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The translated Whitney–Lah numbers: generalizations and q-analogues

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Abstract: In this paper, we derive some combinatorial formulas for the translated Whitney–Lah numbers which are found to be generalizations of already-existing identities of the classical Lah numbers, including the well-known Qi's formula. Moreover, we obtain q-analogues of the said formulas and identities by establishing similar properties for the translated q-Whitney numbers.
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1 Introduction

The (unsigned) Lah numbers, denoted by $L(n, k)$, count the number of partitions of a set X with n elements into k nonempty linearly ordered subsets. These numbers are known to satisfy the following basic combinatorial properties:

- explicit formula
$$L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}; \tag{1}$$
- recurrence relation
$$L(n+1,k) = L(n,k-1) + (n+k)L(n,k); \tag{2}$$
- exponential generating function
$$\sum_{n=0}^{\infty} L(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(\frac{t}{1-t}\right)^k. \tag{3}$$

The numbers $L(n,k)$ are often defined as coefficients of rising factorials in terms of falling factorials. That is

$$\langle t\rangle_n = \sum_{k=0}^{n} L(n,k)(t)_k, \tag{4}$$

where

$$\langle t\rangle_n = t(t+1)(t+2)\cdots(t+n-1)$$

is the rising factorial of t of order n and

$$(t)_k = t(t-1)(t-2)\cdots(t-k+1)$$

is the falling factorial of t of order k with $\langle t\rangle_0 = (t)_0 = 1$ and $(-t)_n = (-1)^n \langle t\rangle_n$. The Lah numbers are actually closely-related with the well-known Stirling numbers. To illustrate this, we first recall that the Stirling numbers of the first and second kinds, denoted by $\left[{n \atop j}\right]$ and $\left\{{n \atop j}\right\}$, respectively, are defined as coefficients in the expansions of the relations

$$(t)_n = \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} t^j \tag{5}$$

and

$$t^n = \sum_{j=0}^{n} \begin{Bmatrix} n \\ j \end{Bmatrix} (t)_j. \tag{6}$$

Notice that putting $-t$ in place of t in (5) yields

$$\langle t\rangle_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} t^j. \tag{7}$$

By substituting (6) in the right-hand side of (7), we get

$$\begin{aligned} \langle t\rangle_n &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \sum_{k=0}^{j} \begin{Bmatrix} j \\ k \end{Bmatrix} (t)_k \\ &= \sum_{k=0}^{n} \left(\sum_{j=k}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \begin{Bmatrix} j \\ k \end{Bmatrix} \right) (t)_k. \end{aligned}$$

By combining this with (4) and comparing the coefficients of $(t)_k$, we are able to write

$$L(n,k) = \sum_{j=k}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \begin{Bmatrix} j \\ k \end{Bmatrix}. \tag{8}$$

It is important to note that here, the numbers $\left[{n \atop j}\right]$ particularly refer to the "unsigned" Stirling numbers of the first kind which count the number of permutations of the n-element set X into j disjoint cycles. Similarly, the Stirling numbers of the second kind $\left\{{n \atop j}\right\}$ can be combinatorially interpreted as the number of partitions of X into j nonempty blocks. With this, the Bell numbers B_n are defined as the total number of partitions of the n-element set X. That is,

$$B_n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}. \tag{9}$$

The paper of Petkovšek and Pisanski [20], and the books of Comtet [4] and Chen and Kho [2] contain detailed discussions on the Lah, Stirling and Bell numbers, including their respective combinatorial properties and interpretations. In addition to these, Qi [21] recently obtained an explicit formula for the Bell numbers expressed in terms of both the Lah numbers and the Stirling numbers of the second kind, viz.

$$B_n = \sum_{k=1}^{n} (-1)^{n-k} \left(\sum_{\ell=1}^{k} L(k, \ell) \right) \left\{ {n \atop k} \right\}. \tag{10}$$

The results of this paper are organized as follows. In Section 2, we present the translated Whitney numbers and derive some formulas which generalize already-existing identities for the classical Lah numbers, including one that will generalize (10). In Section 3, we establish the q-analogues of some of the results in Section 2 using as framework the translated q-Whitney numbers.

