Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 26, 2020, No. 4, 74–79
DOI: 10.7546/nntdm.2020.26.4.74-79

A study on some identities involving (s_k, t)-Jacobsthal numbers

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Received: 21 October 2019 **Revised:** 19 August 2020 **Accepted:** 25 August 2020

Abstract: In this study, we examined the generalization of Pakapongpun for Jacobsthal numbers. With respect to this generalization, we have given some known basic identities, which have an important place in the literature.
Keywords: Jacobsthal numbers, (s_k, t)-Jacobsthal numbers.
2010 Mathematics Subject Classification: 11B37, 11B39.

1 Introduction

The n-th element of the sequence $\{J_n\}$ is given by the following formula:

$$J_n = \frac{2^n - (-1)^n}{3}. \tag{1}$$

For detailed information on Jacobsthal numbers, see [4, 5]. In [6], Horadam defined the k-th associated sequences $\{J_n^k\}$ of $\{J_n\}$ as follows:

$$J_n^k = J_{n+1}^{k-1} + 2J_{n-1}^{k-1}. \tag{2}$$

The author gave recurrence relation, generating function, Binet form, Cassini formula and determinantal evaluations.

Atanassov generalized these numbers as follows [1,2]:

$$J_n^s = \frac{s^n - (-1)^n}{s+1}, \tag{3}$$

where n is a non-negative integer and s is a positive real number. The author again generalized these numbers as follows:

$$J_n^{s,t} = \frac{s^n - (-t)^n}{s+t}, \tag{4}$$

where t is a real number and the real number s is different from $-t$.

In 2014, Bueno defined s_k-Jacobsthal numbers and examined some of their properties. The author defined the n-th element of sequence as follows [3]:

$$J_n^{s_k} = \frac{s_k^n - (-1)^n}{s_k+1}. \tag{5}$$

It should be noted that Bueno wrote the denominator of this formula as s_{k+1} but did all the operations according to the formula (5). If we take 2 as a real number $s_k \neq -1$ and $s_k + 1 \neq 0$, then we get the well known the classic Jacobsthal number sequence.

In 2019, Pakapongpun gave a different generalization by considering this generalization of Bueno. The author defined (s_k, t)-Jacobsthal number as follows [7]:

$$J_n^{s_k,t} = \frac{s_k^n - (-t)^n}{s_k+t}. \tag{6}$$

In the study by Pakapongpun, there are some equations, such as double-indexed elements, consecutive terms and their sums. The sequence $\{J_n^{s_k,t}\}$ which is described above provides the following recurrence relation:

$$J_{n+2}^{s_k,t} = (s_k - t)J_{n+1}^{s_k,t} + s_k t J_n^{s_k,t}; \;\; J_0^{s_k,t} = 0, \;\; J_1^{s_k,t} = 1. \tag{7}$$

Then, the elements of sequence $\{J_n^{s_k,t}\}$ are as follows:

$$\left\{0, 1, s_k - t, s_k^2 - s_k t + t^2, s_k^3 - s_k^2 t + s_k t^2 - t^3, \cdots, (s_k - t)J_{n+1}^{s_k,t} + s_k t J_n^{s_k,t}, \cdots\right\}. \tag{8}$$

And writing $s_k = 2, \; t = 1$, the classic Jacobsthal sequence is obtained.

In this study, we were inspired by the work of Pakapongpun [7] and other references. And we have calculated some identities that are not included in the study of Pakapongpun but are considered important in the literature.

2 Main results

In this section, we will give identities such as Cassini, Catalan and d'Ocagne which have an important place in the literature for integer sequences.

First, let us give the generating function of the sequence $\{J_n^{s_k,t}\}$.

