 1$ is divisible by an odd prime $\neq 307$. On the contrary, let $\frac{17^k-1}{16} = 307^\alpha$. If $\alpha \geq 2$, then $307^2|17^k - 1$; this holds if and only if $921 = 3.307|k$. In particular, $307|k$. But $7036178437|\frac{17^{307}-1}{16}|\frac{17^k-1}{16} = 307^\alpha$, which is impossible. Hence $\alpha = 1$ so that $\frac{17^k-1}{16} = 307$ or $k = 3$. But $k \geq 5$, by hypothesis. Hence we can find an odd prime $p'|\frac{17^k-1}{16}$ and $p' \neq 307$. Also, $p' > 519$.

The proof of (I) is complete.

(II) Consider the factor $17^{k+1} + 1$, where k is odd and ≥ 3. Let

$$T_{17} = \{q|17^{k+1} + 1 \ : q \in [3, 519] - \{5, 29\} \text{ and } s = \frac{1}{2} ord_p 17 \text{ is even}\}.$$

By Lemma 2.6 (b), if T_{17} is non-empty, then the statement in (II) holds. So we may assume that T_{17} is empty. Since $q \nmid 17^{k+1} + 1$ if $s = \frac{1}{2} ord_p 17$ is not even, it follows that $17^{k+1} + 1$ is not divisible by any prime in $[3, 519]$ except for possibly 5 and 29.

We may note that $5|17^{k+1} + 1 \iff k + 1 = 2u \iff 29|17^{k+1} + 1$. Hence if $5 \nmid 17^{k+1} + 1$, then $29 \nmid 17^{k+1} + 1$ so that $\frac{17^{k+1}+1}{2}$ is not divisible by any prime in $[3, 519]$. So if $q'|\frac{17^{k+1}+1}{2}$, then $q' > 519$.

Suppose that $5|17^{k+1} + 1$. Hence $29|17^{k+1} + 1$. We now claim that $\frac{17^{k+1}+1}{2}$ is divisible by an odd prime $q' \notin \{5, 29\}$. On the contrary, let $\frac{17^{k+1}+1}{2} = 5^\alpha.29^\beta$. If $\alpha \geq 2$, then $5^2|17^{k+1} + 1$; this holds if and only if $k + 1 = 10u$. Hence $17^{10} + 1|17^{k+1} + 1$.

Also, $17^{10} + 1 = 2.5^2.29.21881.63541$. Thus, $21881|\frac{17^{10}+1}{2}|\frac{17^{k+1}+1}{2} = 5^\alpha.29^\beta$, which is not possible. Hence $\alpha = 1$.

Suppose that $\beta \geq 2$. Then $29^2|17^{k+1} + 1$; this is equivalent to $k + 1 = 58u$. Hence, $17^{58} + 1|17^{k+1} + 1$. Also,

$$17^{58} + 1 = \{\{2, 1\}, \{5, 1\}, \{29, 2\}, \{4908077, 1\},$$
$$\{5627688836691687811685586936872121257317104508544673081805033, 1\}\}.$$

Thus $4908077|\frac{17^{58}+}{2}|\frac{17^{k+1}+1}{2} = 5^\alpha.29^\beta$, which is impossible. Hence $\beta = 1$.

We now have $\frac{17^{k+1}+1}{2} = 5.29$ or $k = 1$. But $k \geq 3$. This contradiction proves that we can find an odd prime $q'|\frac{17^{k+1}+1}{2}$ and $q' \notin \{5, 29\}$. It follows that $q' \notin [3, 519]$. Hence $q' > 519$. □

3 Further results on bi-unitary triperfect numbers of the form $n = 2^7u$

Let n be a bi-unitary triperfect number divisible unitarily by 2^7 so that $\sigma^{**}(n) = 3n$ and $n = 2^7.u$, where u is odd. Since $\sigma^{**}(2^7) = 2^8 - 1 = 255 = 3.5.17$, using $n = 2^7u$ in $\sigma^{**}(n) = 3n$, we get the following equations:

$$n = 2^7.5^b.17^c.v, \tag{3.1a}$$

and

$$2^7.5^{b-1}.17^{c-1}.v = \sigma^{**}(5^b).\sigma^{**}(17^c).\sigma^{**}(v), \tag{3.1b}$$

where $(v, 2.5.17) = 1$. Considering the parity of the function values of σ^{**} and applying multiplicativity of σ^{**}, we conclude that v has at most five odd prime factors.

In part IV(a) we proved that $b \geq 2$ in $(3.1a)$ and solved completely the case $b = 2$. These results were presented in Theorem 3.1.

Theorem 3.1. *(a) If n is as in $(3.1a)$ and n is a bi-unitary triperfect number, then $b \geq 2$.*
(b) If $b = 2$, then $c = 1$ and $n = 44553600 = 2^7.3^2.5^2.7.13.17$.

In part IV(a) we also presented the following remark.

Remark 3.1. Let n be as given in $(3.1a)$ and $b \geq 3$. Assume that n is a bi-unitary triperfect number. Then $(3.1b)$ is valid. Further suppose that n is not divisible by 3. If b is odd or $4|b$, then $3|\sigma^{**}(5^b)$. Also, if c is odd or $4|c$, then $9|\sigma^{**}(17^c)$. These are not possible in $(3.1b)$, and therefore it follows that $b = 2k$ and $c = 2\ell$, where $k \geq 3$ and ℓ are odd. Hence $b \geq 6$ and $c \geq 2$.

In this paper we consider the case $b \geq 3$ with $3 \nmid n$ in more detail. In Theorems 3.2, 3.3 and 3.4 we present some necessary conditions for such triperfect numbers.

Theorem 3.2. *Let n be as given in $(3.1a)$, where $b \geq 3$ and $3 \nmid n$. Then n is not a bi-unitary triperfect number,*

(a) if n is divisible by 7^3 and 17^3;

(b) if $17^2 \| n$ and $7^3 | n$;

(c) if $17^2 \| n$ and $d = 1$ or $d = 2$, where $7^d \| n$;

(d) if $17^3 | n$ and $7 \| n$;

(e) if $17^3 | n$, $7^2 \| n$ and n is not divisible by 11 and 13.

Proof. We assume that n is not divisible by 3, $7^d \| n$ and n is a bi-unitary triperfect number. From $(3.1b)$, it follows that $v = 7^d.w$, from $(3.1a)$ and $(3.1b)$, we have

$$n = 2^7.5^b.17^c.7^d.w, \tag{3.2a}$$

and

$$2^7.5^{b-1}.17^{c-1}.7^d.w = \sigma^{**}(5^b).\sigma^{**}(17^c).\sigma^{**}(7^d).\sigma^{**}(w), \tag{3.2b}$$

where

$$(w, 2.3.5.7.17) = 1 \text{ and } w \text{ has not more than four odd prime factors.} \tag{3.2c}$$

<u>*Proof of (a).*</u> By Remark 3.1, we can assume that $b \geq 6$. We have $\frac{\sigma^{**}(5^b)}{5^b} \geq \frac{19406}{15625}$, $(b \geq 5)$; $\frac{\sigma^{**}(17^c)}{17^c} \geq \frac{88452}{83521}$, $(c \geq 3)$, and $\frac{\sigma^{**}(7^d)}{7^d} \geq \frac{2752}{2401}$, $(d \geq 3)$. Hence for $c \geq 3$ and $d \geq 3$, by $(3.2a)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{19406}{15625}.\frac{88452}{83521}.\frac{2752}{2401} = 3.00348829 > 3,$$

a contradiction. This proves (a).

Remark 3.2. In view of Theorem 3.2(a), we need to investigate the following cases. *Case I:* $c = 2$, $d \geq 3$; *Case II:* $c = 2$, $d = 1$ or 2; *Case III:* $c \geq 3$, $d = 1$ or 2.

<u>*Proof of (b).*</u> By hypothesis, $c = 2$ and $d \geq 3$, we are dealing with Case I mentioned above. We have $\sigma^{**}(17^2) = 290 = 2.5.29$. Taking $c = 2$ in $(3.2b)$, we see that $29 | w$. Let $w = 29^e.w'$. From $(3.2a)$ and $(3.2b)$, we have

$$n = 2^7.5^b.17^2.7^d.29^e.w', \tag{3.3a}$$

and

$$2^6.5^{b-2}.17.7^d.29^{e-1}.w' = \sigma^{**}(5^b).\sigma^{**}(7^d).\sigma^{**}(29^e).\sigma^{**}(w'), \tag{3.3b}$$

where

$$(w', 2.3.5.7.17.29) = 1 \text{ and } w' \text{ has no more than three odd prime factors.} \tag{3.3c}$$

By Lemma 21.1, we have $\frac{\sigma^{**}(5^b)}{5^b} \geq \frac{487656}{390625}, (b \geq 7)$; $\frac{\sigma^{**}(7^d)}{7^d} \geq \frac{136914}{117649}, (d \geq 5)$; and $\frac{\sigma^{**}(29^e)}{29^e} \geq \frac{731700}{707281} (e \geq 3)$.

Hence when $b \geq 7,\ c = 2,\ d \geq 5$ and $e \geq 3$, from $(3.3a)$, we obtain

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{487656}{390625}.\frac{290}{289}.\frac{136914}{117649}.\frac{731700}{707281} = 3.004585627 > 3,$$

a contradiction. Hence when $b \geq 7,\ c = 2,\ d \geq 5$, we have $e = 1$ or $e = 2$.

If $e = 1$, from $(3.3b)$ it follows that its left-hand side is divisible by 3. This is false.