2 Translated Whitney numbers

In 2013, Belbachir and Bousbaa [1] introduced the translated Whitney numbers using a combinatorial approach which involves "mutations" of some elements of a given finite set. To be more precise, the translated Whitney numbers of first kind, denoted by $\widetilde{w}_{(\alpha)}(n, k)$, were defined as the number of permutations of n elements with k cycles such that the elements of each cycle can mutate in α ways, except the dominant one while the translated Whitney numbers of the second kind, denoted by $\widetilde{W}_{(\alpha)}(n, k)$, were defined as the number of partitions of the an n-element set into k subsets such that the elements of each subset can mutate in α ways, except the dominant one. These numbers were shown to satisfy the recurrence relations [1, Theorems 2 and 8]

$$\widetilde{w}_{(\alpha)}(n, k) = \widetilde{w}_{(\alpha)}(n - 1, k - 1) + \alpha(n - 1)\widetilde{w}_{(\alpha)}(n - 1, k) \tag{11}$$

and

$$\widetilde{W}_{(\alpha)}(n, k) = \widetilde{W}_{(\alpha)}(n - 1, k - 1) + \alpha k \widetilde{W}_{(\alpha)}(n - 1, k), \tag{12}$$

and the horizontal generating functions [1, Theorems 4 and 10]

$$(t| - \alpha)_n = \sum_{k=0}^{n} \widetilde{w}_{(\alpha)}(n, k) x^k \tag{13}$$

and

$$x^n = \sum_{k=0}^{n} \widetilde{W}_{(\alpha)}(n, k)(t|\alpha)_k, \tag{14}$$

where $(t|\alpha)_n$ denotes the generalized factorial of t of increment α given by

$$(t|\alpha)_n = \prod_{i=0}^{n-1} (t - i\alpha), \ (t|\alpha)_0 = 1.$$

In the same paper, Belbachir and Bousbaa [1] also defined translated Whitney–Lah numbers, denoted by $\widehat{w}_{(\alpha)}(n, k)$, as the number of ways to distribute the set $\{1, 2, \ldots, n\}$ into k ordered lists such that the elements of each list can mutate with α ways, except the dominant one. The values of the numbers $\widehat{w}_{(\alpha)}(n, k)$ can be computed using the recurrence relation [1, Theorem 13]

$$\widehat{w}_{(\alpha)}(n,k) = \widehat{w}_{(\alpha)}(n-1,k-1) + \alpha(n+k-1)\widehat{w}_{(\alpha)}(n-1,k) \tag{15}$$

and can be generated using [1, Corollary 15]

$$(t|-\alpha)_n = \sum_{k=0}^{n} \widehat{w}_{(\alpha)}(n,k)(t|\alpha)_k. \tag{16}$$

Similar to what is observed in equation (8), the translated Whitney–Lah numbers may also be expressed as sum of products of $\widetilde{w}_{(\alpha)}(n,k)$ and $\widetilde{W}_{(\alpha)}(n,k)$ as follows [1, Corollary 14]

$$\widehat{w}_{(\alpha)}(n,k) = \sum_{j=k}^{n} \widetilde{w}_{(\alpha)}(n,j)\widetilde{W}_{(\alpha)}(j,k). \tag{17}$$

It is evident that the translated Whitney and Whitney–Lah numbers are generalizations of the Stirling and Lah numbers, respectively. This may be verified by simply setting $\alpha = 1$ in the defining relations of the former.

Recently, Mansour et al. [16] defined the recurrence relation

$$u(n,k) = u(n-1,k-1) + (a_{n-1} + b_k)u(n-1,k) \tag{18}$$

for two sequences $(a_i)_{i\geq 0}$ and $(b_i)_{i\geq 0}$ with boundary conditions given by

$$u(n,0) = \prod_{i=0}^{n-1}(a_i + b_0),\ u(0,k) = \delta_{0,k},$$

where

$$\delta_{i,j} = \begin{cases} 0,\ if\ i \neq j \\ 1,\ if\ i = j \end{cases}$$

is the Kronecker delta. Notice that if $a_{n-1} = \alpha(n-1)$ and $b_k = \alpha k$, the above recurrence relation coincides with equation (15). Moreover, the following useful formula was first established in the same paper:

$$u(n,k) = \sum_{j=0}^{k} \left(\frac{\prod_{i=0}^{n-1}(b_j + a_i)}{\prod_{i=0,i\neq j}^{n-1}(b_j - b_i)} \right). \tag{19}$$

In a later paper, Mansour et al. [17] used the identity in (19) to derive an explicit formula for a certain generalization of the translated Whitney numbers (see [17, Equation 19]). We also note of another related paper by Mansour and Shattuck [19] which provide additional insights on Lah numbers.

