Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 27, 2021, No. 1, 32–44
DOI: 10.7546/nntdm.2021.27.1.32-44

A study on some generalized multiplicative and generalized additive arithmetic functions

D. Bhattacharjee

Department of Mathematics, North-Eastern Hill University
Permanent Campus, Shillong-793022, India
e-mails: debashis_bhattacharjee@yahoo.com,
dbhattacharjee@nehu.ac.in

Received: 14 August 2020 **Revised:** 10 March 2020 **Accepted:** 12 March 2020

Abstract: In this paper by an arithmetic function we shall mean a real-valued function on the set of positive integers. We recall the definitions of some common arithmetic convolutions. We also recall the definitions of a multiplicative function, a generalized multiplicative function (or briefly a GM-function), an additive function and a generalized additive function (or briefly a ***GA***-function). We shall study in details some properties of ***GM***-functions as well as ***GA***-functions using some particular arithmetic convolutions namely the Narkiewicz's *A*-product and the author's *B*-product. We conclude our discussion with some examples.
Keywords: Arithmetic function, Multiplicative function, Arithmetic convolution, Dirichlet convolution, Narkiewicz's *A*-product, *B*-product, Multiplicative*B*-product, ***GA***-function, ***GM***-function.
2010 Mathematics Subject Classification:11A25.

1 Introduction

In this paper, by an arithmetic function we shall mean a real valued function on the set of positive integers. The early history of the theory of arithmetic functions is contained in the first volume of Dickson's [13] monumental "History of the Theory of Numbers". The convolution operation played a prominent role from the very beginning. Many results from early times involved the convolution of two or more particular arithmetical functions.

Early in the 19^{th} century the Dirichlet convolution, as well as addition and multiplication, began to be viewed as binary operations on the set of arithmetic functions. In the works of Cipolla [10] and Bell [2] it was recognized that the arithmetic functions form a commutative ring with unity with respect to addition and convolution.

The study of the structure of the ring of arithmetic functions has continued and we point out papers of, Vaidyanathaswamy [25], Cohen [11], Narkiewicz [19], Subbarao [22], Scheid [20], Apostol [1], Davison [12], Gioia [14], McCarthy [18], Sivaramakrishnan [21], Haukkanen [16], Tóth [23, 24], Lehmer [17], Bhattacharjee [3–7], Bhattacharjee and Saikia [8], and others.

We now introduce some of the most common arithmetic convolutions to our readers.

1. **Natural product (usual product).** If f and g are arithmetic functions, we define the natural product fg (also written as $f \times g$) as the arithmetic function whose value at any positive integer n is given by:
$$(fg)(n) = f(n)g(n)$$
2. **Dirichlet product (convolution).** If f and g are arithmetic functions, we define the Dirichlet product (or Dirichlet convolution) $f * g$ as the arithmetic function whose value at any positive integer n is given by:
$$(f * g)(n) = \sum_{d|n} f(d)\, g\left(\frac{n}{d}\right) = \sum_{d_1 d_2 = n} f(d_1)\, g(d_2)$$
3. **Unitary product (convolution).** If f and g are arithmetic functions, we define the unitary product (or unitary convolution) $f \circ g$ as the arithmetic function whose value at any positive integer n is given by:
$$(f \circ g)(n) = \sum_{\substack{d_1 d_2 = n \\ (d_1, d_2) = 1}} f(d_1) g(d_2)$$
where (d_1, d_2) stands for the greatest common divisor of d_1 and d_2.
4. **L.C.M. product (convolution).** If f and g are arithmetic functions, we define the L.C.M product (or L.C.M convolution) $f \oplus g$ as the arithmetic function whose value at any positive integer n is given by
$$(f \oplus g)(n) = \sum_{[d_1, d_2] = n} f(d_1) f(d_2)$$
where $[d_1, d_2]$ stands for the least common multiple of d_1 and d_2.
5. **The *k*-product (convolution).** If f and g are arithmetic functions, we define the *k*-product $f *_k g$ as the arithmetic function whose value at any positive integer n is given by:
$$(f *_k g)(n) = \sum_{ab=n} f(a)g(b)k((a,b))$$
where $k((a,b))$ is a function of (a,b), the greatest common divisor of a and b.
6. **Davison's product (convolution).** If f and g are arithmetic functions, we define the Davison's product $f *_D g$ as the arithmetic function whose value at any positive integer *n* is given by:
$$(f *_D g)(n) = \sum_{ab=n} f(a)g(b)A(a,b)$$
where $A(a,b)$ is a function of two variables a and b, instead of being a function of their greatest common divisor as in **5**.