Theorem 2.1. *The generating function for the sequence $\{J_n^{s_k,t}\}$ is as follows:*

$$g(x) = \frac{1}{1 - (s_k - t)x - s_k t x^2}. \tag{9}$$

Proof. Let the generating function of the sequence $\{J_n^{s_k,t}\}$ be $g(x) = \sum_{i=0}^{\infty} J_{i+1}x^i$. If we multiply this function by the terms $(s_k - t)x$ and $s_k t x^2$, respectively, and if we take the necessary calculations to take advantage of the recurrence relation, we obtain the following equation:

$$g(x) - (s_k - t)xg(x) - s_k t x^2 g(x) = 1. \tag{10}$$

In this case, the desired formula is obtained. □

In the following theorem, we give the Cassini's identity which is an important relationship between number sequences and matrices.

Theorem 2.2 (Cassini's identity). *For $n \geq 1$, the elements of the sequence $\{J_n^{s_k,t}\}$ provide the following formula:*

$$J_{n+1}^{s_k,t} J_{n-1}^{s_k,t} - (J_n^{s_k,t})^2 = (-1)^n (s_k t)^{n-1}. \tag{11}$$

Proof.

$$\begin{aligned} &\left\{\frac{s_k^{n+1} - (-t)^{n+1}}{s_k + t}\right\}\left\{\frac{s_k^{n-1} - (-t)^{n-1}}{s_k + t}\right\} - \left\{\frac{s_k^{n} - (-t)^{n}}{s_k + t}\right\}^2 \\ &= \frac{2(-t)^n s_k^n - s_k^{n+1}(-t)^{n-1} - s_k^{n-1}(-t)^{n+1}}{(s_k + t)^2}. \end{aligned} \tag{12}$$

$$J_{n+1}^{s_k,t} J_{n-1}^{s_k,t} - (J_n^{s_k,t})^2 = \frac{(-t)^n s_k^n [2 - s_k(-t)^{-1} - (s_k)^{-1}(-t)]}{(s_k + t)^2}. \tag{13}$$

$$J_{n+1}^{s_k,t} J_{n-1}^{s_k,t} - (J_n^{s_k,t})^2 = \frac{(-t)^n s_k^n \frac{(s_k^2 + 2s_k t + t^2)}{s_k t}}{(s_k + t)^2} = (-1)^n \frac{(s_k t)^n}{s_k t} = (-1)^n (s_k t)^{n-1}. \tag{14}$$

□

It should be noted that when $s_k = 2,\ t = 1$, this formula reduces to the well-known Cassini formula for the classical Jacobsthal sequence. We have given Cassini identities according to some s_k and t roots in the table below.

s_k	t	$J_{n+1}^{s_k,t} J_{n-1}^{s_k,t} - (J_n^{s_k,t})^2$
1	1	$(-1)^n$
2	1	$(-1)^n 2^{n-1}$
2	2	$(-1)^n 4^{n-1}$
3	1	$(-1)^n 3^{n-1}$

Now, we have defined the following characteristic matrix M using the terms of the sequence.

$$M = \begin{pmatrix} J_2^{s_k,t} & s_k t \\ J_1^{s_k,t} & J_0^{s_k,t} \end{pmatrix}. \tag{15}$$

Corollary 2.2.1. *For $n \geq 1$, we have*

$$\det(M^n) = s_k t \left\{ J_{n+1}^{s_k,t} J_{n-1}^{s_k,t} - (J_n^{s_k,t})^2 \right\}. \tag{16}$$

Proof. If the induction method over n is used instead of calculating the powers of the matrix M, then we get

$$M^n = \begin{pmatrix} J_{n+1}^{s_k,t} & J_n^{s_k,t} s_k t \\ J_n^{s_k,t} & J_{n-1}^{s_k,t} s_k t \end{pmatrix}. \tag{17}$$

So, the claim appears to be correct. Notice that the powers of the characteristic matrix generated from the terms of the sequence give the Cassini's formula. □

In the following theorem, we generalized the Cassini's formula.