Let $e = 2$. Since $\sigma^{**}(29^2) = 842 = 2.421$, taking $e = 2$ in $(3.3b)$, we see that $421 | w'$. Let $w' = 421^f.w''$. From $(2.4a)$ and $(2.4b)$, we have

$$n = 2^7.5^b.17^2.7^d.29^2.421^f.w'', \quad (b \geq 7,\ d \geq 5) \tag{3.4a}$$

and

$$2^5.5^{b-2}.17.7^d.29.421^{f-1}.w'' = \sigma^{**}(5^b).\sigma^{**}(7^d).\sigma^{**}(421^f).\sigma^{**}(w''), \qquad (3.4b)$$

where

$$(w'', 2.3.5.7.17.29.421) = 1 \text{ and } w'' \text{ has not more than two odd prime factors.} \qquad (3.4c)$$

We obtain a contradiction by examining the prime factors of $\sigma^{**}(7^d)$.

(i) If d is odd or $4|d$, then $8|\sigma^{**}(7^d)$. From $(3.3b)$, we at once have $w'' = 1$, so that from $(3.3a)$, $n = 2^7.5^b.17^2.7^d.29^2.421^f$ and so

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{7}{6}.\frac{842}{841}.\frac{421}{420} = 2.925742293 < 3,$$

a contradiction.

(ii) Let $d = 2u$, where u is odd and ≥ 3 since $d \geq 5$. We have

$$\sigma^{**}(7^d) = \left(\frac{7^u - 1}{6}\right).(7^{u+1} + 1), \quad (u \geq 3 \text{ and odd}).$$

We prove that

(A) $\frac{7^u-1}{6}$ is divisible by an odd prime $p'|w''$ and $p' > 73$,

(B)$\frac{7^{u+1}+1}{2}$ is divisible by an odd prime $q'|w''$ and $q' > 73$.

Proof of (A). We apply Lemma 2.3(a) replacing the interval $[3, 2520]$ by $[3, 73]$. Let

$$S_7' = \{p|7^u - 1 \ : p \in [3, 73] - \{3, 19, 37\} \text{ and } ord_p 7 \text{ is odd}\}.$$

By Lemma 2.3 (a), if S_7' is non-empty, then we can find an odd prime $p'|\frac{7^u-1}{6}$ and $p' > 73$; and by $(3.4b)$, $p'|w''$. In this case (A) follows quickly.

(iii) Suppose that S_7' is empty. Since $p \nmid 7^u - 1$ if $ord_p 7$ is even, it follows that $7^u - 1$ is not divisible by any prime p in $[3, 73]$ except for possibly $p \in \{3, 19, 37\}$. We now discuss the divisibility of $7^u - 1$ by $p \in \{3, 19, 37\}$.

(iv) Since u is odd, $2\|7^u - 1$. Also, $3|7^u - 1$ but $9 \nmid 7^u - 1$. In fact, if $9|7^u - 1$, then $3|\frac{7^u-1}{6}|\sigma^{**}(7^d)$. It follows from $(3.4b)$ that $3|w''$. This is false. Hence $3\|7^u - 1$.

(v) We have $19|7^u - 1 \iff 3|u \iff 9|7^u - 1$. By (iii) above since $9 \nmid 7^u - 1$ it follows that $19 \nmid 7^u - 1$.

(vi) We have $37|7^u - 1 \iff 9|u$. Also, $7^9 - 1 = 2.3^3.19.37.1063$. Hence $37|7^u - 1$ implies that $9|7^9 - 1|7^u - 1$. This cannot happen by (iii).

Thus from (iii)–(vi), $\frac{7^u-1}{6}$ is > 1, odd and not divisible divisible by any prime in $[3, 73]$. Hence if $p'|\frac{7^u-1}{6}$, then $p' > 73$ and from $(3.4b)$, $p'|w''$. This proves (A).

Proof of (B). We apply Lemma 2.3(b) replacing the interval $[3, 2520]$ by $[3, 73]$. Let

$$T_7' = \{q|7^{u+1} + 1 \ : \ q \in [3, 73] - \{3, 5, 13\} \text{ and } s = \frac{1}{2} ord_q 7 \text{ is even}\}.$$

If T_7' is non-empty, (B) follows immediately (cf. Lemma 2.3(b)).

(vii) Assume that T_7' is empty. Since $q \nmid 7^{u+1}+1$ if $s = \frac{1}{2} ord_q 7$ is not even, it follows that $\frac{7^{u+1}+1}{2}$ is divisible by any prime q in $[3, 73]$ except for possibly $q = 3, 5, 13$.

(viii) Clearly, $2 \| 7^{u+1}+1$ and $3 \nmid 7^{u+1}+1$.

(ix) We have $13 | 7^{u+1}+1 \Longleftrightarrow u+1 = 6v \Longleftrightarrow 181 | 7^{u+1}+1$, v being odd. Assume that $13 | 7^{u+1}+1$. Hence $181 | 7^{u+1}+1$. From $(3.4b)$, w'' is divisible by 13 and 181. By (A), $p' | w''$. Thus w'' in $(3.4b)$ is divisible by three distinct odd primes p', 13 and 181. This violates $(3.4c)$. Hence $13 \nmid 7^{u+1}+1$.

(x) It remains to discuss the divisibility of $7^{u+1}+1$ by 5. If $5 \nmid 7^{u+1}+1$, then from (vii)–(ix) it follows that $\frac{7^{u+1}+1}{2}$ is not divisible by any prime in $[3, 73]$. If $q' | \frac{7^{u+1}+1}{2}$, then $q' > 73$ and by $(3.4b)$, $q' | w''$. This proves (B) in the case $5 \nmid 7^{u+1}+1$.

We may assume that $5 | 7^{u+1}+1$. This is equivalent to $u+1 = 2v$. Hence $7^2+1 | 7^{u+1}+1$ and so $5^2 | 7^{u+1}+1$. We now prove that $\frac{7^{u+1}+1}{2}$ is not divisible by 5 alone. Let $\frac{7^{u+1}+1}{2} = 5^\alpha$, where $\alpha \geq 2$. If $\alpha \geq 3$, then $5^3 | 7^{u+1}+1$. This is possible if and only if $u+1 = 10v$. Hence $7^{10}+1 | 7^{u+1}+1$. Also, $7^{10}+1 = 2.5^3.281.4021$. It follows that $281 | \frac{7^{10}+1}{2} | \frac{7^{u+1}+1}{2} = 5^\alpha$ and this cannot happen. Hence $\alpha = 2$. Thus $\frac{7^{u+1}+1}{2} = 5^2$ or $u = 1$. But $u \geq 3$. Hence $\frac{7^{u+1}+1}{2}$ will be divisible by an odd prime say $q' \neq 5$. From (vii)–(x), it follows that $q' \notin [3, 73]$. It is clear that $q' | w''$. This proves (B) completely.

The odd primes p' and q' are distinct and hence we may assume that $p' \geq 79$ and $q' \geq 83$. From $(3.4c)$, $w'' = (p')^g.(q')^h$. From $(3.4a)$, we have $n = 2^7.5^b.17^2.7^d.29^2.421^f.(p')^g.(q')^h$. Hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{7}{6}.\frac{842}{841}.\frac{421}{420}.\frac{79}{78}.\frac{83}{82} = 2.999389027 < 3,$$

a contradiction.

Thus when $c = 2$, we cannot have $b \geq 7$ and $d \geq 5$. Hence we have either $\{c = 2, b = 6, d \geq 5\}$ or $\{c = 2, b \geq 7, d = 3 \text{ or } 4\}$ or $\{c = 2, b = 6, d = 3 \text{ or } 4\}$, since already $b \geq 6$ and $d \geq 3$.

Let $\{c = 2, b = 6, d \geq 5\}$. We have $\sigma^{**}(5^6) = 2.31.313$. Taking $b = 6$ in $(3.3b)$, it follows that w' is divisible by 31 and 313. Let $w' = 31^f.313^g.w''$. Hence from $(3.3a)$ and $(3.3b)$ we obtain (after simplification),

$$n = 2^7.5^6.17^2.7^d.29^e.31^f.313^g.w'', \quad (d \geq 5) \qquad (3.5a)$$

and

$$2^5.5^4.17.7^d.29^{e-1}.31^{f-1}.313^{g-1}.w'' = \sigma^{**}(7^d).\sigma^{**}(29^e).\sigma^{**}(31^f).\sigma^{**}(313^g)\sigma^{**}(w''), \quad (3.5b)$$

where

$$(w'', 2.3.5.7.17.29.31.313) = 1 \text{ and } w'' \text{ has not more than one odd prime factors} . \qquad (3.5c)$$

We have $\frac{\sigma^{**}(5^6)}{5^6} = \frac{19406}{15625}$; by Lemma 2.1, $\frac{\sigma^{**}(7^d)}{7^d} \geq \frac{136914}{117649} \quad (d \geq 5)$; $\frac{\sigma^{**}(29^e)}{29^e} \geq \frac{731700}{707281} \quad (e \geq 3)$ and $\frac{\sigma^{**}(31^f)}{31^f} \geq \frac{953344}{923521} \quad (f \geq 3)$.

From $(3.5a)$, for $e \geq 3$ and $f \geq 3$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128} \cdot \frac{19406}{15625} \cdot \frac{290}{289} \cdot \frac{136914}{117649} \cdot \frac{731700}{707281} \cdot \frac{953344}{923521} = 3.08567307873 > 3,$$

a contradiction.

Hence $(e = 1$ or $2)$ or $(f = 1$ or $2)$.

