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Objects generated by an arbitrary natural number

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Abstract: The set $\underline{Set}(n)$, generated by an arbitrary natural number n, is defined. Some arithmetic functions, defined over its elements are introduced. Some of the arithmetic, set-theoretical and algebraic properties of the new objects are studied.
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1 Introduction

In the present research, some new mathematical objects will be described. They are generated by a fixed arbitrary natural number $n > 1$. Let everywhere below it have the canonical form

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where k, α_1, α_2, $\ldots$, $\alpha_k \geq 1$ are natural numbers and $p_1, p_2, ..., p_k$ are different prime numbers. In [1], the following notations related to n that we will use below, are introduced:

$$\underline{set}(n) = \{p_1, p_2, ..., p_k\},$$

$$\underline{mult}(n) = \prod_{i=1}^{k} p_i.$$

We will show that these new objects have properties specific to algebra.

2 Main definitions

For the fixed $n \geq 2$, let us define the set

$$\underline{Set}(n) = \{m | m = \prod_{i=1}^{k} p_i^{\beta_i} \ \& \ \mathrm{h}(n) \leq \beta_i \leq \mathrm{H}(n)\},$$

where

$$\mathrm{h}(n) = \min(\alpha_1, \ldots, \alpha_k),$$

$$\mathrm{H}(n) = \max(\alpha_1, \ldots, \alpha_k)$$

and let $\omega(n) = k$.

For example, $\underline{Set}(12) = \underline{Set}(2^2.3) = \{6, 12, 18, 36\}$ and $\mathrm{h}(12) = 1, \mathrm{H}(12) = 2, \omega(12) = 2$. $\underline{Set}(72) = \underline{Set}(2^3.3^2) = \{36, 72, 108, 216\}$ and $\mathrm{h}(72) = 2, \mathrm{H}(72) = 3, \omega(72) = 2$.

It is suitable to define

$$\underline{Set}(0) = \{0\}.$$

$$\underline{Set}(1) = \{1\}.$$

Therefore, for each natural number n, $\underline{Set}(n) \neq \emptyset$.

3 Properties of $\underline{Set}(n)$

We see immediately that for n being a prime number, and more generally, if $n = \underline{mult}(n)$ and hence, $\mathrm{h}(n) = \mathrm{H}(n)$, then

$$\underline{Set}(n) = \{n\}.$$

Theorem 1. For the natural number n the cardinality $|\underline{Set}(n)|$ of $\underline{Set}(n)$ is equal to

$$|\underline{Set}(n)| = (\mathrm{H}(n) - \mathrm{h}(n) + 1)^{\omega(n)}.$$

Proof. For $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, $\underline{Set}(n)$ will contain all natural numbers m with $\underline{mult}(m) = \underline{mult}(n)$ and with powers between $\mathrm{h}(n)$ and $\mathrm{H}(n)$. Therefore, each p_i will be met with $\mathrm{H}(n) - \mathrm{h}(n) + 1$ different degrees and this is valid for each of the $\omega(n)$ in number divisors of n. Hence, the number of all elements of $\underline{Set}(n)$ is exactly $(\mathrm{H}(n) - \mathrm{h}(n) + 1)^{\omega(n)}$. □

For example,

$$|\underline{Set}(24)| = |\underline{Set}(2^3.3)|$$

$$= |\{2.3, 2^2.3, 2^3.3, 2.3^2, 2^2.3^2, 2^3.3^2, 2.3^3, 2^2.3^3, 2^3.3^3\}|$$

$$= 9 = (3 - 1 + 1)^2,$$

$$|\underline{Set}(36)| = |\underline{Set}(2^2.3^2)| = |\{2^2.3^2\}| = 1 = (2 - 2 + 1)^2,$$

$$|\underline{Set}(60)| = |\underline{Set}(2^2.3.5)|$$

$$= |\{2.3.5, 2^2.3.5, 2.3^2.5, 2^2.3^2.5, 2.3.5^2, 2^2.3.5^2, 2.3^2.5^2, 2^2.3^2.5^2\}|$$

$$= 8 = (2 - 1 + 1)^3.$$

Theorem 2. For two natural numbers m and n, if m is a divisor of n, $\mathrm{h}(m) = \mathrm{h}(n)$ and $\underline{set}(m) = \underline{set}(n)$, then

$$\underline{Set}(m) \subseteq \underline{Set}(n).$$

Proof. Having in mind that m is a divisor of n, we see that $\mathrm{H}(m) \leq \mathrm{H}(n)$.

Let $t \in \underline{Set}(m)$. Therefore, $t = \prod_{i=1}^{k} p_i^{\gamma_i}$, where $\mathrm{h}(m) \leq \gamma_i \leq \mathrm{H}(m)$ for each $i = 1, \ldots, k$. Hence,

$$\mathrm{h}(n) = \mathrm{h}(m) \leq \gamma_i \leq \mathrm{H}(m) \leq \mathrm{H}(n),$$

i.e., $t \in \underline{Set}(n)$. □

It is important to note that without one of the conditions $\mathrm{h}(m) = \mathrm{h}(n)$ and $\underline{set}(m) = \underline{set}(n)$, the Theorem is not valid. For example, 6 is a divisor of 72 and $\underline{set}(6) = \underline{set}(72) = \{2, 3\}$, but $\underline{Set}(6) = \{6\}$, while $\underline{Set}(72)$ mentioned above, does not contain the element 6.

