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On a translated sum over primitive sequences related to a conjecture of Erdős

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Abstract: For x large enough, there exists a primitive sequence $\mathcal{A}$, such that

$$\sum_{a\in\mathcal{A}} \frac{1}{a(\log a + x)} \gg \sum_{p\in\mathcal{P}} \frac{1}{p(\log p + x)},$$

where $\mathcal{P}$ denotes the set of prime numbers.
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1 Introduction

A sequence $\mathcal{A}$ of positive integers is said to be primitive if there is no element of $\mathcal{A}$ which divides any other. We can see directly that the set of primes $\mathcal{P} = (p_n)_{n\geq 1}$ is primitive, as well as the sequences of the form:

$$\mathcal{A}_d^k = \{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}; \alpha_1, \dots, \alpha_k \in \mathbb{N}, \alpha_1 + \dots + \alpha_k = d\}.$$

Erdős [2] showed that for a primitive sequence $\mathcal{A} \neq \{1\}$, the series $\sum\limits_{a\in\mathcal{A}} \frac{1}{a\log a}$ converges. Later, in [3], he conjectured that if $\mathcal{A} \neq \{1\}$ is a primitive sequence, then $\sum\limits_{a\in\mathcal{A}} \frac{1}{a\log a} \leq \sum\limits_{p\in\mathcal{P}} \frac{1}{p\log p}$. Recently, in [4,5], the authors studied a translated sums of the form

$$S(\mathcal{A}, x) = \sum_{a\in\mathcal{A}} \frac{1}{a(\log a + x)},$$

where $x \in \mathbb{R}_+$. In [5], the authors constructed a primitive sequence $\mathcal{A}$, such that for all $x \geq 81$ $S(\mathcal{A}, x) > S(\mathcal{P}, x)$. In this note, we prove the following result.

Theorem 1.1. *Let $\lambda \geq 1$ and $t > 0$, then for any $x \geq 1656\lambda^{2t} \left(\log(\lambda^{2t} + 2)\right)^{3/2}$, there exists a primitive sequence $\mathcal{A}$ such that*

$$S(\mathcal{A}, x) \geq \lambda^t S(\mathcal{P}, x).$$

For a real x, the quantity $\lfloor x \rfloor$ denotes the integer part of x.

2 Lemmas

Lemma 2.1 ([6,8])**.** *We have:*

$$p_n \geq n\log n \qquad (\forall n \geq 2) \tag{1}$$

$$p_n \leq n(\log n + \log\log n) \quad (\forall n \geq 6) \tag{2}$$

$$\sum_{p\in\mathcal{P},\, p\leq x} \frac{1}{p} > \log\log x \qquad (x > 1). \tag{3}$$

Lemma 2.2 ([1])**.** *For $k \geq 463$,*

$$p_{k+1} \leq p_k \left(1 + \frac{1}{2\log^2 p_k}\right).$$

Lemma 2.3. *For any real number $x > 0$ and any integer $k \geq 2$ the following holds*

$$\sum_{n>k} \frac{1}{p_n(\log p_n + x)} \leq \frac{\log(1 + \frac{x}{\log k})}{x}.$$

Proof. Let $x > 0$ be a real number and $k \geq 2$ be an integer. By (1) and since the function $t \mapsto \frac{dt}{t\log t(\log t + x)}$ decreases on $[1, +\infty)$, then we obtain:

$$\begin{aligned}\sum_{n>k} \frac{1}{p_n(\log p_n + x)} &\leq \sum_{n>k} \frac{1}{n\log n(\log n + \log\log n + x)} \\ &\leq \sum_{n>k} \frac{1}{n\log n(\log n + x)} \leq \int_k^{+\infty} \frac{dt}{t\log t(\log t + x)}.\end{aligned}$$

We put $u = \log t$, so

$$\int_k^{+\infty} \frac{dt}{t\log t(\log t + x)} = \int_{\log k}^{+\infty} \frac{du}{u(u+x)} = \frac{1}{x}\int_{\log k}^{+\infty} \left(\frac{1}{u} - \frac{1}{u+x}\right) du = \frac{\log(1 + \frac{x}{\log k})}{x}.$$