Let $f = 1$. We have $\sigma^{**}(31) = 32 = 2^5$. Hence taking $f = 1$ in $(3.5b)$, we see that its right-hand side is divisible by 2^8 whereas 2^5 is a unitary divisor of its left-hand side.

Let $f = 2$. We have $\sigma^{**}(31^2) = 962 = 2.13.37$. Taking $f = 2$ in $(3.5b)$, we find that w'' is divisible by 13 and 37. This contradicts $(3.5c)$.

Let $e = 1$. We have $\sigma^{**}(29) = 30$. Taking $e = 1$ in $(3.5b)$, we see that its left-hand side is divisible by 3. This cannot happen as $3 \nmid n$ by our assumption.

Let $e = 2$. We have $\sigma^{**}(29^2) = 842 = 2.421$. Taking $e = 2$ in $(3.5b)$, we find that $421|w''$. Let $w'' = 421^h$. From $(3.5a)$ and $(3.5b)$, we obtain

$$n = 2^7.5^6.17^2.7^d.29^2.31^f.313^g.421^h, \quad (d \geq 5) \tag{3.6a}$$

and

$$2^4.5^4.17.7^d.29.31^{f-1}.313^{g-1}.421^{h-1} = \sigma^{**}(7^d).\sigma^{**}(31^f).\sigma^{**}(313^g)\sigma^{**}(421^h). \tag{3.6b}$$

We obtain a contradiction by examining $\sigma^{**}(7^d)$ as follows.

If d is odd or $4|d$, then $8|\sigma^{**}(7^d)$. This is not possible as in such a case 2^6 is a factor of the right-hand side of $(3.6b)$, while 2^4 is a unitary divisor of its left-hand side.

Let $d = 2u$, where u is odd. Then $u \geq 3$, since $d \geq 5$. We have

$$\sigma^{**}(7^d) = \left(\frac{7^u - 1}{6}\right).(7^{u+1} + 1).$$

(xi) Note that $\frac{7^u-1}{6} > 1$ and odd. Also, it is not divisible by 5, 17, 313 and 421, since u is odd (see Appendix C in [2]). It is not divisible by 7 trivially.

(xii) Assume that $29|7^u - 1$. This is equivalent to $7|u$. Hence $7^7 - 1|7^u - 1$ so that

$$4733|\frac{7^7 - 1}{6}|\frac{7^u - 1}{6}|\sigma^{**}(7^d).$$

This is not possible from $(3.6b)$. Hence $29 \nmid 7^u - 1$.

(xiii) We have $31|7^u - 1 \iff 15|u$. Hence $31|7^u - 1$ implies that $3|\frac{7^{15}-1}{6}|\frac{7^u-1}{6}|\sigma^{**}(7^d)$. It follows that 3 is a factor of the left-hand side of $(3.5b)$. This is false. Hence $31 \nmid 7^u - 1$.

From (xi)–(xiii) it follows that $\frac{7^u-1}{6} > 1$, is odd and is not divisible by 5, 7, 17, 29, 31, 313 and 421. But this cannot happen from $(3.5b)$, since $\frac{7^u-1}{6}|\sigma^{**}(7^d)$.

Thus $c = 2, b = 6, d \geq 5$ leads to a contradiction.

Let $c = 2$ and $d = 3$. We have $\sigma^{**}(7^3) = 2^4.5^2$. Taking $d = 3$ in $(3.3b)$, we infer that $w' = 1$. Hence from $(3.3a)$ and $(3.3b)$, we have

$$n = 2^7.5^b.17^2.7^3.29^e, \qquad (3.7a)$$

and

$$2^2.5^{b-4}.17^2.7^3.29^e = \sigma^{**}(5^b).\sigma^{**}(29^e). \qquad (3.7b)$$

We obtain a contradiction by examining $\sigma^{**}(5^b)$. By Remark 3.1, we can take $b = 2k$ where k is odd and ≥ 3. We have

$$\sigma^{**}(5^b) = \left(\frac{5^k - 1}{4}\right).(5^{k+1} + 1).$$

The factor $\frac{5^k-1}{4} > 1$, is odd and is not divisible by 7, 17 and 29, since k is odd and ≥ 3; it is not divisible by 5, trivially. Thus $\frac{5^k-1}{4}$ is not divisible by $2, 5, 7, 17$ and 29. This is not possible from $(3.7b)$, since $\frac{5^k-1}{4} | \sigma^{**}(5^b)$. This contradiction proves that $c = 2, d = 3$ is not admissible.

Let $c = 2$ and $d = 4$. We have $\sigma^{**}(7^4) = 2^6.43$. Taking $d = 4$ in $(3.3b)$, we find an imbalance in powers of 2 between both sides of $(3.3b)$. Hence these values of c and d are not admissible.

The proof of part (b) of Theorem 3.2 is complete.

Proof of (c). By hypothesis in this case $c = 2$ and $d = 1$ or 2.

Let $c = 2$ and $d = 1$. By taking $d = 1$ in $(3.3a)$ and $(3.3b)$, we get

$$n = 2^7.5^b.17^2.7.29^e.w', \qquad (3.8a)$$

and

$$2^3.5^{b-2}.17.7.29^{e-1}.w' = \sigma^{**}(5^b).\sigma^{**}(29^e).\sigma^{**}(w'), \qquad (3.8b)$$

where

$$(w', 2.3.5.7.17.29) = 1 \text{ and } w' \text{ has no more than one odd prime factor}, \qquad (3.8c)$$

where $b = 2k,\ k$ is odd and ≥ 3. Also, $\sigma^{**}(5^b) = \left(\frac{5^k-1}{4}\right).(5^{k+1} + 1)$.

The factor $\frac{5^k-1}{4} > 1$ is odd and is not divisible by $7, 17$ and 29, since k is odd and ≥ 3; it is not divisible by 5, trivially. Thus $\frac{5^k-1}{4}$ is not divisible by $2, 5, 7, 17$ and 29. Since $\frac{5^k-1}{4} > 1$, let $p' | \frac{5^k-1}{4}$. Then p' is odd and from $(3.8b)$, $p' | w'$.

Consider the factor $5^{k+1} + 1$. We have (i) $2 \| 5^{k+1} + 1$. (ii) $5^{k+1} + 1$ is not divisible by 7 or 29, since $k + 1$ is even. (iii) Suppose $17 | 5^{k+1} + 1$. Then $k + 1 = 8u$. Hence $5^8 + 1 | 5^{k+1} + 1$. Also, $5^8 + 1 = 2.17.11489$. It follows that $11489 | 5^{k+1} + 1$. From $(3.8b)$, we have $11489 | w'$. Already $p' | w'$. Thus w' is divisible by two odd primes p' and 11489. This contradicts $(3.8c)$ and so $17 \nmid 5^{k+1} + 1$.

From (i)–(iii), it follows that $\frac{5^{k+1}+1}{2} > 1$, is odd and not divisible by 5, 7, 17 and 29. Let $q' | \frac{5^{k+1}+1}{2}$. Then from $(3.8b)$, $q' | w'$.

Thus w' is divisible by two distinct odd primes p' and q'. This contradicts $(3.8c)$.

Hence the case $c = 2, d = 1$ is not admissible.

Let $c = 2$ and $d = 2$. We have $\sigma^{**}(7^2) = 50 = 2.5^2$. Taking $(d = 2)$, in $(3.3a)$ and $(3.3b)$, we get

$$n = 2^7.5^b.17^2.7^2.29^e.w', \tag{3.9a}$$

and

$$2^5.5^{b-4}.17.7^2.29^{e-1}.w' = \sigma^{**}(5^b).\sigma^{**}(29^e).\sigma^{**}(w'), \tag{3.9b}$$

where

$$(w', 2.3.5.7.17.29) = 1 \text{ and } w' \text{ has no more than three odd prime factors.} \tag{3.9c}$$

In $(3.9a)$ and $(3.9b)$, we can assume that $b = 2k$ where k is odd and ≥ 3. We have

$$\sigma^{**}(5^b) = \left(\frac{5^k-1}{4}\right).(5^{k+1}+1).$$

We prove that

(I) there exists an odd prime $p'|\frac{5^k-1}{4}$, $p'|w'$ and $p' > 67$,

(II) there exists an odd prime $q'|\frac{5^{k+1}+1}{2}$, $q'|w'$ and $q' > 67$.

Proof of (I). Let

$$S_5' = \{p|5^k-1 \ : \ p \in [3,67] - \{11,19,31\} \text{ and } ord_p 5 \text{ is odd }\}.$$

By Lemma 2.3(a) (by replacing the interval $[3, 2520]$ by $[3, 67]$), if S_5' is non-empty, then (I) holds. Also, $11|5^k - 1$ if and only if $5|k$. Further, $5^5 - 1 = 2^2.11.71$. Hence if $11|5^k - 1$, then $71|5^k - 1$. Thus $p' = 71|w'$, $p'|\frac{5^k-1}{4}$ and $p' > 67$. Thus (I) holds. In a similar manner, $19|5^k - 1 \iff 9|k$. Also, $829|\frac{5^9-1}{4}|\frac{5^k-1}{4}$. Thus if $19|5^k - 1$, then $p' = 829|\frac{5^k-1}{4}$, $p'|w'$ and $p' > 67$. Hence (I) holds in this case. It follows that if

$$S_5'' = \{p|5^k-1 \ : \ p \in [3,67] - \{31\} \text{ and } ord_p 5 \text{ is odd }\},$$

then S_5'' is non-empty implies that (I) holds.

Assume that S_5'' is empty. Since $5^k - 1$ is not divisible by p if $ord_p 5$ is even, it follows that $5^k - 1$ is not divisible by any prime in $[3, 67]$ except for possibly 31.