On the other hand, 6 is a divisor of 30 and $\mathrm{h}(6) = \mathrm{h}(30) = 1$, but $\underline{Set}(30) = \{30\}$ and hence $\underline{Set}(6) \not\subseteq \underline{Set}(30)$.

For the well-known operations "Greatest Common Divisor" and "Least Common Multiple" over two natural numbers m and n that are marked by (m, n) and $[m, n]$, respectively, the following equalities are valid.

Theorem 3. For two natural numbers m and n so that $\underline{set}(m) = \underline{set}(n)$:

$$\underline{Set}(m) \cap \underline{Set}(n) \subseteq \underline{Set}((m, n)), \tag{1}$$

$$\underline{Set}(m) \cup \underline{Set}(n) \supseteq \underline{Set}([m, n]). \tag{2}$$

Proof. Let $t \in \underline{Set}(m) \cap \underline{Set}(n)$. Therefore, $t = \prod_{i=1}^{k} p_i^{\gamma_i}$, where $\gamma_1, \ldots, \gamma_k \geq 1$ are natural numbers. From the fact that $t \in \underline{Set}(m)$ it follows that $\mathrm{h}(m) \leq \gamma_i \leq \mathrm{H}(m)$ and from the fact that $t \in \underline{Set}(n)$ it follows that $\mathrm{h}(n) \leq \gamma_i \leq \mathrm{H}(n)$ for $i = 1, \ldots, k$. Therefore

$$\max(\mathrm{h}(m), \mathrm{h}(n)) \leq \gamma_i \leq \min(\mathrm{H}(m), \mathrm{H}(n)).$$

Obviously, when $\max(\mathrm{h}(m), \mathrm{h}(n)) > \min(\mathrm{H}(m), \mathrm{H}(n))$, the number t does not exist. For example,

$$\underline{Set}(6) \cap \underline{Set}(36) = \{6\} \cap \{36\} = \emptyset.$$

Therefore, (1) is valid.

Having in mind that

$$(m, n) = \prod_{i=1}^{k} p_i^{\min(\alpha_i, \beta_i)},$$

for $\underline{Set}((m, n))$ we see that

$$\underline{Set}((m, n)) = \{u | u = \prod_{i=1}^{k} p_i^{\varepsilon_i} \ \& \ \min(\mathrm{h}(m), \varepsilon(n)) \leq \delta_i \leq \min(\mathrm{H}(m), \mathrm{H}(n))\}.$$

Hence, when $\max(\mathrm{h}(m), \mathrm{h}(n)) \leq \min(\mathrm{H}(m), \mathrm{H}(n))$, for t it is valid that

$$\min(\mathrm{h}(m), \mathrm{h}(n)) \leq \max(\mathrm{h}(m), \mathrm{h}(n)) \leq \gamma_i \leq \min(\mathrm{H}(m), \mathrm{H}(n)),$$

i.e., $t \in Set((m, n))$.

In the opposite case, if $t \in \underline{Set}((m,n))$, then

$$\min(\mathrm{h}(m), \mathrm{h}(n)) \leq \gamma_i \leq \min(\mathrm{H}(m), \mathrm{H}(n)).$$

If $\mathrm{h}(m) \leq \mathrm{h}(n)$, then it will be certain that $t \in \underline{Set}(m)$, but only if $\mathrm{h}(n) \leq \gamma_i$ for each $i = 1, \ldots, k$, then $t \in \underline{Set}(n)$ and therefore, $t \in \underline{Set}(m) \cap \underline{Set}(n)$.

Hence (1) is valid. The validity of (2) is proved in the same manner. □

4 Algebraic objects generated by an arbitrary natural number

Let us define for the fixed n:

$$\boxcircle\, n = (\underline{mult}(n))^{\mathrm{h}(n)},$$

$$\boxast\, n = (\underline{mult}(n))^{\mathrm{H}(n)},$$

and for each $m \in \underline{Set}(n)$:

$$\neg m = \prod_{i=1}^{k} p_i^{\mathrm{H}(n)+\mathrm{h}(n)-\beta_i}.$$

We see immediately, that $\boxcircle\, n$, $\boxast\, n \in \underline{Set}(n)$, and for each $m \in \underline{Set}(n)$: $\neg m \in \underline{Set}(n)$. Moreover,

$$\neg m = \frac{\underline{mult}(n)^{\mathrm{H}(n)+\mathrm{h}(n)}}{m} = \frac{\boxcircle\, n . \boxast\, n}{m}.$$

Therefore

$$\neg \boxcircle\, n = \boxast\, n,$$

$$\neg \boxast\, n = \boxcircle\, n.$$

Following [3], we will mention that if S is a fixed set with unit element e_S and if * is an operation over S, then $\langle S, *, e_S \rangle$ is a commutative monoid, if:

1. $(\forall u, v \in S)(u * v \in S)$,
2. $(\forall u, v, w \in S)(u * (v * w) = (u * v) * w)$,
3. $(\forall a \in S)(u * e_S = u = e_S * u)$,
4. $(\forall u, v \in S)(u * v = v * u)$.