7. **The Lehmer ψ-product (convolution).** Let $\psi(x, y)$ be a positive integral-valued function defined for a prescribed set T of ordered pairs (x, y) such that $x, y \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. Then the ψ-product $f \odot g$ of f and g is defined by:

$$(f \odot g)(n) = \sum_{\psi(a,b)=n} f(a)g(b),$$

where $n \in \mathbb{N}$.

8. **Narkiewicz's *A*-product (convolution).** For every positive integer n fix the set A_n of some divisors of n. For arithmetic functions f and g their A-product $f *_A g$ is given by

$$(f *_A g)(n) = \sum_{d \in A_n} f(d)\, g\left(\frac{n}{d}\right)$$

for n = 1, 2, 3, … .

We now recall the definitions of a multiplicative function, a generalized multiplicative function (or briefly a ***GM***-function), an additive function and a generalized additive function (or briefly a ***GA***-function). We shall study in details some properties of ***GM***-functions as well as ***GA***-functions using some particular arithmetic convolutions namely the Narkiewicz's A-product and the author's B-product as multiplicative binary operations.

An arithmetic function F is said to be multiplicative if $F(1) = 1$ and $F(mn) = F(m)F(n)$, whenever $(m, n) = 1$, where (m, n) stands for the greatest common divisor of m and n.

There are several generalizations of multiplicative functions which can be found in McCarthy [18], Sivaramkrishnan [21]. Zafrullah in his paper [26] introduced a new generalization where he defined an arithmetic function F to be a Generalized Multiplicative function (or briefly a ***GM***-function) if $F(1) = 1$ and there exists a multiplicative function f such that:

$$F(mn) = F(m)^{f(n)}F(n)^{f(m)} \tag{1}$$

whenever $(m, n) = 1$.

An arithmetic function F is said to be an additive function if $F(mn) = F(m) + F(n)$ whenever $(m, n) = 1$. We call an arithmetic function F to be a Generalized Additive function (or briefly a ***GA***-function) if there exists a multiplicative function f such that

$$F(mn) = F(m)f(n) + F(n)f(m) \tag{2}$$

whenever $(m, n) = 1$.

Haukkanen in his paper [15] studied some properties of ***GM***-functions and ***GA***-functions considering Dirichlet convolution as the multiplicative binary operation. The present author in this paper tries to study some properties of ***GM***-functions and ***GA***-functions by considering the Narkiewicz's A-product [19] and the author's B-product [3] as multiplicative binary operation.

We conclude our discussion with some examples.

As mentioned earlier, Narkiewicz in his paper [19] defined A-product as follows: For every positive integer n fix the set A_n of some divisors of n. For arithmetic functions f and g their A-product $f *_A g$ is given by:

$$(f *_A g)(n) = \sum_{d \in A_n} f(d)\, g\left(\frac{n}{d}\right) \tag{I}$$

for n = 1, 2, 3, … .

The author in his paper [3] defined *B*-product as follows: For every positive integer n, fix the set B_n of some pairs of divisors of *n*. For arithmetic functions f and g their *B*-product $f *_B g$ is given by:

$$(f *_B g)(n) = \sum_{(u,v)\in B_n} f(u)g(v) \tag{II}$$

for *n* = 1, 2, 3, … .

This *B*-product generalizes simultaneously the *A*-product of Narkiewicz [19] and the L.C.M. product and it has a non-void intersection with the Ψ-product of Lehmer [17]. The τ-product of Scheid [20] is also a particular case of *B*-product.

Narkiewicz in his paper [19] called an *A*-product to be multiplicative if and only if A_{mn}=$A_m \times A_n$ for every pair (*m*, *n*) of relatively prime positive integers (here $B \times C$ denotes the set of all integers, which can be represented in the form $bc, b \in B, c \in C$).

The present author in his paper [4] defined a *B*-product to be multiplicative if and only if B_{mn} = $\{(r_1r_2,\ s_1s_2) \mid (r_1, s_1)\in B_m, (r_2, s_2)\in B_n\}$ for every pair (*m*, *n*) of relatively prime positive integers; in other words a *B*-product is multiplicative if and only if the following conditions hold: For every pair (*m*, *n*) of relatively prime positive integers we have (*r*, *s*)$\in B_{mn}$ if and only if $(r^{(m)}, s^{(m)})\in B_m$ and $(r^{(n)}, s^{(n)})\in B_n$, where $r^{(k)}$ stands for the greatest common divisor of *r* and *k*.