If $31 \nmid 5^k - 1$, it follows that $5^k - 1$ is not divisible by any prime in $[3, 67]$. The same is true with respect to $\frac{5^k-1}{4}$ which is odd and > 1. Let $p'|\frac{5^k-1}{4}$. Then $p' \notin [3, 67]$ and so $p' > 67$. Also, from $(3.9b)$, $p'|w'$. This proves (I) in this case.

We assume that $31|5^k - 1$. We claim that $\frac{5^k-1}{4}$ is divisible by a prime $p' \neq 31$. If this is not so, then we must have $\frac{5^k-1}{4} = 31^\alpha$ for some positive integer α. If $\alpha \geq 2$, then $31^2|5^k - 1$. Hence $93|k$. In particular $31|k$. Hence $1861|5^{31} - 1|5^k - 1$. Thus $1861|\frac{5^k-1}{4} = 31^\alpha$. This is not possible. Hence $\alpha = 1$ and so $\frac{5^k-1}{4} = 31$ or $k = 3$ or $b = 6$.

We prove that $b = 6$ is not admissible. Since $\sigma^{**}(5^8) = 2.31.313$, taking $b = 6$ in $(3.9b)$, it follows that w' is divisible by 31 and 313. Let $w' = 31^f.313^g.w''$ Hence from $(3.9a)$, $n = 2^7.5^b.17^2.7^2.29^e.31^f.313^g.w''$, where w'' is 1 or a prime power. Let $w'' = p^h$, where $p \geq 11$.

We have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{19406}{15625}.\frac{290}{289}.\frac{50}{49}.\frac{29}{28}.\frac{31}{30}.\frac{313}{312}.\frac{11}{10} = 2.992148375 < 3,$$

a contradiction. This proves that $b = 6$ is not admissible.

Thus if $31|5^k - 1$, then $\frac{5^k-1}{4}$ is divisible by a prime $p' \neq 31$. It follows that $p' \notin [3, 67]$ and $p'|w'$ from $(3.9b)$.

The proof of (I) is complete.

Proof of (II). Let

$$T_5' = \{q|5^{k+1} + 1 \ : q \in [3, 67] - \{13\} \text{ and } s = \frac{1}{2} ord_q 5 \text{ is even}\}.$$

In Lemma 2.3(b), if we replace the interval $[3, 2520]$ by $[3, 67]$, it follows that (II) holds whenever T_5' is non-empty.

Let T_5' be empty. Since $q \nmid 5^{k+1} + 1$ if $s = \frac{1}{2} ord_q 5$ is not even, it follows that $5^{k+1} + 1$ is not divisible by any prime in $[3, 67]$ except for possible 13.

Suppose $13 \nmid 5^{k+1} + 1$. Then $\frac{5^{k+1}+1}{2} > 1$, is odd and is not divisible by any prime in $[3, 67]$. If $q'|\frac{5^{k+1}+1}{2}$, then $q' > 67$ and $q'|w'$ by $(3.9b)$. Thus (II) holds in this case.

Suppose that $13|5^{k+1}+1$. We claim that $\frac{5^{k+1}+1}{2}$ is divisible by an odd prime $q' \neq 13$. On the other hand, assume that $\frac{5^{k+1}+1}{2} = 13^\alpha$, for some positive integer α. If $\alpha \geq 2$, then $13^2|5^{k+1}+1$. This is equivalent to $k + 1 = 26u$. Hence $5^{26} + 1|5^{k+1} + 1$, and so $53|\frac{5^{26}+1}{2}|\frac{5^{k+1}+1}{2} = 13^\alpha$. This is not possible. Hence $\alpha = 1$ so that $\frac{5^{k+1}+1}{2} = 13$ or $k = 1$. But $k \geq 3$. Hence we can find an odd prime $q'|\frac{5^{k+1}+1}{2}$ and $q' \neq 13$. It now follows that $q' \notin [3, 67]$. Also, from $(3.9b)$, $q'|w'$.

The proof of (II) is complete.

Since w' is divisible by p', q' and $p' \neq q'$, we can assume that $p' \geq 71$ and $q' \geq 73$. If r denotes the possible third prime factor of w', we can assume that $r \geq 11$. From $(3.9c)$, $w' = (p')^f.(q')^g.r^h$. From $(3.9a)$, we have $n = 2^7.5^b.17^2.7^2.29^e.(p')^f.(q')^g.r^h$. Hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{50}{49}.\frac{29}{28}.\frac{71}{70}.\frac{73}{72}.\frac{11}{10} = 2.98742924 < 3,$$

a contradiction.

The case $c = 2, d = 2$ is complete. The proof of Theorem 3.2(c) is complete.

Proof of (d). By hypothesis $c \geq 3$ and $d = 1$. Since c is even and $4 \nmid c$, we may assume that $c \geq 5$. By Lemma 2.1, we have $\frac{\sigma^{**}(17^c)}{17^c} \geq \frac{25641254}{24137569}$ $(c \geq 5)$. From $(3.2a)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{19406}{15625}.\frac{25641254}{24137569}.\frac{8}{7} = 3.003889074 > 3,$$

a contradiction.

This completes the proof of (d).

Proof of (e). By hypothesis, $c \geq 3$ and $d = 2$. Taking $d = 2$ in $(3.2a)$ and $(3.2b)$, we get

$$n = 2^7.5^b.17^c.7^2.w, \qquad (3.10a)$$

and

$$2^6.5^{b-3}.17^{c-1}.7^2.w = \sigma^{**}(5^b).\sigma^{**}(17^c).\sigma^{**}(w), \qquad (3.10b)$$

and

$$(w, 2.3.5.7.11.13.17) = 1 \text{ and } w \text{ has no more than four odd prime factors.} \qquad (3.10c)$$

By Remark 3.1, we can assume that $b = 2k$, where k is odd and ≥ 3. Also, $c = 2\ell$, where ℓ is odd and ≥ 3, since $c \geq 3$.

We have

$$\sigma^{**}(5^b) = \left(\frac{5^k - 1}{4}\right).(5^{k+1} + 1).$$

We prove that

(C) there exists an odd prime $p'|\frac{5^k-1}{4}$, $p'|w$ and $p' \geq 2521$,

(D) there exists an odd prime $q'|\frac{5^{k+1}+1}{2}$, $q'|w$ and $q' \geq 163$.

First we prove (D).

Proof of (D). Let

$$T_5' = \{q|5^{k+1} + 1 \; : \; q \in [3, 157] - \{13\} \text{ and } s = \frac{1}{2}ord_q 5 \text{ is even}\}.$$

Applying Lemma 2.3(b) (replacing the interval $[3, 2520]$ by $[3, 157]$) we conclude that if T_5' is non-empty, then (D) holds.

Assume that T_5' is empty. Since $q \nmid 5^{k+1} + 1$ when $s = \frac{1}{2}ord_q 5$ is not even, it follows that $\frac{5^{k+1}+1}{2}$ is not divisible by any prime q in $[3, 157]$ except for possibly $q = 13$. Since (by hypothesis) $13 \nmid n$, it follows that $13 \nmid 5^{k+1} + 1$ and thus $\frac{5^{k+1}+1}{2}$ is not divisible by any prime in $[3, 157]$. Let $q'|\frac{5^{k+1}+1}{2}$. Then $q' > 157$ (and so $q' \geq 163$) and $q'|w$.

This proves (D).

Proof of (C). Let

$$S_5 = \{p|5^k - 1 \; : \; p \in [3, 2520] - \{11, 19, 31, 71, 181, 829, 1741\} \text{ and } ord_p 5 \text{ is odd}\}.$$

If S_5 is non-empty, then (C) holds.

Let S_5 be empty. Since $p \nmid 5^k - 1$ if $ord_p 5$ is even, it follows that $\frac{5^k-1}{4}$ is not divisible by any prime $p \in [3, 2520]$ except for possibly $p \in \{11, 19, 31, 71, 181, 829, 1741\}$.

By hypothesis, $11 \nmid n$. Hence $11 \nmid 5^k - 1$; also, $11|5^k - 1 \iff 71|5^k - 1$. Hence $71 \nmid 5^k - 1$.

Further, $181|5^k - 1$ if and only if $15|k$. Hence $181|5^k - 1$ implies $11|5^{15} - 1|5^k - 1$. But $11 \nmid 5^k - 1$. Hence $181 \nmid 5^k - 1$. Since $181|5^k - 1 \iff 1741|5^k - 1$, it follows that $1741 \nmid 5^k - 1$.

We may note that $19|5^k - 1 \iff 9|k \iff 829|5^k - 1$. Suppose $19|5^k - 1$. Hence $9|k$ and consequently $5^9 - 1|5^k - 1$. Also, $5^9 - 1 = 2^2.19.31.829$. Hence $\frac{5^k-1}{4}$ and consequently $\sigma^{**}(5^b)$

is divisible by 19, 31 and 829. From, $(3.10b)$, it follows that w is divisible by 19, 31 and 829. Hence $w = 19^e.31^f.(829)^g.w'$, so that from $(3.10a)$ and $(3.10b)$, we have

$$n = 2^7.5^b.17^c.7^2.19^e.31^f.829^g.w', \tag{3.11a}$$

and

$$2^6.5^{b-3}.17^{c-1}.7^2.19^e.31^f.829^g.w' = \sigma^{**}(5^b)\sigma^{**}(17^c)\sigma^{**}(19^e)\sigma^{**}(31^f)\sigma^{**}(829^g)\sigma^{**}(w'), \tag{3.11b}$$

and

$$(w', 2.3.5.7.11.13.17.19.31.829) = 1 \text{ and } w' \text{ has no more than one odd prime factor.} \tag{3.11c}$$

By what we have proved in (D), $q'|w$ and $q' > 157$ (that is, $q' \geq 163$). By $(3.11c)$, $w' = (q')^h$. Hence $n = 2^7.5^b.17^c.7^2.19^e.31^f.(829)^g.(q')^h$, so that by Lemma 2.1,

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{17}{16}.\frac{50}{49}.\frac{19}{18}.\frac{31}{30}.\frac{829}{828}.\frac{163}{162} = 2.966616483 < 3,$$

a contradiction.