Theorem 2.3 (Catalan's identity). *For $n \geq k$, the elements of the sequence $\{J_n^{s_k,t}\}$ provide the following formula.*

$$J_{n+k}^{s_k,t} J_{n-k}^{s_k,t} - (J_n^{s_k,t})^2 = (-1)^{n+1} (s_k t)^{n-k} \left\{ \frac{s_k^k - t^k}{s_k + t} \right\}^2, \ k \ even. \tag{18}$$

$$J_{n+k}^{s_k,t} J_{n-k}^{s_k,t} - (J_n^{s_k,t})^2 = (-1)^{n} (s_k t)^{n-k} \left\{ \frac{s_k^k + t^k}{s_k + t} \right\}^2, \ k \ odd. \tag{19}$$

Proof. For proof we will use the formula $J_n^{s_k,t} = \frac{s_k^n - (-t)^n}{s_k + t}$. Let us first take the number k to be even.

$$J_{n+k}^{s_k,t} J_{n-k}^{s_k,t} - (J_n^{s_k,t})^2 = \left\{ \frac{s_k^{n+k} - (-t)^{n+k}}{s_k + t} \right\} \left\{ \frac{s_k^{n-k} - (-t)^{n-k}}{s_k + t} \right\} - \left\{ \frac{s_k^n - (-t)^n}{s_k + t} \right\}^2. \tag{20}$$

$$J_{n+k}^{s_k,t} J_{n-k}^{s_k,t} - (J_n^{s_k,t})^2 = \frac{2(-t)^n s_k^n - s_k^{n+k}(-t)^{n-k} - s_k^{n-k}(-t)^{n+k}}{(s_k + t)^2}. \tag{21}$$

$$J_{n+k}^{s_k,t} J_{n-k}^{s_k,t} - (J_n^{s_k,t})^2 = \frac{(-t)^n s_k^n \left\{ 2 - s_k^k(-t)^{-k} - s_k^{-k}(-t)^k \right\}}{(s_k + t)^2}. \tag{22}$$

Considering that k is an even number and making the necessary simplifications, the following equation is obtained:

$$J_{n+k}^{s_k,t} J_{n-k}^{s_k,t} - (J_n^{s_k,t})^2 = \frac{(-t)^n s_k^n \frac{(-s_k^{2k} + 2(s_k t)^k - t^{2k})}{(s_k t)^k}}{(s_k + t)^2}. \tag{23}$$

$$J_{n+k}^{s_k,t} J_{n-k}^{s_k,t} - (J_n^{s_k,t})^2 = (-1)^{n+1} (s_k t)^{n-k} \left\{ \frac{s_k^k - t^k}{s_k + t} \right\}^2. \tag{24}$$

If the number k is odd, the proof can be made similarly. Also, when $k = 1$, this identity is reduced to Cassini's identity. □

Theorem 2.4 (Honsberger's identity). *For $k > 1$, the elements of the sequence $\{J_n^{s_k,t}\}$ provide the following formula.*

$$J_{k-1}^{s_k,t} J_n^{s_k,t} + J_k^{s_k,t} J_{n+1}^{s_k,t} = t J_k^{s_k,t} \left\{ J_n^{s_k,t}(s_k^2 + 1) + s_k(-t)^n \right\} + (-t)^k J_n^{s_k,t}. \tag{25}$$

Proof. For proof we use the equation in Theorem 2.2 in the Pakapongpun's paper [7]. Accordingly, when a is a non-negative number, the author obtained the following formula:

$$J_{n+a}^{s_k,t} = s_k^a J_n^{s_k,t} + (-t)^n J_a^{s_k,t}. \tag{26}$$

Then, with the help of this formula,

$$J_{k-1}^{s_k,t} = \frac{tJ_k^{s_k,t} + (-t)^k}{s_k t} \tag{27}$$

is obtained. Using this formula we get

$$J_{k-1}^{s_k,t} J_n^{s_k,t} + J_k^{s_k,t} J_{n+1}^{s_k,t} = \frac{tJ_k^{s_k,t} + (-t)^k}{s_k t} J_n^{s_k,t} + J_k^{s_k,t} \left\{ s_k^a J_n^{s_k,t} + (-t)^n \right\}. \tag{28}$$