Now, for $a_i = \alpha i$ and $b_j = \alpha j$, we utilize equation (19) to obtain an explicit formula for $\widehat{w}_{(\alpha)}(n,k)$ given in the next theorem.

Theorem 2.1. *The translated Whitney–Lah numbers satisfy the following explicit formula:*

$$\widehat{w}_{(\alpha)}(n,k) = \frac{\alpha^{n-k}}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \langle j \rangle_n . \tag{20}$$

This theorem allows us to write the numbers $\widehat{w}_{(\alpha)}(n,k)$ in a closed form similar to (1). It is implied in the proof of the succeeding corollary.

Corollary 2.1.1. *The translated Whitney–Lah numbers satisfy the following relation:*

$$\widehat{w}_{(\alpha)}(n,k) = \alpha^{n-k} L(n,k). \tag{21}$$

Proof. Since $\langle j \rangle_n = (j+n-1)_n$, then

$$\begin{aligned}
\widehat{w}_{(\alpha)}(n,k) &= \frac{\alpha^{n-k}}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (j+n-1)_n \\
&= \alpha^{n-k} \frac{n!}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{j+n-1}{n}.
\end{aligned}$$

From [9, Identity 5.24], it is known that the binomial coefficients satisfy the following useful identity:

$$\sum_j \binom{\ell}{m+j} \binom{s+j}{n} (-1)^j = (-1)^{\ell+m} \binom{s-m}{n-\ell}. \tag{22}$$

Hence, with $m = 0$, $\ell = k$ and $s = n-1$, we obtain

$$\widehat{w}_{(\alpha)}(n,k) = \alpha^{n-k} \frac{n!}{k!} \binom{n-1}{n-k}. \tag{23}$$

This completes the proof. □

Corollary 2.1.2. *The translated Whitney–Lah numbers satisfy the following exponential generating function:*

$$\sum_{n=k}^{\infty} \widehat{w}_{(\alpha)}(n,k) \frac{t^n}{n!} = \frac{1}{k!} \left(\frac{t}{1-\alpha t} \right)^k. \tag{24}$$

Proof. Applying (20), and both the binomial and negative binomial expansions,

$$\begin{aligned}
\sum_{n=k}^{\infty} \widehat{w}_{(\alpha)}(n,k) \frac{t^n}{n!} &= \frac{1}{\alpha^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{n=k}^{\infty} (\alpha t)^n \binom{j+n-1}{n} \\
&= \frac{1}{\alpha^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (1-\alpha t)^{-j} \\
&= \frac{1}{\alpha^k} \left[(1-\alpha t)^{-1} - 1 \right]^k \\
&= \frac{1}{k!} \left(\frac{t}{1-\alpha t} \right)^k.
\end{aligned}$$ □

Clearly, the results shown in the previous corollaries give back identities (1) and (3) for the classical Lah numbers when $\alpha = 1$. The binomial identity in (22) can also be utilized to derive another interesting formula for the translated Whitney–Lah numbers. By setting $s = n$, $\ell = k-1$ and $m = -1$,

$$\sum_{j=1}^{k} \binom{k-1}{j-1} \binom{n+j}{n} (-1)^j = (-1)^{k-2} \binom{n+1}{n-k+1}.$$

Multiplying both sides by $k!$ gives

$$\sum_{j=1}^{k} \binom{k-1}{j-1} \binom{n+j}{n} (-1)^j = \sum_{j=1}^{k} \widehat{w}_{(\alpha)}(k,j) \frac{(n+j)!(-1)^j}{n! \alpha^{k-j}}$$

in the left-hand side after using (23).

