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On intercalated Fibonacci sequences

Krassimir T. Atanassov[1] and Anthony G. Shannon[2]

[1] Department of Bioinformatics and Mathematical Modelling
IBPhBME, Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105, Sofia--1113, Bulgaria
e-mail: krat@bas.bg

[2] Warrane College, The University of New South Wales
356 Anzac Parade, Kensington, NSW 2033, Australia
e-mail: t.shannon@warrane.unsw.edu.au; tshannon38@gmail.com

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Abstract: We construct three intercalated sequences and develop their essential properties which are generalizations of the three basic Fibonacci sequences. They are extensions of pulsated sequences described at previous Fibonacci conferences. We relate these sequences to the sequence $\{y_n\}_{n \geq 0} = \{0, 1, 4, 15, 56, \ldots\}$.

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1 Introduction

In a series of joint papers, the authors introduced more than ten different Fibonacci-type sequences (see, [1 – 20, 24]). In the present paper, we introduce a new Fibonacci type sequence which is related to the pulsated Fibonacci sequence [7, 12, 19].

We begin with the construction of the following three sequences in Table 1.

These sequences are obtained from the following initial terms and recurrence relations

$$\alpha_0 = a,\ \beta_0 = b,\ \gamma_0 = c$$

$$\alpha_{n+1} = \beta_n + \gamma_n + \alpha_n,\ \ \beta_{n+1} = \alpha_{n+1} + \gamma_n;\ \ \gamma_{n+1} = \alpha_{n+1} + \beta_n$$

for each natural number $n \geq 0$. It is the purpose of this note to show how they are interrelated.

n	β_n	α_n	γ_n
0		a	
	b		c
1		$a + b + c$	
	$a + b + 2c$		$a + 2b + c$
2		$3a + 4b + 4c$	
	$4a + 6b + 5c$		$4a + 5b + 6c$
3		$11a + 15b + 15c$	
	$15a + 20b + 21c$		$15a + 21b + 20c$
4		$41a + 56b + 56c$	
	$56a + 77b + 76c$		$56a + 76b + 77c$
…	…	…	…

Table 1. $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$

2 Preliminary results

Let us define

$$\alpha_n = x_n a + y_n b + y_n c$$

$$\beta_n = p_n a + q_n b + r_n c$$

$$\gamma_n = p_n a + r_n b + q_n c$$

We see and can prove, for example, by induction, that for each natural number $n \geq 0$:

$$x_n = y_n - y_{n-1}$$

$$p_n = y_n$$

$$q_n = r_n + (-1)^n$$

$$r_n = q_n + (-1)^{n-1}$$

$$q_n + r_n = y_{n+1} - y_n .$$

Therefore,

$$q_n = \frac{1}{2}(y_{n+1} - y_n + (-1)^n),$$

$$r_n = \frac{1}{2}(y_{n+1} - y_n + (-1)^{n+1}).$$

Hence, all coefficients can be represented by coefficients y_n for each natural number $n \geq 0$, and we see that the sequence $\{y_n\}_{n\geq 0} = \{0, 1, 4, 15, 56, \ldots\}$ has a common member that is obtained from the equation

$$y^2 - 4y + 1 = 0,$$

which has squares $2 + \sqrt{3}$ and $2 - \sqrt{3}$. Hence,

$$y_n = \frac{1}{2\sqrt{3}}\left((2 + \sqrt{3})^n - (2 - \sqrt{3})^n\right)$$

and

$$y_{n+2} = 4\, y_{n+1} - y_n .$$

Therefore, we can represent all other coefficients in terms of the coefficients in y_n as follows

$$\begin{aligned}
x_n &= y_n - y_{n-1} \\
&= \frac{1}{2\sqrt{3}}((2+\sqrt{3})^n - (2-\sqrt{3})^n) - \frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n-1} - (2-\sqrt{3})^{n-1}) \\
&= \frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n-1}(1+\sqrt{3}) - (2-\sqrt{3})^{n-1}(1-\sqrt{3})) \\
&= \frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n-1} - (2-\sqrt{3})^{n-1} + \sqrt{3}((2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1})) \\
&= \frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n-1} - (2-\sqrt{3})^{n-1} + \frac{1}{2}((2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1}));
\end{aligned}$$

and

$$\begin{aligned}
p_n &= \frac{1}{2+\sqrt{3}}((2+\sqrt{3})^n - (2-\sqrt{3})^n); \\
q_n &= \frac{1}{2}(\frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n-1} - (2-\sqrt{3})^{n-1}) + \frac{1}{2}((2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1})) + (-1)^n); \\
r_n &= \frac{1}{2}(\frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n-1} - (2-\sqrt{3})^{n-1}) + \frac{1}{2}((2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1})) + (-1)^{n+1}).
\end{aligned}$$

Finally, we can formulate and prove by induction the following result.

