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Odd/even cube-full numbers

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Abstract: In this paper we use an elementary method to give an asymptotical ratio of odd to even cube-full numbers and show that it is asymptotically $1 : 1 + 2^{-1/3} + 2^{-2/3}$.
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1 Introduction and result

Let $k > 1$ be a fixed integer. A positive integer n is said to be k-full if each of its prime factors appears to the power at least k. For $k = 2, 3$, these numbers are called square-full and cube-full respectively. Let $N_k(x)$ be the number of k-full integers $\leq x$. In 1935, Erdős and Szekeres [1] proved that for k fixed

$$N_k(x) = x^{1/k} \prod_p \Big(1 + \sum_{m=k+1}^{2k-1} p^{-m/k}\Big) + O(x^{1/(k+1)}). \tag{1}$$

For a study of these asymptotic formulae, we refer to [2, Chapter 14.4].

In this paper, we study the odd/even dichotomy for the set of cube-full numbers. The motivation follows from work by Scott [4] and Jameson [3], where it was shown that the ratio of odd to even square-free numbers is asymptotically $2 : 1$. (A positive integer n is called square-free if it is not divisible by the square of any prime). Very recently, Srichan [5] used an elementary method to prove that the ratio of odd to even square-full numbers is asymptotically $1 : 1+\frac{\sqrt{2}}{2}$. Then, it would be interesting to consider the odd/even dichotomy for the set of cube-full numbers.

Let G be the set of all cube-full numbers. Let $G(x)$, $G_{odd}(x)$ and $G_{even}(x)$ be the set of all cube-full numbers, odd cube-full numbers and even cube-full numbers in the interval $[1, x]$, respectively. We denote by $N(x)$, $N_{odd}(x)$ and $N_{even}(x)$ the number of members of $G(x)$, $G_{odd}(x)$ and $G_{even}(x)$, respectively. We prove the following theorem.

Theorem 1.1. *As $x \to \infty$, we have*

$$\frac{N_{odd}(x)}{N_{even}(x)} \sim 2 - 2^{2/3}. \tag{2}$$

2 Proof of Theorem 1.1

First, we assume that

$$N_{odd}(x) \sim ax^{1/3} \qquad \text{and} \qquad N_{even}(x) \sim bx^{1/3}, \qquad \text{for some } a, b \in \mathbb{R}^+. \tag{3}$$

We wish to show that,

$$\frac{a}{b} = 2 - 2^{2/3}. \tag{4}$$

For an even cube-full number n, we have $2 \mid n$, then also $8 \mid n$. Thus, there are no cube-full numbers n such that $n \equiv 2,\ 4,\ 6 \pmod 8$. Then we write $G_{even}(x) = \{n \le x,\ n \in G \text{ and } 8 \mid n\}$ and $G_{odd}(x) = \{n \le x,\ n \in G \text{ and } n \equiv 1, 3, 5, 7 \,(\text{mod } 8)\}$. Next, we spilt $G_{even}(x)$ into the set $G_{even1}(x)$ and the set $G_{even2}(x)$, where $G_{even1}(x) = \{n \le x,\ n \in G_{even}(x) \text{ and } \frac{n}{8} \in G\}$ and $G_{even2}(x) = \{n \le x,\ n \in G_{even}(x) \text{ and } \frac{n}{8} \notin G\ \}$. Let $N_{even1}(x)$ and $N_{even2}(x)$ be the number of members of $G_{even1}(x)$ and $G_{even2}(x)$, respectively. It is easy to prove that

$$N_{even1}(x) = N(x/8). \tag{5}$$

Now we will show that

$$N_{even2}(x) = N_{odd}(x/16) + N_{odd}(x/32). \tag{6}$$

A positive integer $n \in G_{even2}(x)$ has the form as $2^r m$, with m being an odd cube-full number and $r = 4, 5$. Thus, we write

$$G_{even2}(x) = G_{even21}(x) \cup G_{even22}(x),$$

where

$$G_{even21}(x) = \{n \le x,\ n \in G_{even2}(x) \text{ and } n = 16m \text{ with } m \text{ being odd cube-full } \},$$

and

$$G_{even22}(x) = \{n \le x,\ n \in G_{even2}(x) \text{ and } n = 32m \text{ with } m \text{ being odd cube-full } \}.$$

Formula (6) follows at once.