This ends the proof. $\square$

Lemma 2.4. *For any integer $n \neq 0$, we have:*

$$n! \leq n^n e^{1-n}\sqrt{n} \ .$$

Proof. For $n = 1$, the inequality is verified. For $n \geq 2$, the result follows from

$$n! \leq n^n e^{-n}\sqrt{2\pi n} e^{1/12n}$$

(see [7]). $\square$

3 Proof of Theorem 1.1

Let $\lambda \geq 1$ and let $t > 0$. To prove this theorem, we need the parameters α, c and β which satisfy:

$$c\alpha \geq e^{\beta} + \log 1.008,\ 0 < \alpha \leq \frac{5}{12} \tag{C1}$$

$$\beta \geq 1.950 \tag{C2}$$

those parameters will be chosen later, the real c is chosen to be the smallest possible value so that; for any $x \geq c\lambda^{2t}\left(\log(\lambda^{2t}+2)\right)^{3/2}$, there exists a primitive sequence $\mathcal{A} \neq \{1\}$ such that $\sum\limits_{a\in\mathcal{A}} \frac{1}{a(\log a+x)} > \lambda^t \sum\limits_{p\in\mathcal{P}} \frac{1}{p(\log p+x)}$. Let p_k be the largest prime satisfying $p_k \leq e^{\alpha x}$, then according to Lemma 2.2 and (1), we obtain

$$p_k \leq e^{\alpha x} < p_{k+1} < 1.008 p_k. \tag{4}$$

Assume that $d = \left\lfloor \beta + \log\lambda^{2t} + \frac{3}{2}\log\log\left(\lambda^{2t}+2\right)\right\rfloor$, then from (C1) and (C2), we have $x \geq \frac{1}{\alpha}\left(e^d + \log 1.008\right)$ and from (3) and (4), we obtain

$$\sum_{n=1}^{k} \frac{1}{p_n} > \log\log p_k > \log\log\frac{p_{k+1}}{1.008} > \log\log\frac{e^{\alpha x}}{1.008} \geq d. \tag{5}$$

Now, we define the following sets of positive integers:

$$\mathcal{P}^k = \left\{p_n \,|p_n \in \mathcal{P}, p_n > p_k\right\}, \mathcal{A} = \mathcal{A}_d^k \cup \mathcal{P}^k.$$

It is clear that $\mathcal{A}_d^k \cap \mathcal{P}^k = \varnothing$ and the sets $\mathcal{A}_d^k$, $\mathcal{P}^k$, $\mathcal{A}$ are primitive sequences. Then, according to the multinomial formula and (5), we have

$$\begin{aligned}\sum_{a\in\mathcal{A}_d^k} \frac{1}{a} &= \sum_{\alpha_1+\ldots+\alpha_k=d} \frac{1}{p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}} \geq \sum_{\alpha_1+\ldots+\alpha_k=d} \frac{(1/p_1)^{\alpha_1}}{(\alpha_1)!}\ldots\frac{(1/p_k)^{\alpha_k}}{(\alpha_k)!} \\ &= \frac{1}{d!}(\sum_{n=1}^{k}\frac{1}{p_n})^d > \frac{d^{d-1}}{d!}\sum_{n=1}^{k}\frac{1}{p_n}.\end{aligned}$$

So,

$$\sum_{a\in\mathcal{A}_d^k} \frac{1}{a} > \frac{d^{d-1}}{d!}\sum_{n=1}^{k}\frac{1}{p_n}. \tag{6}$$

Since $x \geq c\lambda^{2t}\left(\log(\lambda^{2t}+2)\right)^{3/2}$, then from (C1) and (C2) we obtain $e^{\alpha x} \geq 3303 \geq p_{464}$. Hence using (4), we find $p_{464} \leq p_k \leq e^{\alpha x} < p_{k+1} < 1.008 p_k$. By using (2), we get

$$\log p_k \leq \alpha x \leq \log p_k + \log 1.008 \leq \log\left(k\left(\log k + \log\log k\right)\right) + \log 1.008.$$

Now, since the function

$$t \mapsto \frac{\log\left(t\left(\log t + \log\log t\right)\right) + \log 1.008}{\log t}$$

decreases on $[464, +\infty)$, then we have

$$\frac{\log\left(t\left(\log t + \log\log t\right)\right) + \log 1.008}{\log t} \leq \frac{\log\left(464\left(\log 464 + \log\log 464\right)\right) + \log 1.008}{\log 464} \simeq 1.339$$

that is,

$$\alpha x \leq 1.339 \log k. \tag{7}$$

By using inequality (7) and Lemma 2.3, we find

$$\sum_{n>k} \frac{1}{p_n(\log p_n + x)} \leq \frac{\log(1+\frac{x}{\log k})}{x} < \frac{\log(1+\frac{1.339}{\alpha})}{x}. \tag{8}$$