On the other hand, the right-hand side simply becomes

$$(-1)^{k-2}\binom{n+1}{n-k+1} = (-1)^k \frac{(n+1)!}{(n-k+1)!}.$$

Thus, we have derived the following theorem:

Theorem 2.2. *For $k \geq 2$ and $n \geq k-1$, the translated Whitney–Lah numbers satisfy*

$$\sum_{j=1}^{k} (-\alpha)^j \widehat{w}_{(\alpha)}(k,j)(n+j)! = (-\alpha)^k \frac{n!(n+1)!}{(n-k+1)!}. \tag{25}$$

When $\alpha = 1$, we immediately recognize

$$\sum_{j=1}^{k} (-1)^j L(k,j)(n+j)! = (-1)^k \frac{n!(n+1)!}{(n-k+1)!}, \tag{26}$$

an identity for the classical Lah numbers which was proved using six different methods by Guo and Qi [10]. A more direct approach in establishing (25) is as follows.

Alternative proof of Theorem 2.2. The generating function in (16) may be rewritten as

$$(-\alpha)^k (-t)_k = \sum_{j=0}^{k} \alpha^k \widehat{w}_{(\alpha)}(k,j)(t)_j. \tag{27}$$

Since $(-n-1)_j n! = (-1)^j (n+j)!$, then replacing t with $-n-1$ in the previous equation gives

$$(-\alpha)^k n!(n+1)_k = \sum_{j=0}^{k} (-\alpha)^j \widehat{w}_{(\alpha)}(k,j)(n+j)!$$

as desired. □

We now proceed to deriving a generalization of the Bell number formula in (10). In the paper of Qi [21], two methods to prove (10) are presented. The first one employs the Faa di Bruno's formula and the n-th derivative of the exponential function $e^{\pm 1/x}$ given by

$$\left(e^{\pm 1/x}\right)^{(n)} = (-1)^n e^{\pm 1/x} \sum_{k=1}^{n} (\pm 1)^k L(n,k) \frac{1}{t^{n+k}}$$

found in the paper of Daboud et al. [7]. The second is less complicated and requires only the use of the inverse relation

$$f_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} g_j \iff g_n = \sum_{j=0}^{n} (-1)^{n-j} \begin{Bmatrix} n \\ j \end{Bmatrix} f_j. \tag{28}$$

To obtain our next objective, we adopt a process that is similar to the latter since by using the orthogonal relations [13, Corollary 4.2]

$$\sum_{j=m}^{n} (-1)^{j-m} \widetilde{W}_{(\alpha)}(n,j) \widetilde{w}_{(\alpha)}(j,m) = \sum_{j=m}^{n} (-1)^{n-j} \widetilde{w}_{(\alpha)}(n,j) \widetilde{W}_{(\alpha)}(j,m) = \delta_{m,n},$$

it can be easily shown that the following inverse relation for the translated Whitney numbers of the first kind is valid:

$$f_n = \sum_{j=0}^{n} \widetilde{w}_{(\alpha)}(n,j) g_j \iff g_n = \sum_{j=0}^{n} (-1)^{n-j} \widetilde{w}_{(\alpha)}(n,j) f_j. \tag{29}$$

Now, taking $g_j = \widetilde{W}_{(\alpha)}(j,k)$ and $f_n = \widehat{w}_{(\alpha)}(n,k)$, we can apply the above inverse relation to (17) to get

$$\widetilde{W}_{(\alpha)}(n,k) = \sum_{j=0}^{n} (-1)^{n-j} \widetilde{W}_{(\alpha)}(n,j) \widehat{w}_{(\alpha)}(j,k). \tag{30}$$

We then recall that the translated Dowling numbers [15], denoted by $D_{(\alpha)}(n)$, are defined as the sum of the translated Whitney numbers of the second kind, i.e.

$$D_{(\alpha)}(n) = \sum_{k=0}^{n} \widetilde{W}_{(\alpha)}(n,k). \tag{31}$$

So by summing both sides of (30) up to n and appyling (31),

$$D_{(\alpha)}(n) = \sum_{k=0}^{n} \sum_{j=0}^{n} (-1)^{n-j} \widetilde{W}_{(\alpha)}(n,j) \widehat{w}_{(\alpha)}(j,k).$$

Thus, we have proved the result in the next theorem.