In view of (5) and (6), we have

$$N_{even}(x) = N(x/8) + N_{odd}(x/16) + N_{odd}(x/32). \tag{7}$$

Then,

$$N_{even}(x) = (N_{even}(x/8) + N_{odd}(x/8)) + N_{odd}(x/16) + N_{odd}(x/32).$$

In view of (3), we have

$$bx^{1/3} = \frac{b}{2}x^{1/3} + \frac{a}{2}x^{1/3} + \frac{a}{2^{4/3}}x^{1/3} + \frac{a}{2^{5/3}}x^{1/3}.$$

This proves (4).

Now it remains to prove the existence of a and b.

In view of (7), we write

$$\begin{aligned} N(x) - N_{odd}(x) &= N(x/8) + N_{odd}(x/16) + N_{odd}(x/32) \\ N(x) - N(x/8) &= N_{odd}(x) + N_{odd}(x/16) + N_{odd}(x/32). \end{aligned}$$

We write $f(x) = N(x) - N(x/8)$, then we have

$$f(x) = N_{odd}(x) + N_{odd}(x/16) + N_{odd}(x/32). \tag{8}$$

In view of (1), we have

$$f(x) \sim cx^{1/3}, \tag{9}$$

for a certain $c > 0$. By the mathematical induction on $m \geq 0$ and (8), we have

$$N_{odd}(x) = \sum_{j=0}^{m}(-1)^j \sum_{i=0}^{j} \binom{j}{i} f\Big(\frac{x}{2^{4j+i}}\Big) - (-1)^m \sum_{i=0}^{m+1} \binom{m+1}{i} N_{odd}\Big(\frac{x}{2^{4m+4+i}}\Big).$$

For $m > \log_2 x^{1/4} - 1$, we have

$$\begin{aligned} N_{odd}(x) &= \sum_{j=0}^{\infty}(-1)^j \sum_{i=0}^{j} \binom{j}{i} f\Big(\frac{x}{2^{4j+i}}\Big) \\ &= \sum_{j=0}^{\infty} \sum_{i=0}^{2j} \binom{2j}{i} f\Big(\frac{x}{2^{8j+i}}\Big) - \sum_{j=0}^{\infty} \sum_{i=0}^{2j+1} \binom{2j+1}{i} f\Big(\frac{x}{2^{8j+i+4}}\Big). \end{aligned}$$

In view of (9), we know that, for $\epsilon > 0$, and for some x_0,

$$(c-\epsilon)x^{1/3} \leq f(x) \leq (c+\epsilon)x^{1/3}, \quad \text{for } x > x_0.$$

We note that the inequality $f(y) \leq (c+\epsilon)y^{1/3}$ only applies to the terms $y = x/2^{4j+i}$ if $x/2^{5j} \geq x_0$. There exists a positive M such that $f(y) \leq My^{1/3}$ for all $y \geq 1$. Suppose that k and x are such that $x \geq 2^{5k}x_0$. For $j > k$, we have

$$\sum_{i=0}^{j} \binom{j}{i} f\Big(\frac{x}{2^{4j+i}}\Big) \leq Mx^{1/3}2^{-4j/3} \sum_{i=0}^{j} \binom{j}{i} 2^{-i/3} = M\alpha^j x^{1/3},$$

with $\alpha = 16^{-1/3} + 32^{-1/3}$. Now we choose $k \geq \log_\alpha \frac{\epsilon(1-\alpha)}{M} - 1$, we have