On the other hand, according to (4) and (5), we have for $x \neq 0$

$$\sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} \geq \sum_{n=1}^{k} \frac{1}{p_n(\alpha x + x)} \geq \frac{1}{(\alpha+1)\,x} \sum_{n=1}^{k} \frac{1}{p_n} \geq \frac{d}{(\alpha+1)\,x},$$

and from (8) we obtain

$$\sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} \geq \frac{d}{(\alpha+1)\log(1+\frac{1.339}{\alpha})} \sum_{n>k} \frac{1}{p_n(\log p_n + x)}.$$

Now we put $h(\alpha) = (\alpha+1)\log(1+\frac{1.339}{\alpha})$, then we obtain

$$\sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} \geq \frac{d}{h(\alpha)}\left(\sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)} - \sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)}\right),$$

therefore,

$$\left(1+\frac{d}{h(\alpha)}\right)\sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} \geq \frac{d}{h(\alpha)} \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)}.$$

Thus

$$\sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} \geq \frac{d}{d+h(\alpha)} \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)}. \tag{9}$$

Since p_k^d is the largest element in $\mathcal{A}_d^k$, then according to (4), we have for any $a \in \mathcal{A}_d^k$

$$\log a \leq d \log p_k \leq d\alpha x,$$

hence, from (6), we obtain:

$$\begin{aligned}
\sum_{a\in\mathcal{A}}\frac{1}{a(\log a+x)} &= \sum_{a\in\mathcal{A}_d^k\cup\mathcal{P}^k}\frac{1}{a(\log a+x)}=\sum_{a\in\mathcal{A}_d^k}\frac{1}{a(\log a+x)}+\sum_{a\in\mathcal{P}^k}\frac{1}{a(\log a+x)}\\
&\geq \frac{1}{(d\alpha x+x)}\sum_{a\in\mathcal{A}_d^k}\frac{1}{a}+\sum_{n>k}\frac{1}{p_n(\log p_n+x)}\\
&> \frac{d^{d-1}}{d!(d\alpha+1)}\sum_{n=1}^{k}\frac{1}{p_n(\log p_n+x)}+\sum_{n>k}\frac{1}{p_n(\log p_n+x)}\\
&= \left(\frac{d^{d-1}}{d!(d\alpha+1)}-1\right)\sum_{n=1}^{k}\frac{1}{p_n(\log p_n+x)}+\sum_{n=1}^{+\infty}\frac{1}{p_n(\log p_n+x)}.
\end{aligned}$$

According to (C1), and since the sequence $(u_n)_{n\geq 2}$ where $u_n=\dfrac{n^{n-1}-n!}{nn!}$ increases, then we have $\dfrac{d^{d-1}}{d!(d\alpha+1)}-1\geq 0$ for $d\geq 4$. By using this last inequality and (9), we obtain

$$\left(\frac{d^{d-1}}{d!(d\alpha+1)}-1\right)\sum_{n=1}^{k}\frac{1}{p_n(\log p_n+x)}\geq\left(\frac{d^{d-1}}{d!(d\alpha+1)}-1\right)\frac{d}{d+h(\alpha)}\sum_{n=1}^{+\infty}\frac{1}{p_n(\log p_n+x)}.$$

Therefore,

$$\begin{aligned}
\sum_{a\in\mathcal{A}}\frac{1}{a(\log a+x)} &> \left(\left(\frac{d^{d-1}}{d!(d\alpha+1)}-1\right)\frac{d}{d+h(\alpha)}+1\right)\sum_{n=1}^{+\infty}\frac{1}{p_n(\log p_n+x)}\\
&= \frac{d^d+d!(d\alpha+1)h(\alpha)}{d!(d\alpha+1)\,(d+h(\alpha))}\sum_{n=1}^{+\infty}\frac{1}{p_n(\log p_n+x)},
\end{aligned}$$

by applying Lemma 2.4, we get

$$\sum_{a\in\mathcal{A}}\frac{1}{a(\log a+x)}>\left(\frac{e^{d-1}+\sqrt{d}(d\alpha+1)h(\alpha)}{\sqrt{d}(d\alpha+1)\,(d+h(\alpha))}\right)\sum_{n=1}^{+\infty}\frac{1}{p_n(\log p_n+x)}. \tag{10}$$

It follows from the expression of d, that: $d>\beta-1+\log\lambda^{2t}+\frac{3}{2}\log\log(\lambda^{2t}+2)$, then $e^{d-1}>e^{\beta-2}\lambda^{2t}\left(\log(\lambda^{2t}+2)\right)^{3/2}$.