Theorem 2.3. *The translated Dowling numbers satisfy the explicit formula given by*

$$D_{(\alpha)}(n) = \sum_{j=0}^{n} (-1)^{n-j} \left(\sum_{k=0}^{j} \widehat{w}_{(\alpha)}(j,k) \right) \widetilde{W}_{(\alpha)}(n,j). \tag{32}$$

To close this section, notice that by (21), we may write

$$D_{(\alpha)}(n) = \sum_{j=0}^{n} (-1)^{n-j} \left(\sum_{k=0}^{j} \alpha^{j-k} L(j,k) \right) \widetilde{W}_{(\alpha)}(n,j).$$

Since it is known that [13, 15] $\widetilde{W}_{(1)}(n,j) = \left\{ {n \atop j} \right\}$ and $D_{(1)}(n) = B_n$, it means that the formula in (32) reduces to the one in (10) when $\alpha = 1$. Moreover, we acknowledge a generalization of (32) that can be seen in the paper of Corcino et al. [6]. The result in the said paper involves an explicit formula for the (r, β)-Bell numbers (or r-Dowling numbers). Readers are also directed to another paper by Corcino et al. [5] which contain more related results.

3 Translated q-Whitney–Lah numbers

Let $[n]_q$ denote the q-analogue of an integer n defined by

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}$$

and let $[t|\alpha]_n$ denote the product

$$[t|\alpha]_n = \prod_{i=0}^{n-1} [t - i\alpha]_q.$$

The translated q-Whitney numbers of the first and second kinds [14], denoted by $w^1_{(\alpha)}[n,k]_q$ and $w^2_{(\alpha)}[n,k]_q$, respectively, are defined in terms of the following horizontal generating functions:

$$[t|\alpha]_n = \sum_{k=0}^{n} w^1_{(\alpha)}[n,k]_q [t]_q^k \tag{33}$$

and

$$[t]_q^n = \sum_{k=0}^{n} w^2_{(\alpha)}[n,k]_q [t|\alpha]_k. \tag{34}$$

Various combinatorial properties of the numbers $w^1_{(\alpha)}[n,k]_q$ and $w^2_{(\alpha)}[n,k]_q$ and a certain combinatorial interpretation in the context of A-tableaux have already been established in the same paper. The properties include the inverse relation [14, Corollary 2.10]

$$f_n = \sum_{j=0}^{n} w^1_{(\alpha)}[n,j]_q g_j \iff g_n = \sum_{j=0}^{n} w^2_{(\alpha)}[n,j]_q f_j. \tag{35}$$

In general, the term "q-analogue" refers to a mathematical expression in terms of a parameter q such that as $q \to 1$, it reduces to a known identity or formula. For instance, it is clear that

$$\lim_{q\to 1} [n]_q = n.$$

Other examples are the q-binomial coefficient

$$\binom{n}{k}_q = \prod_{j=1}^{k} \frac{q^{n-j+1}-1}{q^j-1} = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

and the q-falling factorial of n of order k

$$[n]_{q,k} = \prod_{j=0}^{k-1} \frac{q^{n-j}-1}{q-1} = \frac{[n]_q!}{[n-k]_q!},$$

where $[n]_q! = \prod_{i=1}^{n} [i]_q$ is the q-factorial of n. See for instance the following limits which are easy to verify:

$$\lim_{q\to 1} [n]_q! = n!, \ \lim_{q\to 1} \binom{n}{k}_q = \binom{n}{k}, \ \lim_{q\to 1} [n]_{q,k} = (n)_k.$$

The book of Kac and Cheung [11] is a rich source for further discussions on q-analogues. The study of q-analogues of mathematical identities has been the interest of many mathematicians over a long period of time. For the case of the Lah numbers, Lindsay et al. [12] defined a q-analogue $\mathcal{L}_q(n,k)$ in terms of the following relation:

$$t(t+[1]_q)\cdots(t+[n-1]_q) = \sum_{k=0}^{n} \mathcal{L}_q(n,k) t(t-[1]_q)\cdots(t-[k-1]_q). \tag{36}$$