In the next section, we shall study in details some properties of ***GM***-functions and ***GA***-functions using Narkiewicz's *A*-product and the author's *B*-product as multiplicative binary operations.

2 Main results

2.1 *GM*-Function and its properties

As mentioned in our discussion earlier, Zafrullah in his paper [26] introduced a new generalization of multiplicative functions where he defined an arithmetic function *F* to be a generalized multiplicative function (or ***GM***-function) if *F*(1) = 1 and there exists a multiplicative function f such that $F(mn) = F(m)^{f(n)}F(n)^{f(m)}$, whenever $(m, n) = 1$. We now prove some results related to ***GM***-functions using Narkeiwicz's *A*-product and the author's *B*-product as the multiplicative binary operations.

Theorem 2.1.1. Let *F* be a ***GM***-function and *h* be a multiplicative function and let the *A*-product be multiplicative, then the arithmetic function *H* defined by

$$H(n) = \prod_{d\in A_n} F(d)^{h(\frac{n}{d})}$$

for all $n \in \mathbb{N}$ is a ***GM***-function.

Proof: We have

$$H(1) = \prod_{d\in A_1} F(d)^{h(\frac{n}{d})} = F(1)^{h(\frac{1}{1})} = F(1) = 1$$

Let $(m, n) = 1$, then

$$H(mn) = \prod_{d \in A_{mn}} F(d)^{h\left(\frac{mn}{d}\right)}$$

$$= \prod_{d_1 \in A_m, d_2 \in A_n} F(d_1 d_2)^{h\left(\frac{mn}{d_1 d_2}\right)}$$

$$= \prod_{d_1 \in A_m} \prod_{d_2 \in A_n} [F(d_1)^{f(d_2)} F(d_2)^{f(d_1)}]^{h\left(\frac{m}{d_1}\right) h\left(\frac{n}{d_2}\right)}$$

(where *f* is a multiplicative function)

$$= [\prod_{d_1 \in A_m} F(d_1)^{h\left(\frac{m}{d_1}\right)}]^{\sum_{d_2 \in A_n} f(d_2) h\left(\frac{n}{d_2}\right)} [\prod_{d_2 \in A_n} F(d_2)^{h\left(\frac{n}{d_2}\right)}]^{\sum_{d_1 \in A_m} f(d_1) h\left(\frac{m}{d_1}\right)}$$

$$= [\prod_{d_1 \in A_m} F(d_1)^{h\left(\frac{m}{d_1}\right)}]^{(f *_A h)(n)} [\prod_{d_2 \in A_n} F(d_2)^{h\left(\frac{n}{d_2}\right)}]^{(f *_A h)(m)}$$

$$= H(m)^{v(n)} \times H(n)^{v(m)},$$

where $v = f *_A h$.

Since f and h are multiplicative functions and the A-product is multiplicative, therefore v is a multiplicative function.

Thus, H is a ***GM***-function. ☐

Theorem 2.1.2. Let F be a ***GM***-function and h_1 and h_2 be multiplicative functions and let the A-product be multiplicative, then the arithmetic function H defined by

$$H(n) = \prod_{d \in A_n} F(d)^{h_1(d) h_2\left(\frac{n}{d}\right)}$$

for all $n \in \mathbb{N}$ is a ***GM***-function.

Proof: Let $F(d)^{h_1(d)} = K(d)$. Then

$$H(n) = \prod_{d \in A_n} K(d)^{h_2\left(\frac{n}{d}\right)}$$

and $K(1) = F(1)^{h_1(1)} = 1$. Let $(m, n) = 1$, then

$$K(mn) = F(mn)^{h_1(mn)}$$

$$= [F(m)^{f(n)} F(n)^{f(m)}]^{h_1(m) h_1(n)}$$

(where f is a multiplicative function)

$$= K(m)^{f h_1(n)} K(n)^{f h_1(m)}$$
$$= K(m)^{v(n)} K(n)^{v(m)}$$

Since f and h_1 are multiplicative functions, then $v = f h_1$ is also a multiplicative function. Therefore, K is a ***GM***-function.

Therefore, by Theorem 2.1.1 we conclude that H is a ***GM***-function. ☐

Theorem 2.1.3. Let F be a ***GM***-function and h be a multiplicative function and let the B-product be multiplicative. Then the arithmetic function $J(n)$ given by

$$J(n) = \prod_{(r,s)\in B_n} F(r)^{h(s)}$$

for all $n \in \mathbb{N}$ is a ***GM***-function.