And since $\log\lambda^{2t}<\log(\lambda^{2t}+2)$, $\log\log(\lambda^{2t}+2)\leq\log(\lambda^{2t}+2)-1$ and $\beta\geq 1.950$, we have $d<(\beta+1)\log(\lambda^{2t}+2)$, then $d\alpha+1<((\beta+1)\alpha+1)\log(\lambda^{2t}+2)$.

So, the formula (10) becomes

$$\sum_{a\in\mathcal{A}}\frac{1}{a(\log a+x)}>j_{\alpha,\beta}(\lambda)\sum_{n=1}^{+\infty}\frac{1}{p_n(\log p_n+x)}. \tag{11}$$

where

$$j_{\alpha,\beta}(\lambda)=\frac{e^{\beta-2}\lambda^{2t}+\sqrt{\beta+1}((\beta+1)\alpha+1)h(\alpha)}{\sqrt{\beta+1}((\beta+1)\alpha+1)\left((\beta+1)\log(\lambda^{2t}+2)+h(\alpha)\right)}.$$

Now, we must choose α and β so that, for any $\lambda\geq 1$ and any $t>0$, $j_{\alpha,\beta}(\lambda)\geq 1$ and $\dfrac{e^{\beta}+\log 1.008}{\alpha}$ is the smallest possible. That is, for any $\lambda\geq 1$ and for any $t>0$

$$\frac{e^{\beta-2}}{\sqrt{\beta+1}\,(\beta+1)\left((\beta+1)\alpha+1\right)}\geq\frac{\log(\lambda^{2t}+2)}{\lambda^{2t}}.$$

Since, for any $t > 0$ the function $\lambda \mapsto \dfrac{\log(\lambda^{2t}+2)}{\lambda^{2t}}$ decreases on $[1, +\infty)$, then

$$\frac{e^{\beta-2}}{\sqrt{\beta+1}\,(\beta+1)\,((\beta+1)\alpha+1)} \geq \log 3.$$

Hence, $\dfrac{e^{\beta-2}-(\beta+1)^{\frac{3}{2}}\log 3}{(\beta+1)^{\frac{5}{2}}\log 3} \geq \alpha$ and $\dfrac{e^{\beta}+\log 1.008}{\alpha} \geq \dfrac{\left(e^{\beta}+\log 1.008\right)(\beta+1)^{\frac{5}{2}}\log 3}{e^{\beta-2}-(\beta+1)^{\frac{3}{2}}\log 3}$.

Finally, we will choose β so that the quantity $\dfrac{\left(e^{\beta}+\log 1.008\right)(\beta+1)^{\frac{5}{2}}\log 3}{e^{\beta-2}-(\beta+1)^{\frac{3}{2}}\log 3}$ is also the smallest possible.

A computer calculation gives $\beta \simeq 6.264$, $\alpha \simeq 0.317$ and $c \simeq 1655.234$. By replacing α and β in (11) , we get

$$\sum_{a\in\mathcal{A}} \frac{1}{a(\log a + x)} > \frac{71.094\lambda^{2t}+19.381}{64.659\ln(\lambda^{2t}+2)+19.381}\sum_{n=1}^{+\infty}\frac{1}{p_n(\log p_n + x)}.$$

But, for every $\lambda \geq 1$ and every $t > 0$, we have

$$\frac{71.094\lambda^{2t}+19.381}{64.659\ln(\lambda^{2t}+2)+19.381} > \lambda^{t},$$

which leads to the inequality our main theorem. Thus, for $\lambda \geq 1$, $t > 0$ and for any $x \geq 1656.3\lambda^{2t}\left(\log(\lambda^{2t}+2)\right)^{3/2}$, since $d = \left\lfloor 6.264 + \log\lambda^{2t} + \frac{3}{2}\log\log(\lambda^{2t}+2)\right\rfloor$ and k is the greatest integer such that $p_k \leq e^{0.317x}$, the sequence $\mathcal{A}$ is well defined. This ends the proof. □

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