If necessary adjustments are made here, the desired formula is obtained. Furthermore, the formula can be rearranged with respect to whether k is odd or even. □

Theorem 2.5 (Vajda's identity). *For n, m, k non-negative integers, the elements of sequence $\{J_n^{s_k,t}\}$ satisfies the following formula:*

$$J_{n+m}^{s_k,t} J_{n+k}^{s_k,t} - J_n^{s_k,t} J_{n+m+k}^{s_k,t} = (-1)^n J_k^{s_k,t} J_m^{s_k,t} (s_k t)^n. \tag{29}$$

Proof. For proof we will use the formula $J_n^{s_k,t} = \frac{s_k^n - (-t)^n}{s_k + t}$. Then, we write

$$\begin{aligned} J_{n+m}^{s_k,t} J_{n+k}^{s_k,t} - J_n^{s_k,t} J_{n+m+k}^{s_k,t} &= \left\{ \frac{s_k^{n+m} - (-t)^{n+m}}{s_k + t} \right\} \left\{ \frac{s_k^{n+k} - (-t)^{n+k}}{s_k + t} \right\} \\ &\quad - \left\{ \frac{s_k^n - (-t)^n}{s_k + t} \right\} \left\{ \frac{s_k^{n+m+k} - (-t)^{n+m+k}}{s_k + t} \right\}. \end{aligned} \tag{30}$$

If necessary arrangements are made here, we get

$$J_{n+m}^{s_k,t} J_{n+k}^{s_k,t} - J_n^{s_k,t} J_{n+m+k}^{s_k,t} = \frac{(-t)^{n+m} s_k^n ((-t)^k - s_k^k) + s_k^{n+m} (-t)^n (-(-t)^k + s_k^k)}{(s_k + t)^2}. \tag{31}$$

And so, we have

$$J_{n+m}^{s_k,t} J_{n+k}^{s_k,t} - J_n^{s_k,t} J_{n+m+k}^{s_k,t} = \frac{(s_k^k - (-t)^k)(s_k^{n+m} (-t)^n - (s_k^n)(-t)^{n+m}}{(s_k + t)^2}. \tag{32}$$

Again, after some necessary arrangements are made, then

$$J_{n+m}^{s_k,t} J_{n+k}^{s_k,t} - J_n^{s_k,t} J_{n+m+k}^{s_k,t} = (-1)^n J_k^{s_k,t} J_m^{s_k,t} (s_k t)^n \tag{33}$$

is obtained, which is the desired result. □

Theorem 2.6 (d'Ocagne's identity). *For n, m non-negative integers, the elements of sequence $\{J_n^{s_k,t}\}$ satisfies the following formula:*

$$J_m^{s_k,t} J_{n+1}^{s_k,t} - J_n^{s_k,t} J_{m+1}^{s_k,t} = \frac{s_k^n (-t)^m + s_k^m (-t)^n}{s_k + t}. \tag{34}$$

With the help of the last formula, many different and useful formulas can be obtained by using the roots of the recurrence relation of the sequence.

Theorem 2.7 (Gelin–Cesàro identity). *For $n > 1$, the elements of the sequence $\{J_n^{s_k,t}\}$ provide the following formula.*

$$\begin{aligned} &J_{n-2}^{s_k,t} J_{n-1}^{s_k,t} J_{n+1}^{s_k,t} J_{n+2}^{s_k,t} - (J_n^{s_k,t})^4 \\ &= (-1)^n (s_k t)^{n-1} \left\{ (J_n^{s_k,t})^2 - (J_2^{s_k,t})^2 (s_k, t)^n - (J_n^{s_k,t})^2 (s_k, t)^{-n} \right\}. \end{aligned} \tag{35}$$

Conclusion

In this study, we first examined the generalization of Pakapongpun [7] for Jacobsthal numbers. Then, with respect to this generalization, we have given some basic identities which are a worthy addition to the literature. The results obtained in this study can be used in other number sequence generalizations.

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