Hence 19 and consequently 829 cannot divide $5^k - 1$.

Till now, $\frac{5^k-1}{4} > 1$, is odd and is not divisible by any prime in $[3, 2520]$ except for possibly by 31. If $31 \nmid 5^k - 1$, then $\frac{5^k-1}{4}(> 1)$ will not be divisible by any prime in $[3, 2520]$. If $p'|\frac{5^k-1}{4}$, then $p' \notin [3, 2520]$ and $p'|w$. This would prove (C).

Suppose that $31|5^k - 1$. We claim that we can find an odd prime $p'|\frac{5^k-1}{4}$ and $p' \neq 31$. If this is not so, then we must have $\frac{5^k-1}{4} = 31^\alpha$, where α is a positive integer. If $\alpha \geq 2$, then $31^2|5^k-1$; this is equivalent to $93|k$. In particular, $31|k$ and so $5^{31}-1|5^k-1$. Thus $1861|\frac{5^{31}-1}{4}|\frac{5^k-1}{4} = 31^\alpha$. This is impossible. Hence $\alpha = 1$ and consequently $\frac{5^k-1}{4} = 31$ or $k = 3$ or $b = 6$.

We now show that $b = 6$ is not possible. We have $\sigma^{**}(5^6) = 2.31.313$. Taking $b = 6$ in $(3.10b)$, we find that 31 and 313 are factors of w. Let $w = 31^e.313^f.w'$. Now, from $(3.10a)$ and $(3.10b)$, we have

$$n = 2^7.5^b.17^c.7^2.31^e.313^f.w', \tag{3.12a}$$

and

$$2^5.5^{b-3}.17^{c-1}.7^2.31^{e-1}.313^{f-1}.w' = \sigma^{**}(17^c).\sigma^{**}(31^e).\sigma^{**}(313^f).\sigma^{**}(w'), \tag{3.12b}$$

and

$$(w', 2.3.5.7.11.13.17.31.313) = 1 \text{ and } w' \text{ has no more than two odd prime factors.} \tag{3.12c}$$

From Remark 3.1, we can assume that $c = 2\ell$, where ℓ is odd and ≥ 3 (since $c \geq 3$). We have $\sigma^{**}(17^c) = \left(\frac{17^\ell-1}{16}\right).(17^\ell + 1)$. We obtain a contradiction by showing that

(E) $\frac{17^\ell-1}{16}$ is divisible by an odd prime $p' > 127$.

Let

$$S_{17} = \{p|17^\ell - 1 \ : \ p \in [3, 127] \text{ and } ord_p 17 \text{ is odd}\}.$$

By Lemma 2.6(a), if S_{17} is non-empty, then we can find an odd prime $p'|\frac{17^\ell-1}{16}$ and $p' > 127$. Hence (E) follows in this case.

Suppose that S_{17} is empty. Since $p \nmid 17^\ell - 1$ if $ord_p 17$ is even, it follows that $17^\ell - 1$ is not divisible by any prime in $[3, 127]$. Since $\frac{17^\ell-1}{16} > 1$, is odd and is not divisible by any prime in $[3, 127]$, if $p'|\frac{17^\ell-1}{16}$, then $p' > 127$ or $p' \geq 131$. Further, since $313|17^\ell - 1 \iff 312|\ell$, it follows that $313 \nmid 17^\ell - 1$ since ℓ is odd. Hence $p' \neq 313$. From $(3.12b)$, it is clear that $p'|w'$. By $(3.12c)$, w' has no more than two odd prime factors. If $r \neq p'$ denotes the possible second prime factor of w', then $r \geq 19$. From $(3.12a)$, we have $n = 2^7.5^b.17^c.7^2.31^e.313^f.(p')^g.r^h$ so that by Lemma 2.1,

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{17}{16}.\frac{50}{49}.\frac{31}{30}.\frac{313}{312}.\frac{131}{130}\frac{19}{18} = 2.977023814 < 3,$$

a contradiction. Hence (E) holds.

We thus finally proved that $b = 6$ is not admissible. This means that $\frac{5^k-1}{4}$ is divisible by an odd prime $p' \neq 31$. Also, as $\frac{5^k-1}{4}$ is not divisible by any prime in $[3, 2520] - \{31\}$, it now follows that $p' \notin [3, 2520]$. Also, $p'|w$ by $(3.10b)$. This proves (C).

By $(3.10c)$, w has no more than four odd prime factors; p' and q' are two prime factors of w. Let the other two possible prime factors be r and t. We can assume that $r \geq 19$ and $s \geq 23$. From $(3.10a)$, we have $n = 2^7.5^b.17^c.7^2.(p')^e.(q')^f.r^f.t^h$ and so by Lemma 2.1,

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{17}{16}.\frac{50}{49}.\frac{2521}{2520}.\frac{163}{162}.\frac{19}{18}.\frac{23}{22} = 2.998984579 < 3,$$

a contradiction.

This completes the proof of Theorem 3.2 (e). □

Theorem 3.3. *Let n be as given in $(3.1a)$, where $b \geq 3$ and $3 \nmid n$. Assume that $7^2\|n$ and $17^3|n$. Let n be a bi-unitary triperfect number. Let $s = 11$ or $s = 13$. Assume that $s|n$, so that from $(3.10b)$, $w = s^e.w'$ and consequently from $(3.10a)$,*

$$n = 2^7.5^b.17^c.7^2.s^e.w'. \tag{3.13a}$$

Then

(a) $c = 6$ is not admissible.

If $c = 2\ell$, where ℓ is odd, then

(b) $\frac{17^\ell-1}{16}$ is divisible by an odd prime $p' > 519$ and $p'|w'$,

(c) $\frac{17^\ell+1}{2}$ is divisible by an odd prime $q' > 519$ and $q'|w'$.

Proof. Substituting $w = s^e.w'$ in $(3.10b)$, we obtain

$$2^6.5^{b-3}.17^{c-1}.7^2.s^e.w' = \sigma^{**}(5^b).\sigma^{**}(17^c).\sigma^{**}(s^e).\sigma^{**}(w'); \tag{3.13b}$$

also,

$$(w', 2.3.5.7.s.17) = 1 \text{ and } w' \text{ has no more than three odd prime factors.} \qquad (3.13c)$$

By Remark 3.1, we can assume that $b = 2k$, where k is odd and ≥ 3. Also, $c = 2\ell$, where ℓ is odd and ≥ 3 since $c \geq 3$.

<u>*Proof of (a).*</u> We have $\sigma^{**}(17^6) = 2.307.41761$. Hence taking $c = 6$ in $(3.13b)$, we obtain

$$2^5.5^{b-3}.17^5.7^2.s^e.w' = 307.41761.\sigma^{**}(5^b).\sigma^{**}(s^e).\sigma^{**}(w'). \qquad (3.13d)$$

From $(3.13d)$, w' is divisible by 307 and 41761. Hence we may assume that

$$w' = (307)^f.(41761)^g.w'';$$

using this in $(3.13a)$ and $(3.13d)$, we get

$$n = 2^7.5^b.17^6.7^2.s^e.(307)^f.(41761)^g.w'', \qquad (3.14a)$$

and

$$2^5.5^{b-3}.17^5.7^2.s^e.(307)^{f-1}.(41761)^{g-1}.w''$$
$$= \sigma^{**}(5^b).\sigma^{**}(s^e).\sigma^{**}((307)^f).\sigma^{**}((41761)^g)\sigma^{**}(w''), \qquad (3.14b)$$

where

$$(w'', 2.3.5.7.s.17.307.41761) = 1 \text{ and } w'' \text{ is 1 or a prime power.} \qquad (3.14c)$$

We obtain a contradiction by examining the factors of $\sigma^{**}(5^b)$ as follows.

We have $b = 2k$, where k is odd and ≥ 3. Also, $\sigma^{**}(5^b) = \left(\frac{5^k-1}{4}\right).(5^{k+1}+1)$. We claim that we can find two distinct odd primes p and q such that

(I) $p|\frac{5^k-1}{4}$ and $p|w''$ and (II) $q|\frac{5^{k+1}+1}{2}$ and $q|w''$.

- Proof of (I).

  (i) Since k is odd, we have $4\|5^k-1$. Hence $\frac{5^k-1}{4}$ is odd; also, it is > 1, since $k \geq 3$.

  (ii) We have $7|5^t-1 \iff 6|t$; $17|5^t-1 \iff 16|t$; $13|5^t-1 \iff 4|t$; $307|5^t-1 \iff 306|t$ and $41761|5^t-1 \iff 4176|t$. In all these cases, first of all t must be even. Since k is odd, 5^k-1 is not divisible by any of the primes in $\{7, 13, 17, 307, 41761\}$; and is not divisible by 5 trivially.