Proof: We have

$$J(1) = \prod_{(r,s)\in B_1} F(r)^{h(s)} = F(1)^{h(1)} = 1$$

Let $(m,n) = 1$. Then

$$J(mn) = \prod_{(r,s)\in B_{mn}} F(r)^{h(s)}$$

$$= \prod_{(r^{(m)},s^{(m)})\in B_m} \prod_{(r^{(n)},s^{(n)})\in B_n} F(r^{(m)}r^{(n)})^{h(s^{(m)}s^{(n)})}$$

$$= \prod_{(r^{(m)},s^{(m)})\in B_m} \prod_{(r^{(n)},s^{(n)})\in B_n} [F(r^{(m)})^{f(r^{(n)})} F(r^{(n)})^{f(r^{(m)})}]^{h(s^{(m)})h(s^{(n)})}$$

(where f is a multiplicative function)

$$= \prod_{(r^{(m)},s^{(m)})\in B_m} \prod_{(r^{(n)},s^{(n)})\in B_n} \left[F(r^{(m)})^{h(s^{(m)})}\right]^{f(r^{(n)})h(s^{(n)})} \times \left[F(r^{(n)})^{h(s^{(n)})}\right]^{f(r^{(m)})h(s^{(m)})}$$

$$= \left[\left(\prod_{(r^{(m)},s^{(m)})\in B_m} \left[F(r^{(m)})^{h(s^{(m)})}\right] \right)^{\sum_{(r^{(n)},s^{(n)})\in B_n} f(r^{(n)})h(s^{(n)})}\right]$$

$$\times \left[\left(\prod_{(r^{(n)},s^{(n)})\in B_n} \left[F(r^{(n)})^{h(s^{(n)})}\right] \right)^{\sum_{(r^{(m)},s^{(m)})\in B_m} f(r^{(m)})h(s^{(m)})}\right]$$

$$= J(m)^{(f*_B h)(n)} \times J(n)^{(f*_B h)(m)}$$

$$= J(m)^{v(n)} \times J(n)^{v(m)}$$

where $v = f *_B h$.

Since f and h are multiplicative functions and the B-product is multiplicative, therefore v is a multiplicative function.

Hence, J is a ***GM***-function. □

Theorem 2.1.4. Let F be a ***GM***-function and let h_1 and h_2 be multiplicative functions and let the B-product be multiplicative. Then the arithmetic function J defined by

$$J(n) = \prod_{(r,s)\in B_n} F(r)^{h_1(r)h_2(s)}$$

for all $n \in \mathbb{N}$ is a ***GM***-function.

Proof: Let $U(r) = F(r)^{h_1(r)}$. Therefore,

$$J(n) = \prod_{(r,s)\in B_n} U(r)^{h_2(s)}$$

Clearly,

$$U(1) = F(1)^{h_1(1)} = 1.$$

Let $(m,n) = 1$.Then

$$U(mn) = F(mn)^{h_1(mn)} = \left(F(m)^{f(n)}F(n)^{f(m)}\right)^{h_1(m)h_1(n)}$$

(where f is a multiplicative function)

$$= [F(m)^{f(n)}]^{h_1(m)h_1(n)} [F(n)^{f(m)}]^{h_1(m)h_1(n)}$$

$$= [F(m)^{h_1(m)}]^{f(n)h_1(n)} [F(n)^{h_1(n)}]^{f(m)h_1(m)}$$

$$= U(m)^{v(n)}U(n)^{v(m)}$$

where $v = fh_1$, is a multiplicative function.

Therefore, U is a ***GM***-function.

Hence, by Theorem 2.1.3 we conclude that J is a ***GM***-function. □

2.2 *GA*-Function and its properties

An arithmetic function *F* is said to be additive if $F(mn) = F(m) + F(n)$ whenever $(m,n) = 1$. We define an arithmetic function *F* to be a generalized additive function (or briefly a ***GA***-function) if there exists a multiplicative function f such that $F(mn) = F(m)f(n) + F(n)f(m)$ whenever $(m,n) = 1$. Chawla [9] calls such a function a distributive function. One can notice that the notion of ***GA***-function is an additive analogue of ***GM***-function. We now prove some results on ***GA***-function in the context of Narkiewicz's *A*-product and the author's *B*-product.

Theorem 2.2.1. Let *F* be a ***GA***-function and *h* be a multiplicative function and let the *A*-product be multiplicative. Then the arithmetic function *H* defined by

$$H(n) = \sum_{d\in A_n} F(d)\, h(d)$$

for all $n \in \mathbb{N}$, is a ***GA***-function.