3 Main result

Theorem 1. For each natural number $n \geq 1$ the general terms (n-th members) of the three sequences $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ are respectively

$$\begin{aligned}
\alpha_n = {} & (\frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n-1} - (2-\sqrt{3})^{n-1}) + \frac{1}{2}((2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1}))a \\
& + \frac{1}{2\sqrt{3}}((2+\sqrt{3})^n - (2-\sqrt{3})^n)(b+c);
\end{aligned}$$

$$\begin{aligned}
\beta_n = {} & \frac{1}{2\sqrt{3}}((2+\sqrt{3})^n - (2-\sqrt{3})^n)\, a + \frac{1}{2}(\frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n-1} - (2-\sqrt{3})^{n-1}) \\
& + \frac{1}{2}((2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1}) + (-1)^n)b + \frac{1}{2}(\frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n-1} - (2-\sqrt{3})^{n-1}) \\
& + \frac{1}{2}((2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1}) + (-1)^{n+1})c;
\end{aligned}$$

$$\begin{aligned}
\gamma_n = {} & \frac{1}{2\sqrt{3}}((2+\sqrt{3})^n - (2-\sqrt{3})^n)\, a + \frac{1}{2}(\frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n-1} - (2-\sqrt{3})^{n-1}) \\
& + \frac{1}{2}((2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1}) + (-1)^{n+1})b + \frac{1}{2}(\frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n-1} - (2-\sqrt{3})^{n-1}) \\
& + \frac{1}{2}((2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1}) + (-1)^n))c.
\end{aligned}$$

Proof. When $n = 1, 2$, the validity of the assertion is seen from the definition of the sequences. Let assume that the assertion is valid for some $n \geq 2$. Now, we shall check its validity for $n + 1$, using the shorter form of the representations of α_n, β_n and γ_n:

$$\begin{aligned}
\alpha_{n+1} &= \beta_n + \gamma_n + \alpha_n \\
&= p_n\, a + q_n\, b + r_n\, c + p_n\, a + r_n\, b + q_n\, c + x_n\, a + y_n\, b + y_n\, c \\
&= (2p_n + x_n)\, a + (q_n + r_n + y_n)\, b + (q_n + r_n + y_n)\, c \\
&= (2y_n + y_n - y_{n-1})\, a + (y_{n+1} - y_n + y_n)\, b + (y_{n+1} - y_n + y_n)\, c \\
&= (3y_n - y_{n-1})\, a + y_{n+1}\, b + y_{n+1}\, c \\
&\qquad\qquad [\text{from } x_{n+1} = y_{n+1} - y_n = 4y_n - y_{n-1} - y_n = 3y_n - y_{n-1}] \\
&= y_{n+1}\, a + y_{n+1}\, b + y_{n+1}\, c;
\end{aligned}$$

$$\begin{aligned}
\beta_{n+1} &= \alpha_{n+1} + \gamma_n \\
&= x_{n+1}\, a + y_{n+1}\, b + y_{n+1}\, c + p_n\, a + r_n\, b + q_n\, c \\
&= (x_{n+1} + p_n)\, a + (y_{n+1} + r_n)\, b + (y_{n+1} + q_n)\, c \\
&= (p_{n+1} - p_n + p_n)\, a + (y_{n+1} + r_n)\, b + (y_{n+1} + q_n)\, c \\
&= p_{n+1}\, a + (y_{n+1} + \frac{1}{2}(y_{n+1} - y_n + (-1)^{n+1}))\, b + (y_{n+1} + \frac{1}{2}(y_{n+1} - y_n + (-1)^n))\, c \\
&= p_{n+1}\, a + \frac{1}{2}(3y_{n+1} - y_n + (-1)^{n+1})\, b + \frac{1}{2}(3y_{n+1} - y_n + (-1)^n)\, c \\
&= p_{n+1}\, a + \frac{1}{2}(4y_{n+1} - y_{n+1} - y_n + (-1)^{n+1})\, b + \frac{1}{2}(4y_{n+1} - y_{n+1} - y_n)\, c \\
&\qquad\qquad [\text{from } y_{n+2} = 4\, y_{n+1} - y_n] \\
&= p_{n+1}\, a + \frac{1}{2}(y_{n+2} - y_{n+1} + (-1)^{n+1})\, b + \frac{1}{2}(y_{n+2} - y_{n+1} + (-1)^n)\, c \\
&= p_{n+1}\, a + q_{n+1}\, b + r_{n+1}\, c.
\end{aligned}$$

The check for γ_{n+1} is identical to that for β_{n+1}.
This completes the proof. □

4 Conclusion

We can see some of the foregoing with a simplistic numerical example, in which $a = 1$, $b = 2$ and $c = 3$, as set out in Table 2. The examples are trivial, but their interdependence can be clearly seen as part of their 'basic' nature, and some of those in [25; A001075, A001835] have a rich history. $\{x_n\}$ satisfies the same second order linear homogeneous recurrence relation as $\{y_n\}$ but with different initial conditions as in Horadam sequences [22].

The $\{y_n\}$ sequence is shown in [25] to have applications in other parts of number theory (such as prime-free sequences), geometry, combinatorics, special functions, numerical analysis and Hessenberg matrices. It is related to many other classes of sequences [21, 23].

n	β_n	α_n	γ_n	y_n	x_n	$x_n + y_n$	$\gamma_n - \alpha_n$	$\gamma_n - \beta_n$
0	2	1	3	0	1	1	2	1
1	9	6	8	1	1	2	2	–1
2	31	23	32	4	3	7	9	1
3	118	86	117	15	11	26	31	–1
4	438	321	439	56	41	97	118	1

Table 2. Numerical examples of sequences, $n = 0, 1, 2, 3, 4$

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