  (iii) We have $11|5^k-1 \iff 5|k \iff 71|5^k-1$. Hence if $11|5^k-1$ from $(3.14b)$, it follows that $71|w''$. Hence (I) holds in this case. We may assume that $11 \nmid 5^k-1$. Then $\frac{5^k-1}{4}$ is odd, is > 1 and is not divisible by $5, 7, 11, 13, 17, 307$ and 41761 or it is not divisible by $5, 7, s, 17, 307$ and 41761. It follows that if $p|\frac{5^k-1}{4}$, then $p \notin \{5, 7, s, 17, 307, 41761\}$. From $(3.14b)$, we conclude that $p|w''$.

  This proves (I).

- Proof of (II). Consider the factor $5^{k+1}+1$, where k is odd and ≥ 3.

  (iv) $2\|5^{k+1}+1$ and so $\frac{5^{k+1}+1}{2}$ is odd and clearly > 1.

  (v) For any positive integer $t,\ 5^t+1$ is not divisible by 11 and trivially not divisible by 5. In particular, $5^{k+1}+1$ is not divisible by 5 and 11.

  (vi) $7|5^{k+1}+1 \Longleftrightarrow k+1=3u;\ 307|5^{k+1}+1 \Longleftrightarrow k+1=153u$. Since $k+1$ is even $5^{k+1}+1$ is not divisible by 7 or 307.

  (vii) $17|5^{k+1}+1 \Longleftrightarrow k+1=8u;\ 41761|5^{k+1}+1 \Longleftrightarrow k+1=2088u=8u'$, where u' is odd. If either $17|5^{k+1}+1$ or $41761|5^{k+1}+1$, it follows that $5^8+1|5^{k+1}+1$. Also, $5^8+1=2.17.11489$. Hence $11489|5^{k+1}+1$. From $(3.14b)$, it follows that $11489|w''$. In both the cases it follows that (II) holds with $q=11489$. In what follows, assume that $5^{k+1}+1$ is neither divisible by 17 nor 41761.

  (viii) Assume that $13\nmid 5^{k+1}+1$. From (iv) to (vii), it follows that $\frac{5^{k+1}+1}{2}$ is not divisible by $5,7,11,13,17,307$ and 41761; in particular, it is not divisible by any of the primes $5,7,s,17,307$ and 41761. Hence if $q|\frac{5^{k+1}+1}{2}$, then $q\notin\{5,7,s,17,307,41761\}$. From $(3.14b)$, $q|w''$. This proves (II) in this case.

  (ix) Assume that $13|5^{k+1}+1$. Assume that $\frac{5^{k+1}+1}{2}$ is divisible by 13 alone so that $\frac{5^{k+1}+1}{2}=13^{\alpha}$, α being a positive integer. Suppose $\alpha\geq 2$. Hence $13^2|5^{k+1}+1$; this holds if and only if $k+1=26u$ and so $5^{26}+1|5^{k+1}+1$. Also, $5^{26}+1=2.13^2.53.83181652304609$. Thus $53|\frac{5^{26}+1}{2}|\frac{5^{k+1}+1}{2}=13^{\alpha}$, which is impossible. Hence $\alpha=1$ and so $\frac{5^{k+1}+1}{2}=13$ so that $k=1$. But $k\geq 3$. It follows that $\frac{5^{k+1}+1}{2}$ is divisible by an odd prime $q\neq 13$. Clearly, $q\notin\{5,7,s,17,307,41761\}$. From $(3.14b)$, $q|w''$.

  This proves (II).

Thus p and q are factors of w''. This violates $(3.14c)$. This proves that $c=6$ is not admissible.
The proof of Theorem 3.3(a) is complete.

Proof of (b). By (a), $c\neq 6$. Hence $\ell\geq 5$. Now from (I) of Corollary 2.1, it follows that $\frac{17^{\ell}-1}{16}$ is divisible by an odd prime $p'>519$; that $p'|w'$ readily follows from $(3.13b)$. Hence (b) follows.

Proof of (c). By (II) of Corollary 2.1, it follows that $\frac{17^{\ell+1}+1}{2}$ is divisible by an odd prime $q'>519$. By $(3.34b)$, $p'|w'$. Hence (c) follows.

This completes the proof of Theorem 3.3. □

Theorem 3.4. *Let n be as given in $(3.1a)$, where $b\geq 3$ and $3\nmid n$. Assume that $7^2\|n$ and $17^3|n$. Let n be a bi-unitary triperfect number.*

(a) Then n is not divisible by 11 and 13 simultaneously.

(b) Suppose that n is divisible by 11 or 13. Let $s=11$ or 13. Then we have

$$n>\begin{cases}4.5349\times 10^{169}, & \textit{if } s=11,\\ 3.43\times 10^{114}, & \textit{if } s=13.\end{cases}$$

Proof. The relevant equations are $(3.10a)$ and $(3.10b)$.
Proof of (a). We assume that n is divisible by 11 and 13. From $(3.10a)$, we find that 11 and 13 divide w. Let $w = 11^e.13^f.w'$, where $(w', 2.3.5.7.11.13.17) = 1$. From $(3.10a)$ and $(3.10b)$, we obtain

$$n = 2^7.5^b.17^c.7^2.11^e.13^f.w', \tag{3.15a}$$

and

$$2^6.5^{b-3}.17^{c-1}.7^2.11^e.13^f.w' = \sigma^{**}(5^b).\sigma^{**}(17^c).\sigma^{**}(11^e).\sigma^{**}(13^f).\sigma^{**}(w'), \tag{3.15b}$$

where

$$(w', 2.3.5.7.11.13.17) = 1 \text{ and } w' \text{ has no more than two odd prime factors.} \tag{3.15c}$$

We recall that $b = 2k$ and $c = 2\ell$, where $k,\ \ell$ are both ≥ 3 and odd. We can take $b \geq 5$ and $c \geq 3$ by hypothesis. We have by Lemma 2.1,

$$\frac{\sigma^{**}(5^b)}{5^b} \geq \frac{19406}{15625}, \quad (b \geq 5); \quad \frac{\sigma^{**}(17^c)}{17^c} \geq \frac{88452}{83521}, \quad (c \geq 3);$$
$$\frac{\sigma^{**}(11^e)}{11^e} \geq \frac{15984}{14641}, \quad (e \geq 3); \quad \frac{\sigma^{**}(13^f)}{13^f} \geq \frac{30772}{28561}, \quad (f \geq 3).$$

Hence for $e \geq 3$ and $f \geq 3$, from $(3.15a)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{19406}{15625}.\frac{88452}{83521}.\frac{50}{49}.\frac{15984}{14641}.\frac{30772}{28561} = 3.145061575,$$

a contradiction.

Thus $e \geq 3$ and $f \geq 3$ cannot hold. The following cases arise:
(i) $\{e \geq 3,\ f = 1, 2\}$ (ii) $\{e = 1, 2,\ f \geq 3\}$ and (iii) $\{e = 1, 2,\ f = 1, 2\}$.

(i) Let $e \geq 3$ and $f = 1$. From $(3.15a)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{19406}{15625}.\frac{88452}{83521}.\frac{50}{49}.\frac{15984}{14641}.\frac{14}{13} = 3.143630701 > 3,$$

a contradiction.

Let $f = 2$. We will not be using that $e \geq 3$. By (b) and (c) of Theorem 3.3, w' is divisible by two distinct odd primes p' and q' exceeding 519. We may assume that $p' \geq 521$ and $q' \geq 523$. Also, by $(3.15c)$, $w' = (p')^g.(q')^h$. Hence from $(3.15a)$, $n = 2^7.5^b.17^c.7^2.11^e.13^2.w'$ and so we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{17}{16}.\frac{50}{49}.\frac{11}{10}.\frac{170}{169}.\frac{521}{520}.\frac{523}{522} = 2.998910842 < 3,$$

a contradiction.

(ii) Let $e = 1$. Since $3|12 = \sigma^{**}(11)$, taking $e = 1$ in $(3.15b)$, it follows that 3 is a factor of its left-hand side. This cannot happen. Hence $e = 1$ is not admissible (independent of $f \geq 3$).

Let $e = 2$. We have $\sigma^{**}(11^2) = 122 = 2.61$. Taking $e = 2$, in $(3.15b)$, we find that $61|w'$. By (b) and (c) of Theorem 3.3, w' is already divisible by two odd primes p' and q' exceeding 61. Thus w' is divisible by three odd primes, namely, 61, p' and q'. This cannot happen in view of $(3.15c)$. Thus $e = 2$ is not admissible (independent of $f \geq 3$). Thus (ii) cannot hold.

(iii) This case will not occur as neither $e = 1$ nor $e = 2$ is admissible.

This proves part (a) of Theorem 3.4.

Proof of (b). We have $n = 2^7.5^b.17^c.7^2.s^e.w'$, and n satisfies $(3.13b)$ and $(3.13c)$.

We prove that $b \geq 54$ and $c \geq 54$. We recall that $b = 2k$ and $c = 2\ell$, where k and ℓ are odd and ≥ 3.

(1) We have $\sigma^{**}(5^6) = 2.31.313$. Taking $b = 6$ in $(3.13b)$, we see that w' is divisible by 31 and 313. By Theorem 3.3 (b) and (c), w' is divisible by p' and q' both exceeding 519. Hence w' must be divisible by four odd prime factors namely, 31, 313, p' and q'. This contradicts $(3.13c)$. Hence $b = 6$ is not possible.

(2) We have $\sigma^{**}(5^{10}) = 2.11.13.71.601$. Taking $b = 10$ in $(3.13b)$, we infer that w' is divisible by 11 and 13. By (a) of the present theorem this is not possible. Hence $b = 10$ is not admissible.