Proof: Let $U(d) = F(d)h(d)$ and $(m,n) = 1$. Then

$$U(mn) = F(mn)h(mn)$$

$$= [F(m)f(n) + F(n)f(m)]h(m)h(n)$$

(where f is a multiplicative function)

$$= F(m)h(m)f(n)h(n) + F(n)h(n)f(m)h(m)$$

$$= U(m)v(n) + U(n)v(m)$$

(where $v = fh$ is a multiplicative function).

Therefore, U is a ***GA***-function. Now

$$H(mn) = \sum_{d \in A_{mn}} U(d) = \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} U(d_1 d_2)$$

As $d_1 \in A_m, d_2 \in A_n$ so $(d_1, d_2) = 1$, and since U is a ***GA***-function, therefore there exists a multiplicative function *f* such that $U(d_1 d_2) = U(d_1)f(d_2) + U(d_2)f(d_1)$. Therefore,

$$H(mn) = \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} [U(d_1)f(d_2) + U(d_2)f(d_1)]$$

$$= \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} U(d_1)f(d_2) + \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} U(d_2)f(d_1)$$

$$= H(m)w(n) + H(n)w(m),$$

where

$$w(m) = \sum_{d_1 \in A_m} f(d_1), \quad w(n) = \sum_{d_2 \in A_n} f(d_2)$$

are multiplicative functions.

Hence, H is a ***GA***-function. □

Theorem 2.2.2. Let F be a ***GA***-function and h_1 and h_2 be multiplicative functions and let the *A*-product be multiplicative. Then the arithmetic function *H* defined by

$$H(n) = \sum_{d \in A_n} F(d)\, h_1(d) h_2(\frac{n}{d})$$

for all $n \in \mathbb{N}$, is a ***GA***-function.

Proof: Let *U*(*d*) = *F*(*d*)$h_1(d)$. Then from the proof of Theorem 2.2.1, *U* is a ***GA***-function. Now

$$H(n) = \sum_{d \in A_n} U(d)\, h_2\left(\frac{n}{d}\right)$$

Let $(m, n) = 1$. Then

$$H(mn) = \sum_{d \in A_{mn}} U(d)\, h_2(\frac{mn}{d})$$

$$= \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} U(d_1 d_2)\, h_2\left(\frac{mn}{d_1 d_2}\right)$$

As $d_1 \in A_m, d_2 \in A_n$ so $(d_1, d_2) = 1$, and since U is a ***GA***-function, therefore there exists a multiplicative function *f* such that

$$U(d_1 d_2) = U(d_1)f(d_2) + U(d_2)f(d_1).$$

Therefore,

$$H(mn) = \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} [U(d_1)f(d_2) + U(d_2)f(d_1)] \times [h_2(\frac{m}{d_1})\ h_2(\frac{n}{d_2})]$$

$$= \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} [U(d_1)f(d_2)h_2(\frac{m}{d_1})\ h_2(\frac{n}{d_2}) + U(d_2)f(d_1)\, h_2(\frac{m}{d_1})\ h_2(\frac{n}{d_2})]$$

$$= \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} U(d_1)h_2(\frac{m}{d_1})f(d_2)h_2(\frac{n}{d_2}) + \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} U(d_2)h_2(\frac{n}{d_2})f(d_1)h_2(\frac{m}{d_1})$$

$$= \sum_{d_1 \in A_m} U(d_1) h_2\left(\frac{m}{d_1}\right) \sum_{d_2 \in A_n} f(d_2) h_2(\frac{n}{d_2}) + \sum_{d_2 \in A_n} U(d_2) h_2\left(\frac{n}{d_2}\right) \sum_{d_1 \in A_m} f(d_1) h_2(\frac{m}{d_1})$$

$$= [(U*_A h_2)(m)].[(f*_A h_2)(n)] + [(U*_A h_2)(n)].[(f*_A h_2)(n)]$$

$$= H(m).v\,(n) + H(n).v\,(m)$$

where $v\,(k) = (f*_A h_2)(k)$.

Now since f and h_2 are multiplicative functions and the A-product is multiplicative, therefore $v = f*_A h_2$ is a multiplicative function. Consequently H is a ***GA***-function. □

Theorem 2.2.3. Let F be a ***GA***-function and let h_1 and $h_2$be multiplicative functions and let the B-product be multiplicative, then the arithmetic function H defined by

$$H(n) = \sum_{(r,s) \in B_n} F(r)\, h_1(r) h_2(s)$$

for all $n \in \mathbb{N}$ is a ***GA***-function.

