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On the quantity $m^2 - p^k$ where $p^k m^2$ is an odd perfect number

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Abstract: We prove that $m^2 - p^k$ is not a square, if $n = p^k m^2$ is an odd perfect number with special prime p, under the hypothesis that $\sigma(m^2)/p^k$ is a square. We are also able to prove the same assertion without this hypothesis. We also show that this hypothesis is incompatible with the set of assumptions $\big(m^2 - p^k$ is a power of two $\big) \land \big(p$ is a Fermat prime$\big)$. We end by stating some conjectures.

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1 Introduction

Let $\sigma(x)$ denote the sum of the divisors of $x \in \mathbb{N}$, the set of positive integers. Denote the deficiency [13] of x by $D(x) = 2x - \sigma(x)$, and the sum of the aliquot divisors [14] of x by $s(x) = \sigma(x) - x$. Note that we have the identity $D(x) + s(x) = x$.

If a positive integer n is odd and $\sigma(n) = 2n$, then n is said to be an odd perfect number [17]. Euler proved that an odd perfect number, if one exists, must have the form $n = p^k m^2$, where p is the special prime satisfying $p \equiv k \equiv 1 \pmod 4$ and $\gcd(p, m) = 1$.

Descartes, Frenicle, and subsequently Sorli conjectured that $k = 1$ always holds [1]. Sorli conjectured $k = 1$ after testing large numbers with eight distinct prime factors for perfection [15]. Dris [7], and Dris and Tejada [12], call this conjecture as the Descartes–Frenicle–Sorli Conjecture, and derive conditions equivalent to $k = 1$.

Since m is odd, then $m^2 \equiv 1 \pmod 4$. Likewise, $p \equiv k \equiv 1 \pmod 4$, which implies that $p^k \equiv 1 \pmod 4$. It follows that $m^2 - p^k \equiv 0 \pmod 4$. Since

$$p^k < \frac{2m^2}{3}$$

(by a result of Dris [8]), we know *a priori* that

$$m^2 - p^k > \frac{p^k}{2}$$

so that we are sure that $m^2 - p^k > 0$. In particular, since $m^2 - p^k \equiv 0 \pmod 4$, we infer that $m^2 - p^k \geq 4$.

The index $i(p)$ of the odd perfect number $n = p^k m^2$ at the prime p is then equal to

$$i(p) := \frac{\sigma(m^2)}{p^k} = \frac{m^2}{\sigma(p^k)/2} = \frac{D(m^2)}{s(p^k)} = \frac{s(m^2)}{D(p^k)/2} = \gcd(m^2, \sigma(m^2)).$$

The term *index of an odd perfect number (at a certain prime)* was coined by Chen and Chen [4].

In this paper, we will refer continually to the following result by Broughan et al., which we will state without proof:

Lemma 1.1 ([2, Lemma 8, p. 7])**.** *If $n = p^k m^2$ is an odd perfect number and $\sigma(m^2)/p^k$ is a square, then $k = 1$ holds.*

2 Summary

We now present a summary of our results in this section.

The first proposition allows us to rule out $m^2 - p^k = s^2$ (where $s \geq 2$), under the assumption that $\sigma(m^2)/p^k$ is a square.

Theorem 2.1. *If $n = p^k m^2$ is an odd perfect number and $\sigma(m^2)/p^k$ is a square, then $m^2 - p^k$ is not a square.*

In the second proposition, we remove the requirement that $\sigma(m^2)/p^k$ is a square and prove unconditionally that $m^2 - p^k$ is not a square, with some help from MSE user FredH (`https://math.stackexchange.com/users/82711`).

Theorem 2.2. *If $n = p^k m^2$ is an odd perfect number, then $m^2 - p^k$ is not a square.*

Finally, in the third proposition, we use the hypothesis that $\sigma(m^2)/p^k$ is a square to prove that $m^2 - p^k$ is not a power of two when p is a Fermat prime.

