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Markov equation with components of some binary recurrent sequences

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Abstract: The generalized Lucas sequence $\{U_n\}_{n\geq 0}$ is defined by $U_{n+1} = rU_n + sU_{n-1};\ n \geq 0$ with $U_0 = 0,\ U_1 = 1$ of which the Fibonacci sequence (F_n) is the particular case $r = s = 1$. In 2018, F. Luca and A. Srinivasan searched for the solutions $x, y, z \in F_n$ of the Markov equation $x^2 + y^2 + z^2 = 3xyz$ and proved that $(F_1, F_{2n-1}, F_{2n+1});\ n \geq 1$ is the only solution. In this paper, we extend this work from the Fibonacci sequence to any generalized Lucas sequence U_n for the case $s = \pm 1$.
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1 Introduction

The Fibonacci and Pell sequences are most famous examples of linear recursive sequences. Further, the generalized Lucas sequence $\{U_n\}_{n\geq 0}$ and its companion sequence $\{V_n\}_{n\geq 0}$ are defined by

$$U_{n+1} = rU_n + sU_{n-1} \quad \text{and} \quad V_{n+1} = rV_n + sV_{n-1}$$

for all $n \geq 0$ with initial terms $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, $V_1 = r$, where r and s are integers such that $r^2 + 4s > 0$. The Binet formula for these sequences are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \tag{1}$$

where $\alpha = \frac{r + \sqrt{r^2 + 4s}}{2}$ and $\beta = \frac{r - \sqrt{r^2 + 4s}}{2}$ are roots of their characteristic equation $x^2 - rx - s = 0$. Clearly, $\alpha - \beta = \sqrt{r^2 + 4s}$, $\alpha + \beta = r$ and $\alpha\beta = -s$.

With $(r, s) = (1, 1)$ we obtain the Fibonacci (F_n) sequence and the Lucas (L_n) sequence respectively. Similarly, with $(r, s) = (2, 1)$ we obtain the Pell (P_n) and Pell–Lucas (Q_n) sequence respectively. The sequence $q_n = Q_n/2$ is known as the associated Pell sequence and is recursively defined by

$$q_0 = q_1 = 1, \quad q_{n+1} = 2q_n + q_{n-1};\ n \geq 0.$$

Thus, from (1) it is obvious to have the Binet formula

$$q_n = \frac{\alpha^n + \beta^n}{2}, \quad \text{where } \alpha = 1 + \sqrt{2} \text{ and } \beta = 1 - \sqrt{2}. \tag{2}$$

Behera and Panda [1] defined balancing numbers n as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r),$$

where r is the balancer corresponding to n. Further, the sequence of balancing numbers (B_n) satisfies the recurrence relation for U_n with $(r, s) = (6, -1)$. The sequence of balancing numbers are closely related to the sequence of Pell and associated Pell numbers. For instance, $B_n = P_n q_n$ and $B_{2n} = 2P_n$ (see [5]).

In the year 1880, A. Markoff [4] studied the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz \tag{3}$$

in positive integers $x \leq y \leq z$ and the components of the solution triple (x, y, z) corresponding to the Markov equation (3) are known as Markov numbers. Further, if (x, y, z) is a Markov triple then $(x, z, 3xz - y)$ and $(y, z, 3yz - x)$ are also Markov triples. The first few Markov numbers are

$$1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, \ldots$$

(sequence A002559 in [6]), which appears as the coordinates of the Markov triples

$$(1, 1, 1), (1, 1, 2), (1, 2, 5), (1, 5, 13), (2, 5, 29), (1, 13, 34), (1, 34, 89), (2, 29, 169),$$

$$(5, 13, 194), (1, 89, 233), (5, 29, 433), (1, 233, 610), (2, 169, 985), (13, 34, 1325), \ldots .$$

One can see that, all Markov numbers on the regions adjacent to 2's region are odd-indexed Pell numbers whereas all Markov numbers on the regions adjacent to 1's region are odd-indexed Fibonacci numbers. Thus, there are infinitely many Markov triples of the form $(1, F_{2n-1}, F_{2n+1})$ and $(2, P_{2n-1}, P_{2n+1})$ for arbitrary positive integers n. In a recent paper [3], F. Luca and A. Srinivasan proved that there is no Markov triple (F_i, F_j, F_k) other than $(F_1, F_{2n-1}, F_{2n+1})$; $n \geq 1$.