(3) Let $b = 14$. We have $\sigma^{**}(5^{14}) = 2.17.11489.19531$; and $11489|17^\ell - 1 \iff 11488|\ell$. This is not possible since ℓ is odd. Hence $11489 \nmid 17^\ell - 1$. Let $p = 11489$ and $r = ord_p 17$ so that $r = 11488$. Hence $r/2 = 5744 = 2^4.379$. Hence $p|17^{\ell+1} + 1$ if and only if $\ell + 1 = \frac{r}{2}.u = 16.379.u = 16.u'$, where u' is odd. Thus $p|17^{\ell+1} + 1$ implies that $17^{16} + 1$ is a factor of $17^{\ell+1} + 1$. We have

$$17^{16} + 1 = \{\{2, 1\}, \{257, 1\}, \{1801601, 1\}, \{52548582913, 1\}\}.$$

It now follows from $(3.13b)$ that w' is divisible by five odd prime factors, namely, 257, 1801601, 52548582913, 11489 and 19531. This contradicts $(3.13c)$. Hence $p \nmid 17^{\ell+1} + 1$ and so $p \nmid \sigma^{**}(17^c)$.

Let $q = 19531$. Then $r' = ord_q 17 = 9765 = 3^2.5.7.31$. Hence $31|r'$ and so $17^{31} - 1|17^\ell - 1$ if $q|17^\ell - 1$. Also,

$$17^{31} - 1 = \{\{2,4\}, \{4093,1\}, \{6123493,1\}, \{347340647626008901939025023,1\}\}.$$

It now follows from $(3.13b)$ that w' is divisible by five odd prime factors, namely, 11489, 19531, 4093, 6123493 and 347340647626008901939025023. This contradicts $(3.13c)$. Hence $q \nmid 17^\ell - 1$.

Since $r' = 9765$ is odd, $q \nmid 17^t + 1$ for any positive integer t. In particular, $q \nmid 17^{\ell+1} + 1$. Hence from the above discussion it follows that $q \nmid \sigma^{**}(17^c)$. Let p' and q' be as given in (b) and (c) of Theorem 3.3. Then the four distinct primes p, q, p' and q' are factors of w' by $(3.13b)$. This is a contradiction to $(3.13c)$. Hence $b = 14$ is not admissible.

(4) We have $\sigma^{**}(5^{18}) = 2.13.19.31.41.829.9161$. Taking $b = 18$ in $(3.13b)$, we see that w' is divisible by five odd primes, namely, 19, 31, 41, 829 and 9161. This violates $(3.13c)$. Hence $b = 18$ is not admissible.

(5) We have $\sigma^{**}(5^{22}) = 2.313.390001.12207031$. Taking $b = 22$ in $(3.13b)$, we find that $w' = (313)^f.(390001)^g.(12207031)^h$. Hence from $(3.13a)$,

$$n = 2^7.5^b.17^c.7^2.s^e.(313)^f.(390001)^g.(12207031)^h,$$

so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{17}{16}.\frac{50}{49}.\frac{11}{10}.\frac{313}{312}.\frac{390001}{390000}.\frac{12207031}{12207030} = 2.979385259 < 3,$$

a contradiction. Hence $b = 22$ is not admissible.

(6) Let $b = 26$. We have $\sigma^{**}(5^{26}) = \{\{2,1\},\{13,1\},\{234750601,1\},\{305175781,1\}\}$. Let $p = 234750601$ and $q = 305175781$. We now prove that $\sigma^{**}(17^c)$ is not divisible by either p or q. We have

$$\sigma^{**}(17^c) = \left(\frac{17^\ell - 1}{16}\right).(17^{\ell+1} + 1).$$

Let $r = ord_p 17$. Then $r|p-1 = 234750600 = \{\{2,3\},\{3,2\},\{5,2\},\{7,1\},\{31,1\},\{601,1\}\}$ and hence r takes 288 choices. Verifying these choices, we can show that $r = 5868765$.

Since $45|r$, we have $17^{45} - 1|17^r - 1$. Assume that $p|17^\ell - 1$. Hence $r|\ell$ so that $17^{45} - 1|17^r - 1|17^\ell - 1$. Also,

$$17^{45} - 1 = \{\{2,4\},\{19,1\},\{307,1\},\{3691,1\},\{33931,1\},\{88741,1\},\{316531,1\},$$
$$\{1270657,1\},\{1674271,1\},\{5113320301,1\},\{6566760001,1\}\}.$$

It follows from $(3.13b)$ that w' will be divisible by ten odd prime factors. This violates $(3.13c)$.Thus $p \nmid 17^\ell - 1$. Since r is odd, $p \nmid 17^t + 1$ for any positive integer t. In particular, $p \nmid 17^{\ell+1} + 1$.

Let $r' = ord_q 17$. Then $r'|q-1 = 305175180 = \{\{2,2\},\{3,1\},\{5,1\},\{367,1\},\{13859,1\}\}$. Hence r' takes 48 choices. Verifying these choices it can be shown that $r' = 21798270$. As r' is even and ℓ is odd, $q \nmid 17^\ell - 1$. Also, if $s' = r'/2 = 10899135$, then s' is odd. Hence $q \nmid 17^{\ell+1} + 1$ since $\ell + 1$ is even.

It follows that $\sigma^{**}(17^c)$ is not divisible by either p or q. If p' and q' are as in (b) and (c) of Theorem 3.3, it is clear from $(3.13b)$ that p, p', q and q' are four distinct odd prime factors of w'. This is a contradiction to $(3.13c)$. Thus $b = 26$ is not admissible.

(7) Since $\sigma^{**}(5^{30}) = 2.11.31.71.181.1741.2593.29423041$, taking $b = 30$ in $(3.13b)$, we infer that w' will be divisible by six odd prime factors and this violates $(3.13c)$. Hence $b = 30$ is not admissible.

In a similar way we have

$$\sigma^{**}(5^{34}) = 2\,.\,13\,.\,37\,.\,409\,.\,601\,.\,6597973\,.\,466344409,$$
$$\sigma^{**}(5^{38}) = 2\,.\,191\,.\,241\,.\,313\,.\,6271\,.\,3981071\,.\,632133361,$$
$$\sigma^{**}(5^{42}) = 2\,.\,13\,.\,31\,.\,89\,.\,379\,.\,19531\,.\,519499\,.\,1030330938209,$$
$$\sigma^{**}(5^{46}) = 2\,.\,17\,.\,8971\,.\,11489\,.\,152587500001\,.\,332207361361,$$
$$\sigma^{**}(5^{50}) = 2\,.\,11\,.\,13\,.\,53\,.\,71\,.\,101\,.\,251\,.\,401\,.\,9384251\,.\,83181652304609.$$

Hence if $b = 34, 38, 42, 46$ or 50, from $(3.13b)$, we infer that $(3.13c)$ is violated. Hence $b \geq 54$.

We now show that $c \geq 56$.

(8) We recall that $c = 2\ell$, where ℓ is odd and ≥ 3. By (a) of Theorem 3.3, $c = 6$ is not admissible.

(9) Let $c = 10$. We have $\sigma^{**}(17^{10}) = 2.5.29.83233.88741$. Let $p = 83233$ and $q = 88741$. Then $r = ord_p 5 = 9248$. Since r is even, $p \nmid 5^k - 1$. Let $s' = r/2 = 4624$. Suppose that $p|5^{k+1} + 1$. This is equivalent to $k + 1 = s'u = (4624)u = 16u'$, where u' is odd. Hence $5^{16} + 1|5^{k+1} + 1$. Also, $5^{16} + 1 = 2.2593.29423041$. From $(3.13b)$, we infer that w' will be divisible by the five odd primes $2593, 29423041, 29, 83233$ and 88741. This is not possible in virtue of $(3.13c)$. Hence $p \nmid 5^{k+1} + 1$.

We have $r' = ord_q 5 = 44370$. Since r' is even, $q \nmid 5^k - 1$. Also, $s' = r'/2 = 22185$ and so s' is odd. Hence $q \nmid 5^{k+1} + 1$.

Thus neither p nor q divides $\sigma^{**}(5^b)$.

We now prove that we can find a prime $t|\frac{5^k-1}{4}$ and $t \notin \{5, 7, 11, 13, 29\}$. Since k is odd, $5^k - 1$ is not divisible by 7, 13 and 29. Suppose that $11 \nmid 5^k - 1$. If t is any prime factor of $\frac{5^k-1}{4}$ which is > 1 and odd, then $t \notin \{5, 7, 11, 13, 29\}$. Suppose that $11|5^k - 1$. This is equivalent to $71|5^k - 1$. Hence we can take $t = 71$. From $(3.13b)$, in both the cases $t|w'$ and $t \notin \{p, q\}$. Hence $29, t, p, q$ would be four prime factors of w' in $(3.13b)$ and this violates $(3.13c)$.

Thus $c = 10$ is not admissible.

(10) We have $\sigma^{**}(17^{14}) = 2.18913.184417.25646167$. Taking $c = 14$ in $(3.13b)$, we find that $w' = (18913)^f.(184417)^g.(25646167)^h$. Hence

$$n = 2^7.5^b.17^c.7^2.s^e.(18913)^f.(184417)^g.(25646167)^h$$

so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{17}{16}.\frac{50}{49}.\frac{11}{10}.\frac{18913}{18912}.\frac{184417}{184416}.\frac{25646167}{25646166} = 2.970031854 < 3,$$

a contradiction. Hence $c = 14$ is not admissible.