Proof: Let $U(d) = F(d) h_1(d)$. Then, from the proof of Theorem 2.2.1, U is a ***GA***-function. Therefore,

$$H(n) = \sum_{(r,s) \in B_n} U(r)\, h_2(s)$$

Let $(m, n) = 1$.Then

$$H(mn) = \sum_{(r,s) \in B_{mn}} U(r)\, h_2(s)$$

$$= \sum_{(r^{(m)},s^{(m)}) \in B_m} \sum_{(r^{(n)},s^{(n)}) \in B_n} U\left(r^{(m)} r^{(n)}\right) h_2\left(s^{(m)} s^{(n)}\right)$$

As $\left(r^{(m)}, s^{(m)}\right) \in B_m$ and $(r^{(n)}, s^{(n)}) \in B_n$, so $(r^{(m)}, r^{(n)}) = 1$ and $(s^{(m)}, s^{(n)}) = 1$ and since U is a ***GA***-function, therefore, there exists a multiplicative function f such that

$$U\left(r^{(m)} r^{(n)}\right) = U\left(r^{(m)}\right) f\left(r^{(n)}\right) + U\left(r^{(n)}\right) f\left(r^{(m)}\right)$$

Therefore

$$H(mn) = \sum_{(r^{(m)},s^{(m)}) \in B_m} \sum_{(r^{(n)},s^{(n)}) \in B_n} [U\left(r^{(m)}\right) f\left(r^{(n)}\right) + U\left(r^{(n)}\right) f\left(r^{(m)}\right)]\, [h_2(s^{(m)} s^{(n)})\,]$$

$$= \sum_{(r^{(m)},s^{(m)}) \in B_m} \sum_{(r^{(n)},s^{(n)}) \in B_n} [U\left(r^{(m)}\right) f\left(r^{(n)}\right) h_2\left(s^{(m)}\right) h_2\left(s^{(n)}\right) + U\left(r^{(n)}\right) f\left(r^{(m)}\right) h_2\left(s^{(m)}\right) h_2\left(s^{(n)}\right)]$$

$$= \sum_{(r^{(m)},s^{(m)}) \in B_m} \sum_{(r^{(n)},s^{(n)}) \in B_n} U\left(r^{(m)}\right) f\left(r^{(n)}\right) h_2\left(s^{(m)}\right) h_2(s^{(n)}))$$

$$+ \sum_{(r^{(m)},s^{(m)}) \in B_m} \sum_{(r^{(n)},s^{(n)}) \in B_n} U\left(r^{(n)}\right) f\left(r^{(m)}\right) h_2\left(s^{(m)}\right) h_2(s^{(n)}))$$

$$= \sum_{(r^{(m)},s^{(m)})\in B_m} U(r^{(m)})h_2(s^m) \sum_{(r^{(n)},s^{(n)})\in B_n} f(r^{(n)}h_2(s^{(n)}))$$

$$+ \sum_{(r^{(n)},s^{(n)})\in B_n} U(r^{(n)})h_2(s^{(n)}) \sum_{(r^{(m)},s^{(m)})\in B_m} f(r^{(m)}h_2(s^{(m)}))$$

$$= H(m)(f *_B h_2)(n) + H(n)(f *_B h_2)(m).$$

Now since f and h_2 are multiplicative functions and the B-product is multiplicative, therefore, $f *_B h_2$ is a multiplicative function. Consequently, H is a ***GA***-function. ☐

We conclude our discussion with some examples.

3 Examples

Example 1. If

$$H(n) = \prod_{d\in A_n} d^d$$

for all $n \in \mathbb{N}$, and the A-product is multiplicative, then H is a ***GM***-function.

Proof. We have

$$H(n) = \prod_{d\in A_n} d^d .$$

Let $F(d) = d^d$, then for every relatively prime positive integers m, n we have

$$F(mn) = (mn)^{mn} = m^{mn}n^{mn} = (m^m)^n(n^n)^m = F(m)^nF(n)^m = F(m)^{f(n)}F(n)^{f(m)},$$

Where $f(u) = u$ for every $u \in \mathbb{N}$, is a multiplicative function. Therefore, F is a ***GM***-function. Now

$$H(mn) = \prod_{d\in A_{mn}} F(d) = \prod_{d_1d_2\in A_{mn}} F(d_1d_2),$$

(where $d = d_1d_2$, $d_1|m$ and $d_2|n$)

$$= \prod_{d_1d_2\in A_{mn}} F(d_1)^{g(d_2)}F(d_2)^{g(d_1)}$$

(where g is a multiplicative function)

$$= \{ [\prod_{d_1\in A_m} F(d_1)]^{\sum_{d_2\in A_n} g(d_2)}\} \times \{ [\prod_{d_2\in A_n} F(d_2)]^{\sum_{d_1\in A_m} g(d_1)}\}$$

$$= H(m)^{u(n)} \times H(n)^{u(m)}$$

where $u(m) = \sum_{d_1\in A_m} g(d_1)$ and $u(n) = \sum_{d_2\in A_n} g(d_2)$

Since g is a multiplicative function and the A-product is multiplicative, therefore, u is a multiplicative function. Hence, H is a ***GM***-function.