Theorem 2.3. *If $n = p^k m^2$ is an odd perfect number and $\sigma(m^2)/p^k$ is a square, then either $m^2 - p^k \neq 2^{2t+1}$ for integers $t \geq 1$ or p is not a Fermat prime.*

3 A proof of Theorem 2.1

Suppose that $n = p^k m^2$ is an odd perfect number with special prime p, and that $m^2 - p^k = s^2$, for some $s \geq 2$.

Then $m^2 - s^2 = p^k = (m+s)(m-s)$, so that we obtain

$$\begin{cases} p^{k-v} = m + s \\ p^v = m - s \end{cases}$$

where v is a positive integer satisfying $0 \leq v \leq (k-1)/2$. It follows that we have the system

$$\begin{cases} p^{k-v} + p^v = p^v(p^{k-2v} + 1) = 2m \\ p^{k-v} - p^v = p^v(p^{k-2v} - 1) = 2s \end{cases}$$

Since p is a prime satisfying $p \equiv 1 \pmod 4$ and $\gcd(p, m) = 1$, from the first equation it follows that $v = 0$, so that we obtain

$$\begin{cases} p^k + 1 = 2m \\ p^k - 1 = 2s \end{cases},$$

which yields $m = \dfrac{p^k + 1}{2} < p^k$.

Lastly, note that the inequality $p < m$ has been proved by Brown [3], Dris [6], and Starni [16], so that we are faced with the inequality $p < m < p^k$. This implies that $k > 1$.

However, by assumption we have that $\sigma(m^2)/p^k$ is a square. This implies by Lemma 1.1 that $k = 1$, a clear contradiction.

This ends the proof of Theorem 2.1. □

Remark 3.1. *The following shorter proof for Theorem 2.1 was communicated by a referee.*

First, since $\sigma(m^2)/p^k$ is a square by assumption, then $k = 1$ by Lemma 1.1.

Then $m^2 - p^k = m^2 - p$, and it is straightforward to show that $m^2 - p = s^2$ for $s \geq 2$ is impossible: This would imply $m = (p+1)/2$, which contradicts $p < m$.

4 A proof of Theorem 2.2

The following proof is lifted verbatim from [10]:

Here's a way to finish the proof without appealing to any conjecture.

If $n = p^k m^2$ is a perfect number with $\gcd(p, m) = 1$, we have $\sigma(p^k)\sigma(m^2) = 2p^k m^2$. We know that $\sigma(p^k) = (p^{k+1} - 1)/(p - 1)$ and we have shown in Theorem 2.1 that $m = (p^k + 1)/2$, so we can conclude that

$$2(p^{k+1} - 1)\sigma(m^2) = (p-1)p^k(p^k + 1)^2. \qquad (*)$$

Consider the GCD of $p^{k+1} - 1$ with the right-hand side:

$$\gcd\left(p^{k+1} - 1, (p-1)p^k(p^k + 1)^2\right) \leq (p-1)\gcd(p^{k+1} - 1, p^k + 1)^2,$$

since p^k is coprime to $p^{k+1} - 1$.

Noticing that $p^{k+1}-1 = p(p^k+1)-(p+1)$, we find $\gcd(p^{k+1}-1, p^k+1) = \gcd(p+1, p^k+1)$, which is $p+1$ because k is odd. Thus $\gcd(p^{k+1}-1, (p-1)p^k(p^k+1)^2) \le (p-1)(p+1)^2$.

Since $k \equiv 1 \pmod 4$ and we have shown in Theorem 2.1 that $k > 1$, we have $k \ge 5$. If $(*)$ holds, the left-hand side of the inequality must be $p^{k+1}-1$, which is then greater than p^5. But the right-hand side is less than p^4, so this is impossible.

This completes the proof of Theorem 2.2. □

5 A proof of Theorem 2.3

Suppose that $n = p^k m^2$ is an odd perfect number with special prime p, and that $\sigma(m^2)/p^k$ is a square. We show that the assumption $\big(m^2 - p^k = 2^{2t+1}, t \ge 1\big) \wedge \big(p \text{ is a Fermat prime}\big)$ shall contradict Lemma 1.1.

