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The Gauss product and Raabe's integral for k-gamma functions

József Sándor

Department of Mathematics, Babes-Bolyai University
Cluj-Napoca, Romania
e-mail: jsandor@math.ubbcluj.ro

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Abstract: We obtain an extension of the famous Gauss product formula to the case of k-gamma functions. The Sándor–Tóth short product formula [16] is also attended to these functions. An asymptotic formula and Raabe's integral analogue are also considered.
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1 Introduction

As a generalization of the classical Euler gamma function $\Gamma(x)$, in 2007 R. Diaz and E. Pariguan [6] have introduced and studied the notion of k-gamma function.

For $k > 0$, the Γ_k-function is defined by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n! h^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \tag{1}$$

for $x \in \mathbb{C} \setminus k\mathbb{Z}^-$, where $\mathbb{C}$ is the set of complex numbers, $\mathbb{Z}^-$ is the set of negative integers, $(x)_{n,k}$ denotes the classical Pochhammer symbol $(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k)$.

For $x \in \mathbb{C}$, with $\mathrm{Re}(x) > 0$, it can be proved the integral representation [6]

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\, dt. \tag{2}$$

Also, it satisfies the following properties [6]:

$$\begin{aligned} &(i) \quad \Gamma_k(x+k) = x\Gamma_k(x), \\ &(ii) \quad \frac{\Gamma_k(x+nk)}{\Gamma_k(x)} = (x)_{n,k}, \\ &(iii)\ \Gamma_k(k) = 1, \\ &(iv)\ \frac{1}{\Gamma_k(x)} = x.k^{-\frac{x}{k}}.e^{\frac{x}{k}\cdot\gamma} \prod_{n=1}^{\infty} \left(\left(1+\frac{x}{nk}\right) e^{-\frac{x}{nk}} \right). \end{aligned} \tag{3}$$

It is obvious also that $\Gamma_1(x) \equiv \Gamma(x)$.

One of the motivations of introduction of the $\Gamma_k(x)$-function is in its connection with the symbol $(x)_{n,k}$ which appears in a variety of contexts (see [5] and the references). In the recent years, there is an increasing interest about the k-gamma function (see, e.g., [5, 6, 8–11]).

The famous short product formula of Gauss for the Euler gamma function states that one has the identity

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}. \tag{4}$$

In 1989, J. Sándor and L. Tóth [16] studied the short product

$$\prod_{l=1,(l,n)=1}^{n} \Gamma\left(\frac{l}{n}\right) = \frac{(2\pi)^{\frac{\varphi(n)}{2}}}{e^{\frac{\Lambda(n)}{2}}}, \tag{5}$$

where $\varphi(n)$ is the Euler totient function, and $\Lambda(n)$ is the von Mangoldt function. This paper has evoked large interest, see, e.g. [1–4, 12–15]. Particularly, the recent paper by M. E. Bachraoui and J. Sándor [2] offers an extension of (5) to the Γ_q-function, which is a classical extension of gamma function, due to F. H. Jackson (see the references from [2]).

The aim of this paper is to extend (4) and (5) to the case of k-gamma functions.

2 Main results

The main results are contained in the following.

Theorem 2.1. *One has the identity*

$$\prod_{l=1}^{n-1} \Gamma_k\left(\frac{kl}{n}\right) = \left(\frac{2\pi}{k}\right)^{\frac{n-1}{2}} . \frac{1}{\sqrt{n}}. \tag{6}$$

Theorem 2.2. *One has the identity*

$$P_k(n) = \prod_{l=1,(l,n)=1}^{n} \Gamma_k\left(\frac{kl}{n}\right) = \frac{\left(\frac{2\pi}{k}\right)^{\frac{\varphi(n)}{2}}}{\exp\left(\frac{\Lambda(n)}{2}\right)} = \begin{cases} \frac{\left(\frac{2\pi}{k}\right)^{\frac{\varphi(n)}{2}}}{\sqrt{p}}, & \text{for } n = p^m \\ \left(\frac{2\pi}{k}\right)^{\frac{\varphi(n)}{2}} & \text{for } n \neq p^m \end{cases}, \tag{7}$$

where p is an arbitraty prime, and m is an arbitrary positive integer.

Theorem 2.3. *One has the following Raabe type integral formula*

$$\int_0^1 \log \Gamma_k(kx)dx = \log \sqrt{\frac{2\pi}{k}}. \tag{8}$$

Theorem 2.4. *One has the following asymptotic formula*

$$\sum_{n \le x} \log P_k(n) = \frac{3\log\left(\frac{2\pi}{k}\right)}{2\pi^2}x^2 + O(x \log x), \tag{9}$$

where $P_k(n)$ is defined in Theorem 2.2.

First, one needs the following auxiliary result.

Lemma 2.1. *The following extension of the Euler reflexion formula holds true:*

$$\Gamma_k(x)\Gamma_k(k-x) = \frac{\pi}{k \sin\left(\frac{\pi x}{k}\right)}. \tag{10}$$

Proof. By using the fundamental identity (i) of (3) one can write that $\Gamma_k(k-x) = -x\Gamma_k(-x)$. By the Weierstrass type relation (iv) of (3) one gets

$$\frac{1}{\Gamma_k(x)\Gamma_k(k-x)} = x\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2k^2}\right)$$

(where we have omitted some obvious computations). Now, by the classical Euler formula

$$\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right) \tag{11}$$

with the application for $x := \frac{x}{k}$, identity (10) follows. □

The following auxiliary result was stated first by A. Hurwitz ([7, 16]).