$$M \sum_{j>k} \alpha^j \leq \epsilon. \tag{10}$$

Then, for $k \geq \log_\alpha \frac{\epsilon(1-\alpha)}{M} - 1$, and $x \geq 2^{5k}x_0$, we get

$$\begin{aligned}
N_{odd}(x) \geq\ & (c-\epsilon)\sum_{j=0}^{\infty}\sum_{i=0}^{2j}\binom{2j}{i}\frac{x^{1/3}}{2^{(8j+i)/3}} - (c+\epsilon)\sum_{j=0}^{k}\sum_{i=0}^{2j+1}\binom{2j+1}{i}\frac{x^{1/3}}{2^{(8j+i+4)/3}} \\
& - M\sum_{j>k}\sum_{i=0}^{2j+1}\binom{2j+1}{i}\frac{x^{1/3}}{2^{(8j+i+4)/3}} \\
=\ & (c-\epsilon)x^{1/3}\sum_{j=0}^{\infty}2^{-8j/3}\sum_{i=0}^{2j}\binom{2j}{i}2^{-i/3} - (c+\epsilon)x^{1/3}\sum_{j=0}^{k}2^{-(8j+4)/3}\sum_{i=0}^{2j+1}\binom{2j+1}{i}2^{-i/3} \\
& - Mx^{1/3}\sum_{j>k}2^{-(8j+4)/3}\sum_{i=0}^{2j+1}\binom{2j+1}{i}2^{-i/3} \\
=\ & (c-\epsilon)x^{1/3}\sum_{j=0}^{\infty}2^{-8j/3}\Big(2^{-1/3}+1\Big)^{2j} - (c+\epsilon)x^{1/3}\sum_{j=0}^{k}2^{-(8j+4)/3}\Big(2^{-1/3}+1\Big)^{2j+1} \\
& - Mx^{1/3}\sum_{j>k}2^{-(8j+4)/3}\Big(2^{-1/3}+1\Big)^{2j+1} \\
\geq\ & (c-\epsilon)x^{1/3}\sum_{j=0}^{\infty}\Big(2^{-5/3}+2^{-4/3}\Big)^{2j} - (c+\epsilon)x^{1/3}\sum_{j=0}^{\infty}\Big(2^{-5/3}+2^{-4/3}\Big)^{2j+1} \\
& - Mx^{1/3}\sum_{j>k}\Big(2^{-5/3}+2^{-4/3}\Big)^{2j+1} \\
\geq\ & (c-\epsilon)x^{1/3}\sum_{j=0}^{\infty}\alpha^{2j} - (c+\epsilon)x^{1/3}\sum_{j=0}^{\infty}\alpha^{2j+1} - Mx^{1/3}\sum_{j>k}\alpha^{j}. \qquad (11)
\end{aligned}$$

In view of (10), and (11) we have

$$\begin{aligned}
N_{odd}(x) &\geq (c-\epsilon)x^{1/3}\sum_{j=0}^{\infty}\alpha^{2j} - (c+\epsilon)x^{1/3}\sum_{j=0}^{\infty}\alpha^{2j+1} - \epsilon x^{1/3} \\
&= \Big(\frac{c}{1+\alpha} - \frac{\epsilon}{1-\alpha} - \epsilon\Big)x^{1/3}. \qquad (12)
\end{aligned}$$

Similary, we have

$$N_{odd}(x) \leq \Big(\frac{c}{1+\alpha} + \frac{\epsilon}{1-\alpha} + \epsilon\Big)x^{1/3}. \qquad (13)$$

In view of (12) and (13), we have

$$\Big(\frac{c}{1+\alpha} - \frac{\epsilon}{1-\alpha} - \epsilon\Big)x^{1/3} \leq N_{odd}(x) \leq \Big(\frac{c}{1+\alpha} + \frac{\epsilon}{1-\alpha} + \epsilon\Big)x^{1/3}. \qquad (14)$$

The existence of a follows from (14) and by the similar proof the existence of b is obtained.

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