(The following proof is adapted from the proof of Theorem 5 in [11].)

Assume to the contrary that $m^2 - p^k = 2^{2t+1}$ for some integer $t \ge 1$, and that p is a Fermat prime. This means that $p = 2^r + 1$ for some integer $r \ge 2$. Since p is a Fermat prime, we have $r = 2^l$, for some integer $l \ge 1$. In other words, $p = 2^{2^l} + 1$ is a Fermat prime.

Now, note that it is trivial to prove that

$$3 \mid 2^{2^l - 1} + 1 = \frac{p+1}{2}.$$

By assumption, $\sigma(m^2)/p^k$ is a square, which implies that $k = 1$. It follows that

$$m^2 - p = m^2 - (2^{2^l} + 1) = 2^{2t+1}$$

from which we get $m^2 - 2^{2^l} = 2^{2t+1} + 1$, which implies that $3 \mid \big(m^2 - 2^{2^l}\big)$. This means that $3 \nmid m^2$, since $l \ge 1$ and $3 \nmid 2^{2^l}$.

But we know that $3 \mid (p+1)/2 \mid m^2$. This contradicts $3 \nmid m^2$.

This finishes the proof of Theorem 2.3. □

Remark 5.1. *The divisibility constraint* $(p+1)/2 \mid m^2$ *is true in general since*

$$(p+1) = \sigma(p) \mid \sigma(p^k) \mid 2m^2$$

follows from $k \equiv 1 \pmod 4$, $\gcd(p^k, \sigma(p^k)) = 1$, *and the equation*

$$\sigma(p^k)\sigma(m^2) = \sigma(p^k m^2) = \sigma(n) = 2n = 2p^k m^2.$$

6 Concluding remarks and future research

Actually, more stringent conditions on $m^2 - p^k$ can be derived when $\sigma(m^2)/p^k$ is a square. Since $\sigma(m^2)/p^k$ is always odd, and by assumption it is a square, then since $p \equiv k \equiv 1 \pmod 4$ holds, we know that $\sigma(m^2) \equiv 1 \pmod 4$ also holds. This last congruence is known to hold if and only if $p \equiv k \pmod 8$ (see [5, 9]). Since by assumption $\sigma(m^2)/p^k$ is a square, we obtain $k = 1$ by Lemma 1.1. In particular, we know that $p^k \equiv 1 \pmod 8$. But we also know that m is odd. Therefore, we infer that $m^2 \equiv 1 \pmod 8$. It follows that $m^2 - p^k \equiv 0 \pmod 8$.

What follows is an elementary attempt to rule out $m^2 - p^k = 8$.

Lemma 6.1. *If $n = p^k m^2$ is an odd perfect number with special prime p and $\sigma(m^2)/p^k$ is a square, then $m^2 - p^k \neq 8$.*

Proof. Let $n = p^k m^2$ be an odd perfect number with special prime p. Suppose that $\sigma(m^2)/p^k$ is a square.

Assume to the contrary that $m^2 - p^k = 8$. Subtract 9 from both sides, then transfer p^k to the right-hand side:

$$m^2 - 9 = p^k - 1,$$

$$(m+3)(m-3) = p^k - 1.$$

This last equation implies that, in general, we have the divisibility constraint $(m+3) \mid (p^k - 1)$.

This divisibility constraint then implies that $(m+3) \leq (p^k - 1)$, from which we obtain $m < m + 4 \leq p^k$.

Lastly, note that the inequality $p < m$ has been proved by Brown [3], Dris [6], and Starni [16], so that we are faced with the inequality $p < m < p^k$. But this contradicts Lemma 1.1, so we are done.

This ends the proof of Lemma 6.1. □

We end this section with the following conjectures, which we leave for other researchers to investigate.

Conjecture 6.2. *If $n = p^k m^2$ is an odd perfect number and $\sigma(m^2)/p^k$ is a square, then $m^2 - p^k$ is not a cube.*

Conjecture 6.3. *If $n = p^k m^2$ is an odd perfect number, then $m^2 - p^k$ is not a cube.*

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