Example 2. If

$$H(n) = \prod_{(r,s)\in B_n} r^s$$

for all $n \in \mathbb{N}$, and the B-product is multiplicative, then H is a ***GM***-function.

Proof. Let $F(r) = r$ and $h(s) = s$, then $F(mn) = mn = m^{\eta(n)} n^{\eta(m)} = F(m)^{\eta(n)} F(n)^{\eta(m)}$, where $\eta(k) = 1$ for every $k \in \mathbb{N}$ is a multiplicative function and therefore F is a ***GM***-function. Now since $h(s) = s$ is a multiplicative function, and the *B*-product is multiplicative, therefore by Theorem 2.1.3

$$H(n) = \prod_{(r,s)\in B_n} r^s = \prod_{(r,s)\in B_n} F(r)^{h(s)}$$

is a ***GM***-function.

Example 3. If

$$H(n) = \sum_{d\in A_n} d \log d$$

for all $n \in \mathbb{N}$, and the *A*-product is multiplicative, then *H* is a ***GA***-function.

Proof. Let $(m,n) = 1$, then

$$H(mn) = \sum_{d\in A_{mn}} d\log d$$

$$= \sum_{d_1\in A_m, d_2\in A_n} (d_1 d_2) \log(d_1 d_2)$$

$$= \sum_{d_1\in A_m, d_2\in A_n} (d_1 d_2) [\log d_1 + \log d_2]$$

$$= \sum_{d_1\in A_m, d_2\in A_n} (d_1 log\, d_1) d_2 + \sum_{d_1\in A_m, d_2\in A_n} (d_2\, log d_2) d_1$$

$$= \sum_{d_1\in A_m} d_1 log d_1 \sum_{d_2\in A_n} d_2 + \sum_{d_2\in A_n} d_2 log d_2 \sum_{d_1\in A_m} d_1$$

$$= H(m) \sum_{d_2\in A_n} d_2 + H(n) \sum_{d_1\in A_m} d_1$$

$$= H(m)\alpha(n) + H(n)\alpha(m)$$

where

$$\alpha(k) = \sum_{d\in A_k} d = \sum_{d\in A_k} h(d)$$

and $h(s) = s$ for every $s \in \mathbb{N}$. Since h is a multiplicative function and the *A*-product is multiplicative, therefore α is a multiplicative function. Hence H is a ***GA***-function.

Example 4. If

$$H(n) = \sum_{(r,s)\in B_n} r\log s$$

for all $n \in \mathbb{N}$, and the *B*-product is multiplicative, then *H* is a ***GA***-function.

Proof. Let $(m,n) = 1$. Then

$$H(mn) = \sum_{(r,s)\in B_{mn}} r \log s$$

$$= \sum_{(r^{(m)},s^{(m)})\in B_m,(r^{(n)},\, s^{(n)})\in B_n} r^{(m)}r^{(n)} \log\left(s^{(m)}s^{(n)}\right)$$

$$= \sum_{(r^{(m)},s^{(m)})\in B_m,(r^{(n)},\, s^{(n)})\in B_n} r^{(m)}r^{(n)} \,[\log(s^{(m)}) + \log(s^{(n)})]$$

$$= \sum_{(r^{(m)},s^{(m)})\in B_m,(r^{(n)},\, s^{(n)})\in B_n} r^{(m)}\log(s^{(m)})\, r^{(n)} + \sum_{(r^{(m)},s^{(m)})\in B_m,(r^{(n)},\, s^{(n)})\in B_n} r^{(n)} \log(s^{(n)}) r^{(m)}$$

$$= \sum_{(r^{(m)},s^{(m)})\in B_m} r^{(m)} \log\left(s^{(m)}\right) \sum_{(r^{(n)},\, s^{(n)})\in B_n} r^{(n)} + \sum_{(r^{(n)},\, s^{(n)})\in B_n} r^{(n)} \log(s^{(n)}) \sum_{(r^{(m)},s^{(m)})\in B_m} r^{(m)}$$

$$= H(m)\alpha(n) + H(n)\alpha(m)$$

where

$$\alpha(k) = \sum_{(r^{(k)},\, s^{(k)})\in B_k} r^{(k)} = \sum_{(r^{(k)},\, s^{(k)})\in B_k} h(r^{(k)}),$$

and $h(s) = s$ for every $s \in \mathbb{N}$.