Now, we prove the following theorem.

Theorem 4. For the fixed n:

(a) $\langle \underline{Set}(n), (.), \boxast\, n \rangle$,

(b) $\langle \underline{Set}(n), [.], \boxcircle\, n \rangle$

are commutative monoids.

Proof. Let n be fixed. To see the validity of (a), we check sequentially the following equalities.

Let $u, v, w \in \underline{Set}(n)$. Therefore,

$$u = \prod_{i=1}^{k} p_i^{\beta_i}, \qquad v = \prod_{i=1}^{k} p_i^{\gamma_i}, \qquad w = \prod_{i=1}^{k} p_i^{\delta_i},$$

where for each $i = 1, 2, \ldots, k$: $\mathrm{h}(n) \le \beta_i, \gamma_i, \delta_i \le \mathrm{H}(n)$. Hence,

$$(u, v) = \prod_{i=1}^{k} p_i^{\min(\beta_i, \gamma_i)}$$

and from $\mathrm{h}(n) \le \min(\beta_i, \gamma_i) \le \mathrm{H}(n)$ it follows that $(u, v) \in \underline{Set}(n)$.

$$\begin{aligned}
(u, (v, w)) &= (\prod_{i=1}^{k} p_i^{\beta_i}, (\prod_{i=1}^{k} p_i^{\gamma_i}, \prod_{i=1}^{k} p_i^{\delta_i})) \\
&= (\prod_{i=1}^{k} p_i^{\beta_i}, \prod_{i=1}^{k} p_i^{\min(\gamma_i, \delta_i)}) \\
&= \prod_{i=1}^{k} p_i^{\min(\beta_i, \min(\gamma_i, \delta_i)}) = \prod_{i=1}^{k} p_i^{\min(\beta_i, \gamma_i, \delta_i)} = \prod_{i=1}^{k} p_i^{\min(\min(\beta_i, \gamma_i), \delta_i)} \\
&= (\prod_{i=1}^{k} p_i^{\min(\beta_i, \gamma_i)}, \prod_{i=1}^{k} p_i^{\delta_i}) \\
&= ((\prod_{i=1}^{k} p_i^{\beta_i}, \prod_{i=1}^{k} p_i^{\gamma_i}), \prod_{i=1}^{k} p_i^{\delta_i}) \\
&= ((u, v), w).
\end{aligned}$$

$$(u, \boxast(n)) = \prod_{i=1}^{k} p_i^{\min(\beta_i, \mathrm{H}(n))} = \prod_{i=1}^{k} p_i^{\beta_i} = u = \prod_{i=1}^{k} p_i^{\min(\mathrm{H}(n), \beta_i)} = (\boxast(n), u).$$

$$(u, v) = \prod_{i=1}^{k} p_i^{\min(\beta_i, \gamma_i)} = \prod_{i=1}^{k} p_i^{\min(\gamma_i, \beta_i)} = (v, u).$$

The validity of the second assertion is proved in the same manner. □

In [2], the author introduced the following concepts.

We call $\langle M, *, e_*, e_\circ \rangle$ a "(commutative) multi unitary group" (shortly, (c-)μ-group) if and only if $e_0 \in M$, $\langle M, *, e_* \rangle$ is a (commutative) monoid and

$$(\forall a \in M)(\exists a_\circ \in M)(a * a_\circ = e_\circ = a_\circ * a). \tag{3}$$

Two (c-)μ-groups MG_1 and MG_2 are dual, if and only if they have the forms

$$MG_1 = \langle M, *, e_*, e_\circ \rangle \quad \text{and} \quad MG_2 = \langle M, \circ, e_\circ, e_* \rangle$$

for some given operations $*$ and $\circ$, and for the unitary elements $e_*, e_\circ \in M$.

Theorem 5. For the fixed natural number n

$$\langle \underline{Set}(n), (.), \boxast n, \boxcircle n \rangle \quad \text{and} \quad \langle \underline{Set}(n), [.], \boxcircle n, \boxast n \rangle$$

are dual (c-)μ-groups.

Proof. From Theorem 4 we saw that $\langle \underline{Set}(n), (.), \boxtimes n \rangle$ and $\langle \underline{Set}(n), [.], \boxdot n \rangle$ are commutative monoids. Now, we see that for arbitrary $u \in \underline{Set}(n)$:

$$(u, \boxdot n) = \boxdot n = (\boxdot n, n)$$

and

$$[u, \boxtimes n] = \boxtimes n = [\boxtimes n, n],$$

i.e., condition (3) is satisfied and hence $\langle \underline{Set}(n), (.), \boxtimes n, \boxdot n \rangle$ and $\langle \underline{Set}(n), [.], \boxdot n, \boxtimes n \rangle$ are dual (c-)μ-groups. □

5 Conclusion

In a next research, other properties of the introduced here objects will be discussed. In addition, we will show that these objects have properties specific for modal logic.

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