An earlier q-analogue of the Lah numbers can be attributed to Garsia and Remmel [8] who defined the q-Lah numbers, denoted by $L_q(n,k)$, as

$$[t]_q[t+1]_q\cdots[t+n-1]_q = \sum_{k=0}^{n} L_q(n,k)[t]_q[t-1]_q\cdots[t-k+1]_q \tag{37}$$

with the recurrence relation

$$L_q(n+1,k) = q^{n+k-1}L_q(n,k-1) + [n+k]_q L_q(n,k) \tag{38}$$

and explicit formula

$$L_q(n,k) = \binom{n}{k}_q \frac{[n-1]_q!}{[k-1]_q!} q^{k(k-1)}. \tag{39}$$

A more general notion was also introduced in [14, Equation 15] called the translated q-Whitney numbers of the third kind, denoted by $L_{(\alpha)}[n,k]_q$, which are defined as coefficients in the expansion of

$$[t|-\alpha]_n = \sum_{k=0}^{n} L_{(\alpha)}[n,k]_q [t|\alpha]_k. \tag{40}$$

These numbers can be computed recursively using the formula [14, Equation 31]

$$L_{(\alpha)}[n+1,k]_q = q^{\alpha(n+k-1)} L_{(\alpha)}[n,k-1]_q + [\alpha(n+k)]_q L_{(\alpha)}[n,k]_q. \tag{41}$$

Looking at equations (38) and (41), it is easy to see that $L_{(1)}[n,k]_q = L_q(n,k)$.

Theorem 3.1. *The numbers $L_{(\alpha)}[n,k]_q$ satisfy the following:*

$$L_{(\alpha)}[n,k]_q = \sum_{j=0}^{n} w^1_{(-\alpha)}[n,j]_q w^2_{(\alpha)}[j,k]_q. \tag{42}$$

Proof. Putting $-\alpha$ in place of α in (33) and by applying (34),

$$\begin{aligned}[t|-\alpha]_n &= \sum_{k=0}^{n} w^1_{(-\alpha)}[n,k]_q [t]_q^k \\ &= \sum_{j=0}^{n} \left\{ \sum_{k=j}^{n} w^1_{(-\alpha)}[n,k]_q w^2_{(\alpha)}[k,j]_q \right\} [t|\alpha]_j.\end{aligned}$$

Comparing the coefficients of $[t|\alpha]_j$ in the last equation with that of (40) gives the desired result. □

The identity in the previous theorem suggests that the numbers $L_{(\alpha)}[n,k]_q$ may be referred to as the translated q-Whitney–Lah numbers. To establish an explicit formula, we will use a method different from the one used in the previous section. We start by rewriting (40) into the form

$$\begin{aligned}[\alpha k|-\alpha]_n &= \sum_{j=0}^{n} L_{(\alpha)}[n,j]_q [\alpha k|\alpha]_j \\ &= \sum_{j=0}^{k} \binom{k}{j}_{q^\alpha} \left\{ \frac{L_{(\alpha)}[n,j]_q [\alpha k|\alpha]_j}{\binom{k}{j}_{q^\alpha}} \right\}.\end{aligned}$$

Since the well-known q-binomial inversion formula can be expressed as

$$f_k = \sum_{j=0}^{k} \binom{k}{j}_{q^\alpha} g_j \iff g_k = \sum_{j=0}^{k} (-1)^{k-j} q^{\alpha\binom{k-j}{2}} \binom{k}{j}_{q^\alpha} f_j, \tag{43}$$

then with $f_k = [\alpha k|-\alpha]_q$ and $g_j = \frac{L_{(\alpha)}[n,j]_q [\alpha k|\alpha]_j}{\binom{k}{j}_{q^\alpha}}$, we get

$$[\alpha k|\alpha]_k L_{(\alpha)}[n,k]_q = \sum_{j=0}^{k} (-1)^{k-j} q^{\alpha\binom{k-j}{2}} \binom{k}{j}_{q^\alpha} [\alpha j|-\alpha]_n,$$

the result in the next theorem.