Since h is a multiplicative function and the *B*-product is multiplicative, therefore α is a multiplicative function. Hence H is a ***GA***-function.

References

[1] Apostol, T. M. (1976). *Introduction to Analytic Number Theory*, Springer-Verlag, New York.

[2] Bell, E. T. (1915). *Arithmetical Theory of Certain Numerical Functions. Volume 1*, University of Washington.

[3] Bhattacharjee, D. (1997). *B*-product and its properties. *Bulletin of Pure and Applied Sciences*, 16(1), 77–83.

[4] Bhattacharjee, D. (1998). Multiplicative *B*-product and its properties. *Georgian Mathematical Journal*, 5(4), 315–320.

[5] Bhattacharjee, D. (2002). A Generalized B-product and its properties, *Bulletin of the Allahabad Mathematical Society*, 17, 17–21.

[6] Bhattacharjee, D. (2005). Multiplicative K_B-product and its properties.*Bulletin of the Calcutta Mathematical Society*, 97(2), 153–162.

[7] Bhattacharjee, D. (2007). On Some New Arithmetical Convolutions. *Proceedings of the 2007 International Conference on High Performance Computing Networking and Communication Systems (HPCNCS-07)*, 9–12 July 2007, Orlando, FL, USA, 26–30.

[8] Bhattacharjee, D., & Saikia, P. K. (2012). On Some New Class of Arithmetic Convolutions Involving Arbitrary Sets of Integers. *Romanian Journal of Mathematics and Computer Science*, 2(2), 23–35.

[9] Chawla, L. M. (1973). Distributive arithmetical functions of two or more variables. *Journal of Natural Sciences Math.*, 13, 263–270.

[10] Pellegrino, F. (1953) Sviluppi moderni del Calcolo numerico integrale di Michele Cipolla. *IV Congresso dell' Unione Matematica Italiana-Taormina*, 2, 161–168.

[11] Cohen, E. (1959).Arithmetical functions associated with arbitrary sets of integers, *Acta Arithmetica*, 5, 407–415.

[12] Davison, T. M. K. (1966). On Arithmetic Convolutions. *Canadian Mathematical Bulletin*, 9, 287–296.

[13] Dickson, L. E.(1919).*History of the Theory of Numbers*, Washington: Carnegie Institution.

[14] Gioia, A. A. (1965).The *K*-product of the arithmetic functions, *Canadian Journal of Mathematics*, 17, 970–976.

[15] Haukkanen, P. (1992). A note on generalized multiplicative and generalized additive arithmetic functions. *The Mathematics Student*, 61(1–4), 113–116.

[16] Haukkanen, P. (1996). On a Binomial Convolution of Arithmetic Functions. *Nieuw Archief voor Wiskunde*, 14(2), 209–216.

[17] Lehmer, D. H. (1931). A New Calculus of Numerical Functions. *American Journal of Mathematics*, 53, 843–854.

[18] McCarthy, P. J. (1986). *Introduction to Arithmetical Functions*, Springer-Verlag, New York.

[19] Narkiewicz, W. (1963). On a class of arithmetical convolutions. *Colloquium Mathematicum*, 10, 81–94.

[20] Scheid, H. (1969). Einige Ringe zahlentheoretischer Funktionen. *Journal für die reine und angewandte Mathematik*, 1969(237), 1–11.

[21] Sivaramakrishnan, R. (1989). *Classical Theory of Arithmetical Functions,* Monographs and Textbooks in Pure and Applied Mathematics, 126, Marcel-Dekker Inc., New York.

[22] Subbarao, M. V. (1972). On Some Arithmetical Convolutions, *Lecture Notes in Mathematics*, Springer, 251, 247–271.

[23] Tóth, L. (2002). On a Class of Arithmetic Convolutions involving Arbitrary Sets of Integers. *Mathematica Pannonica*, 13, 249–263.

[24] Tóth, L. (2004). On a Certain Arithmetic Functions involving Exponential Divisors. *Annales Univ. Sci. Budapest., Sect. Comp*., 24, 285–294.

[25] Vaidyanathaswamy, R. (1931). The Theory of Multiplicative Arithmetical Functions. *Transactions of the American Mathematical Society*, 33, 579–662.

[26] Zafrullah, M. (1988). On generalized multiplicative functions. *Journal of Natural Sciences and Mathematics*, 28, 257–268.