Lemma 2.2 *Let $s : [0,1] \to \mathbb{C}$ be an arbitrary function, and put*

$$f(n) = \sum_{k \in A(n)} s\left(\frac{k}{n}\right), \qquad g(n) = \sum_{k=1}^{n} s\left(\frac{k}{n}\right),$$

where $A(n) = \{l : 1 \le l \le n, (l,n) = 1\}$. Then one has

$$f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right), \tag{12}$$

where μ is the classical Möbius function.

Corollary 2.1 *If $F(n) = \prod_{k \in A(n)} s\left(\frac{k}{n}\right)$ and $G(n) = \prod_{k=1}^{n} s\left(\frac{k}{n}\right)$, then*

$$F(n) = \prod_{d|n}\left(G\left(\frac{n}{d}\right)\right)^{\mu(d)}. \tag{13}$$

Proof. This follows by letting $f = \ln F$ and $g = \ln G$ in Lemma 2.2. □

3 Proofs of the theorems

Proof of Theorem 2.1. Letting $x = \frac{kl}{n}$ in identity (10), we get

$$\Gamma_k\left(\frac{kl}{n}\right)\Gamma_k\left(k\left(1-\frac{l}{n}\right)\right) = \frac{\pi}{k}.\frac{1}{\sin\frac{\pi l}{n}}. \tag{14}$$

By remarking that, when $l = 1, 2, \ldots, n-1$ one has

$$\prod_{l=1}^{n-1}\Gamma_k\left(k\left(1-\frac{l}{n}\right)\right) = \prod_{l=1}^{n-1}\Gamma_k\left(k.\frac{l}{n}\right),$$

as $1-\frac{l}{n} = \frac{n-l}{n}$, and applying identity (14) to $l = 1, 2, \ldots, n-1$, by term-by-term multiplication of the of the obtained relation, we get

$$\left(\prod_{l=1}^{n-1}\Gamma_k\left(\frac{kl}{n}\right)\right)^2 = \left(\frac{\pi}{k}\right)^{n-1}\frac{1}{\prod\limits_{l=1}^{n-1}\sin\frac{\pi l}{n}} = \left(\frac{\pi}{k}\right)^{n-1}\frac{2^{n-1}}{n}$$

by the well-known trigonometric identity $\prod\limits_{l=1}^{n-1}\sin\frac{\pi l}{n} = \frac{2^{n-1}}{n}$.
Now, relation (6) follows at once from the above. □

Proof of Theorem 2.2. By Theorem 2.1 and Corollary 2.1, the left-hand side of (7) can be written as

$$\frac{\left(\frac{2\pi}{k}\right)^{\frac{1}{2}\sum\limits_{d|n}d\mu\left(\frac{n}{d}\right)-\frac{1}{2}\sum\limits_{d|n}\mu\left(\frac{n}{d}\right)}}{\sqrt{h(n)}},$$

where $h(n) = \prod\limits_{d|n} d^{\mu\left(\frac{n}{d}\right)}$.

Now, it is well-known that (see, e.g., [7]) $\sum\limits_{d|n} d\mu\left(\frac{n}{d}\right) = \sum\limits_{d|n}\frac{\mu(d)}{d} = \frac{\varphi(n)}{n}$ and $\sum\limits_{d|n}\mu\left(\frac{n}{d}\right) = \sum\limits_{d|n}\mu(d) = 0$.

Also, $\log h(n) = \Lambda(n) = \log p$ if $n = p^m$; and it is equal to 0, if $n \neq p^m$. Thus, identity (7) follows. □

Proof of Theorem 2.3. We will use the classical Riemann sum approach, based on the limit formula

$$\int_0^1 f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right). \tag{15}$$

Let $f(x) = \log\Gamma_k(kx)$. By Theorem 2.1, and relation (15) one has

$$\int_0^1 \log\Gamma_k(kx)dx = \lim_{n\to\infty}\left(\frac{n-1}{2n}\log\left(\frac{2\pi}{k}\right) - \frac{1}{2n}\log n\right) = \frac{1}{2}\log\left(\frac{2\pi}{k}\right).$$

Thus gives relation (8). □

Proof of Theorem 2.4. By Theorem 2.2 one can write

$$\begin{aligned}\sum_{n\le x}\log P_k(n) &= \sum_{n\le x}\left(\frac{\varphi(n)}{2}\log\left(\frac{2\pi}{k}\right)-\frac{1}{2}\Lambda(n)\right)\\ &= \frac{1}{2}\log\frac{2\pi}{k}\sum_{n\le x}\varphi(n)-\frac{1}{2}\sum_{n\le x}\Lambda(n)\\ &= \frac{1}{2}\log\frac{2\pi}{k}.\left(\frac{3}{\pi^2}x^2+O(x\log x)\right)-\frac{1}{2}O(x)\\ &= \frac{3\log\frac{2\pi}{k}}{2\pi^2}x^2+O(x\log x),\end{aligned}$$

where we have used the classical asymptotic relations (see, e.g., [7]):

$$\sum_{n\le x}\varphi(n)=\frac{3}{\pi^2}x^2+O(x\log x),$$

and

$$\sum_{n\le x}\Lambda(n)=O(x).$$

This completes the proof. □

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