Theorem 3.2. *The translated q-Whitney–Lah numbers satisfy the following explicit formula:*

$$L_{(\alpha)}[n,k]_q = \frac{1}{[k]_{q^\alpha}![\alpha]_q^k}\sum_{j=0}^{k}(-1)^{k-j}q^{\alpha\binom{k-j}{2}}\binom{k}{j}_{q^\alpha}[\alpha j|-\alpha]_n. \tag{44}$$

Formula (44) is a q-analogue of the explicit formula in (20) since

$$\lim_{q\to 1}[k]_{q^\alpha}! = k!,\ \ \lim_{q\to 1}[\alpha j|\alpha]_n = \alpha^n \langle j\rangle_n$$

and

$$\begin{aligned}\lim_{q\to 1} L_{(\alpha)}[n,k]_q &= \lim_{q\to 1}\left(\frac{1}{[k]_{q^\alpha}![\alpha]_q^k}\sum_{j=0}^{k}(-1)^{k-j}q^{\alpha\binom{k-j}{2}}\binom{k}{j}_{q^\alpha}[\alpha j|-\alpha]_n\right)\\ &= \frac{\alpha^{n-k}}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\langle j\rangle_n .\end{aligned}$$

Furthermore, we may use the above explicit formula in establishing a kind of exponential generating function for the numbers $L_{(\alpha)}[n,k]_q$. But before proceeding, we first mention the following useful identities:

$$[\alpha j|-\alpha]_n = [\alpha]_q^n[j+n-1]_{q^\alpha,n},\ \ \frac{[j+n-1]_{q^\alpha,n}}{[n]_{q^\alpha}!} = \binom{j+n-1}{n}_{q^\alpha} \tag{45}$$

and

$$\prod_{k=0}^{n-1}\frac{1}{1-q^k t} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}_q t^k. \tag{46}$$

Corollary 3.2.1. *The translated q-Whitney–Lah numbers satisfy the following exponential generating function:*

$$\sum_{n=0}^{\infty} L_{(\alpha)}[n,k]_q\frac{t^n}{[n]_{q^\alpha}!} = \frac{1}{[k]_{q^\alpha}![\alpha]_q^k}\sum_{j=0}^{k}(-1)^{k-j}q^{\alpha\binom{k-j}{2}}\binom{k}{j}_{q^\alpha}\prod_{n=0}^{j-1}(1-q^{\alpha n}[\alpha]_q t)^{-1}. \tag{47}$$

Proof. From equations (44) and (45), we have

$$\sum_{n=0}^{\infty} L_{(\alpha)}[n,k]_q\frac{t^n}{[n]_{q^\alpha}!} = \frac{1}{[k]_{q^\alpha}![\alpha]_q^k}\sum_{j=0}^{k}(-1)^{k-j}q^{\alpha\binom{k-j}{2}}\binom{k}{j}_{q^\alpha}\sum_{n=0}^{\infty}\binom{j+n-1}{n}_{q^\alpha}([\alpha]_q t)^n.$$

The result is obtained by applying (46) in the second summation. □

By taking the limit of (47) as $q\to 1$,

$$\lim_{q\to 1}\sum_{n=0}^{\infty} L_{(\alpha)}[n,k]_q\frac{t^n}{[n]_{q^\alpha}!} = \frac{1}{\alpha^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left(\frac{1}{1-\alpha t}\right)^j$$

which in turn simplifies to (24). On the other hand, the next theorem contains a q-analogue of (25).

Theorem 3.3. *The translated q-Whitney–Lah numbers satisfy the following:*

$$\sum_{j=0}^{k}(-[\alpha]_q)^j q^{-nj-\binom{j+1}{2}}L_{(\alpha)}[k,j]_q[n+j]_{q^\alpha}! = \frac{(-[\alpha]_q)^k[n]_{q^\alpha}![n+1]_{q^\alpha}!}{[n-k+1]_{q^\alpha}!}. \tag{48}$$

Proof. The proof is somewhat parallel to the alternative proof of Theorem 2.2. We proceed by rewriting (40) as

$$[-\alpha]_q^k \prod_{i=0}^{k-1} [-t-i]_{q^\alpha} = \sum_{j=0}^{k} [\alpha]_q^j L_{(\alpha)}[k,j]_q \prod_{i=0}^{j-1} [t-i]_{q^\alpha}. \tag{49}$$