(11) We have

$$\begin{aligned}
\sigma^{**}(17^{18}) = {} & \{\{2,1\},\{5,2\},\{19,1\},\{29,1\},\{307,1\},\{21881,1\},\{63541,1\},\\
& \{1270657,1\}\};\\
\sigma^{**}(17^{22}) = {} & \{\{2,1\},\{73,1\},\{1321,1\},\{41761,1\},\{72337,1\},\{2141993519227,1\}\};\\
\sigma^{**}(17^{26}) = {} & \{\{2,1\},\{5,1\},\{29,1\},\{212057,1\},\{5766433,1\},\{100688449,1\},\\
& \{2919196853,1\}\};\\
\sigma^{**}(17^{30}) = {} & \{\{2,1\},\{257,1\},\{307,1\},\{88741,1\},\{1801601,1\},\{6566760001,1\},\\
& \{52548582913,1\}\};\\
\sigma^{**}(17^{34}) = {} & \{\{2,1\},\{5,1\},\{29,1\},\{37,1\},\{109,1\},\{181,1\},\{2089,1\},\{10949,1\},\\
& \{83233,1\},\{382069,1\},\{1749233,1\},\{2699538733,1\}\};\\
\sigma^{**}(17^{38}) = {} & \{\{2,1\},\{41,1\},\{229,1\},\{1103,1\},\{41761,1\},\{202607147,1\},\\
& \{291973723,1\},\{1186844128302568601,1\}\};\\
\sigma^{**}(17^{42}) = {} & \{\{2,1\},\{5,1\},\{29,1\},\{43,1\},\{89,1\},\{307,1\},\{13567,1\},\{25741,1\},\\
& \{25646167,1\},\{256152733,1\},\{940143709,1\},\{6901823633,1\}\};\\
\sigma^{**}(17^{46}) = {} & \{\{2,1\},\{47,1\},\{18913,1\},\{184417,1\},\{48661191868691111041,1\},\\
& \{265526182192280901629774 81,1\}\};\\
\sigma^{**}(17^{50}) = {} & \{\{2,1\},\{5,1\},\{29,1\},\{2551,1\},\{5351,1\},\{88741,1\},\{19825313,1\},\\
& \{26278001,1\},\{1224199237,1\},\{11330289301,1\},\{13938043025453,1\}\}.
\end{aligned}$$

From the above it follows from $(3.13b)$ that for $c = 18, 22, 26, 30, 34, 38, 42, 46, 50$, w' is divisible by at least four odd primes which violates $(3.13c)$. Hence $c \geq 54$.

Let $s = 11$. We note that when e is odd or $4|e$, then $3|\sigma^{**}(11^e)$. From $(3.13b)$, it follows that $3|w'$. But this not possible. Hence we may assume that $e = 2m$, where m is odd.

(12) If $e = 2$. Since $\sigma^{**}(11^2) = 122 = 2.61$, from $(3.13a)$ and $(3.13b)$, we get

$$n = 2^7.5^b.17^c.7^2.11^2.61^f.w'' \tag{3.16a}$$

and

$$2^5.5^{b-3}.17^{c-1}.7^2.11^2.61^{f-1}.w'' = \sigma^{**}(5^b).\sigma^{**}(17^c).\sigma^{**}(61)^f.\sigma^{**}(w''), \tag{3.16b}$$

where

$$(w'', 2.3.5.7.11.17.61) = 1 \text{ and } w'' \text{ has no more than two odd prime factors.} \tag{3.16c}$$

By (b) and (c) of Theorem 3.3, w' and consequently w'' is divisible by primes p' and q' each exceeding 519. Hence from $(3.16c)$, $w'' = (p')^g.(q')^h$, so that

$$n = 2^7.5^b.17^c.7^2.11^2.61^f.(p')^g.(q')^h.$$

We can assume $p' \geq 521$ and $q' \geq 523$. We have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{17}{16}.\frac{50}{49}.\frac{122}{121}.\frac{61}{60}.\frac{521}{520}.\frac{523}{522} = 2.778188424 < 3,$$

a contradiction. Thus, $e = 2$ is not admissible.

(13) Let $e = 6$. We have $\sigma^{**}(11^6) = 2.7.19.7321$. Taking $e = 6$ in $(3.13b)$, we see that w' is divisible by 19 and 7321. We now examine whether $\sigma^{**}(17^c)$ is divisible by 7321. Let $p = 7321$. Then $r = ord_p 17 = 2440$. Since r is even and ℓ is odd, $p \nmid 17^\ell - 1$. Also, $r/2 = 1220 = 20.61$. Hence if $p | 17^{\ell+1} + 1$, then $17^{20} + 1 | 17^{\ell+1} + 1$. We have

$$17^{20} + 1 = 2.41.41761.1186844128302568601.$$

It now follows from $(3.13b)$ that w' is divisible by the four odd primes $19, 7321, 41761$ and 1186844128302568601. This violates $(3.13c)$. So, $p \nmid 17^{\ell+1} + 1$. From (b) and (c) of Theorem 3.3, we infer that $19, 7321, p', q'$ are four distinct odd prime factors of w'. Again this violates $(3.13c)$. Thus $e = 6$ is not admissible.

(14) Let $e = 10$. We have $\sigma^{**}(11^{10}) = 2.5.13.61.1117.3221$. Taking $e = 10$ in $(3.13b)$, we find that w' is divisible by four odd primes, namely, $13, 61, 1117$ and 3221. This contradicts $(3.13c)$. Hence $e = 10$ is not admissible.

(15) When $e = 14$, we have $\sigma^{**}(11^e) = 2.17.43.45319.6304673$. Since $r = ord_{43} 17 = 6$ is even, $43 \nmid 17^\ell - 1$. Also, $r/2 = 3$ is odd. Hence $43 \nmid 17^{\ell+1} + 1$.

Let $p = 45319$. Then $r' = ord_p 17 = 45318$. Hence $p \nmid 17^\ell - 1$ since r' is even. Also, $r'/2 = 22654 = 2.47.241 = 94u'$, where u' is odd. Hence if $p | 17^{\ell+1} + 1$, then $17^{94} + 1 | 17^{\ell+1} + 1$. We have

$$\begin{aligned} 17^{94} + 1 = &\{\{2, 1\}, \{5, 1\}, \{29, 1\}, \{8837, 1\}, \\ &\{179265103693349709880136365395087273880137628400303290349204841 \\ &92380077903123625556163033417839614972209382061, 1\}\}. \end{aligned}$$

Taking into account of the factors of $\sigma^{**}(11^{14})$ and $17^{94} + 1$, from $(3.13b)$, we see that w' is divisible by six odd prime factors contradicting $(3.13c)$. Hence $p \nmid 17^{\ell+1} + 1$. Let p' and q' be as given in (b) and (c) of Theorem 3.3. Then from the above discussion either $p' \notin \{43, 45319, 6304673\}$ or $q' \notin \{43, 45319, 6304673\}$. From $(3.13b)$ it now follows that w' is divisible by at least four odd primes contradicting $(3.13c)$. We conclude that $e = 14$ is not admissible.

(16) We have

$$\sigma^{**}(11^{18}) = \{\{2,1\},\{7,1\},\{19,1\},\{61,1\},\{1772893,1\},\{212601841,1\}\};$$
$$\sigma^{**}(11^{22}) = \{\{2,1\},\{7321,1\},\{10657,1\},\{15797,1\},\{20113,1\},\{1806113,1\}\};$$
$$\sigma^{**}(11^{26}) = \{2,1\},\{29,1\},\{61,1\},\{1093,1\},\{1933,1\},\{55527473,1\},\{3158528101,1\}\};$$
$$\sigma^{**}(11^{30}) = \{\{2,1\},\{5,1\},\{7,1\},\{19,1\},\{3221,1\},\{51329,1\},\{195019441,1\},\{447600088289,1\}\};$$
$$\sigma^{**}(11^{34}) = \{\{2,1\},\{13,1\},\{61,1\},\{1117,1\},\{3138426605161,1\},\{50544702849929377,1\}\};$$
$$\sigma^{**}(11^{38}) = \{\{2,1\},\{41,1\},\{7321,1\},\{1120648576818041,1\},\{6115909044841454629,1\}\};$$
$$\sigma^{**}(11^{42}) = \{\{2,1\},\{7,2\},\{19,1\},\{43,1\},\{61,1\},\{1723,1\},\{8527,1\},\{27763,1\},\{45319,1\},\{251857,1\},\{2649263870814793,1\}\};$$
$$\sigma^{**}(11^{46}) = \{\{2,1\},\{17,1\},\{97,1\},\{241,1\},\{829,1\},\{1777,1\},\{6304673,1\},\{28878847,1\},\{1106131489,1\},\{3740221981231,1\}\};$$
$$\sigma^{**}(11^{50}) = \{\{2,1\},\{5,2\},\{61,1\},\{3001,1\},\{3221,1\},\{24151,1\},\{1856458657451,1\},\{9768997162071483134919121,1\}\}.$$

It now follows that none of $e = 18, 22, 26, 30, 34, 38, 42, 46$ and 50 is admissible as in each case w' will be divisible by at least four odd prime factors (which follows from $(3.13b)$) contradicting $(3.13c)$.

Thus $e \geq 54$.

We may note that by (b) and (c) of Theorem 3.3, $w' \geq 521.523$. Thus if $s = 11$,

$$n = 2^7.5^b.17^c.7^2.11^e.w' \geq 128.5^{54}.17^{54}.11^{54}.521.523 > 4.5349 \times 10^{169},$$

and if $s = 13$,

$$n = 2^7.5^b.17^c.7^2.13^e.w' \geq 128.5^{54}.17^{54}.13.521.523 > 3.43 \times 10^{114}.$$

This proves (b) of Theorem 3.4. The proof of Theorem 3.4 is complete. □

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