We put $-n-1$ in place of t and multiply both sides by $[n]_{q^\alpha}!$ so that the left-hand side becomes

$$\begin{aligned}
[-\alpha]_q^k \prod_{i=0}^{k-1} [n+1-i]_{q^\alpha} [n]_{q^\alpha}! &= [-\alpha]_q^k [n]_{q^\alpha}! [n+1]_{q^\alpha,k} \\
&= \frac{[-\alpha]_q^k [n]_{q^\alpha}! [n+1]_{q^\alpha}!}{[n-k+1]_{q^\alpha}!}
\end{aligned}$$

while the right-hand side is

$$\sum_{j=0}^{k} [\alpha]_q^j L_{(\alpha)}[k,j]_q [n]_{q^\alpha}! \prod_{i=0}^{j-1} [t-i]_{q^\alpha} = \sum_{j=0}^{k} (-[\alpha]_q)^j q^{-nj-\binom{j+1}{2}} L_{(\alpha)}[k,j]_q [n+j]_{q^\alpha}!,$$

where the identity $j(n+1) + \binom{j}{2} = nj + \binom{j+1}{2}$ is used. Combining these equations give the desired result. □

The corollary below is a direct consequence of (48) when we set $\alpha = 1$. This formula is a q-analogue of Guo and Qi's [10] identity in (26) which can easily be verified by taking the limit as $q \to 1$.

Corollary 3.3.1. *The q-Lah numbers satisfy*

$$\sum_{j=0}^{k} (-1)^j q^{-nj-\binom{j+1}{2}} L_q(k,j)[n+j]_q! = \frac{(-1)^k [n]_q! [n+1]_q}{[n-k+1]_q}. \tag{50}$$

The translated q-Dowling numbers [14], denoted by $D_{(\alpha)}[n]_q$, are defined by the following sum:

$$D_{(\alpha)}[n]_q = \sum_{k=0}^{n} w_{(\alpha)}^2 [n,k]_q. \tag{51}$$

The last theorem presents a q-analogue of the explicit formula in (32).

Theorem 3.4. *The translated q-Dowling numbers satisfy the following explicit formula*

$$D_{(\alpha)}[n]_q = \sum_{j=0}^{n} \left(\sum_{j=0}^{k} L_{(\alpha)}[j,k]_q \right) w_{(-\alpha)}^2 [n,j]_q. \tag{52}$$

Proof. We put $-\alpha$ in place of α, and set $g_j = w_{(\alpha)}^2[j,k]_q$ and $f_n = L_{(\alpha)}[n,k]_q$ in the inverse relation in (35) so that when the resulting relation is applied to (42),

$$w_{(\alpha)}^2[n,k]_q = \sum_{j=0}^{n} w_{(-\alpha)}^2[n,j]_q L_{(\alpha)}[j,k]_q.$$

The desired result is obtained by summing over up to n. □

The explicit formula [15, Equation 10]

$$\widetilde{W}_{(\alpha)}(n,k)=\frac{1}{\alpha^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\alpha j)^n$$

shows that $\widetilde{W}_{(-\alpha)}(n,k)=(-1)^{n-k}\widetilde{W}_{(\alpha)}(n,k)$. Hence,

$$\begin{aligned}\lim_{q\to 1}D_{(\alpha)}[n]_q &= \lim_{q\to 1}\sum_{j=0}^{n}\left(\sum_{j=0}^{k}L_{(\alpha)}[j,k]_q\right)w^2_{(-\alpha)}[n,j]_q\\ &= \sum_{j=0}^{n}(-1)^{n-j}\left(\sum_{k=0}^{j}\widehat{w}_{(\alpha)}(j,k)\right)\widetilde{W}_{(\alpha)}(n,j)\end{aligned}$$

which is precisely (32). A similar formula for a q-analogue of the r-Dowling numbers can be seen in the paper of Cillar and Corcino [3]. However, since the definitions of their q-analogue and ours are distinctly motivated, it is difficult to say that their result is a generalization of the one in Theorem 3.4.

As we end, it may be worthwhile to say that the present paper was not able to express the explicit formula of $L_{(\alpha)}[n,k]_q$ in a way similar to that of (23) for the case of $\widehat{w}_{(\alpha)}(n,k)$. Perhaps this can be done by establishing a q-analogue of the binomial identity in (22) and use it to simplify the right-hand side of the explicit formula in (44).

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