The above studies motivate us to search for the existence of Markov triples (r, U_{n-1}, U_{n+1}) and (U_i, U_j, U_k).

In this paper, we completely solve (3) with $s \in \{\pm 1\}$ and $(x, y, z) = (r, U_{n-1}, U_{n+1})$. Further, we prove that there are only two sets of Markov triple of the form (U_i, U_j, U_k) when $s = 1$. We also prove a similar result when $s = -1$ with certain restrictions over i, j and n. The following are our main results.

Theorem 1.1. *For any positive integer n, if (r, U_{n-1}, U_{n+1}) is a Markov triple corresponding to* (3) *with $s \in \{\pm 1\}$, then*

$$(r, U_{n-1}, U_{n+1}) = (1, F_{2k-1}, F_{2k+1}) \text{ and } (2, P_{2k-1}, P_{2k+1}) \text{ for all integer } k \geq 1.$$

Theorem 1.2. *For positive integers i, j and n, if $(x, y, z) = (U_i, U_j, U_n)$ is a Markov triple corresponding to* (3) *with $s = 1$, then*

$$(x, y, z) = (F_2, F_{2k-1}, F_{2k+1}) \text{ and } (P_2, P_{2k-1}, P_{2k+1}) \text{ for all integer } k \geq 1.$$

Theorem 1.3. *Let i, j and n are positive integers such that $i + j \leq n$. Then, the Diophantine equation* (3) *has no solution of the form $(x, y, z) = (U_i, U_j, U_n)$ when $s = -1$.*

2 Preliminaries

To prove our main findings, we need the following results and definitions, which will be used in the forthcoming section with or without further reference.

The following Lemma plays an important role for solving the Markov equation.

Lemma 2.1. *[3, Lemma 2.1] If $(a, b, c) \neq (1, 1, 1)$ satisfies the Markov equation and $a \leq b \leq c$, then $3ab < b + c$.*

Lemma 2.2. *The k-th term of the generalized Lucas sequence $\{U_n\}$ with $s \in \{\pm 1\}$ satisfies the inequality*

$$\alpha^{k-2+\epsilon} \leq U_k \leq \alpha^{k-1+\epsilon},$$

where ϵ satisfies $2\epsilon + s = 1$.

Proof. First consider that $s = 1$, for which $\epsilon = 0$ and we need to prove the inequality $\alpha^{k-2} \leq U_k \leq \alpha^{k-1}$. Since $\alpha\beta = -1$ in the sequence $\{U_n\}$ corresponding to $s = 1$, we have

$$U_k = \frac{\alpha^k - (-1)^k \alpha^{-k}}{\alpha + \alpha^{-1}} < \frac{\alpha^k + \alpha^{-k}}{\alpha + \alpha^{-1}} = \alpha^{k-1} \frac{1 + \frac{1}{\alpha^{2k}}}{1 + \frac{1}{\alpha^2}}. \tag{4}$$

Since $1 + \frac{1}{\alpha^{2k}} < 1 + \frac{1}{\alpha^2}$ for $k \geq 1$, from (4) we conclude that

$$U_k < \alpha^{n-1}. \tag{5}$$

Now, we prove the left bound of U_k. In view of Proposition 6 from [2],

$$\alpha^k = \alpha U_k + U_{k-1}$$

or

$$U_k = \alpha^{k-1} - \frac{U_{k-1}}{\alpha}.$$

Since α satisfies the characteristic equation $x^2 - rx - 1 = 0$, we have $\alpha^2 - 1 = r\alpha$. Applying (5) to the above equation, we get

$$U_k > \alpha^{k-1} - \frac{\alpha^{k-2}}{\alpha} = \alpha^{n-3}(\alpha^2 - 1) = \alpha^{n-3}(r\alpha) > \alpha^{n-2}.$$

Next, for $s = -1$, $\epsilon = 1$ and thus, we need to show that $\alpha^{k-1} \leq U_k \leq \alpha^k$. So let us now assume that $s = -1$. Since, $\frac{\beta^k}{\alpha - \beta} > 0$ for each positive integer k, we have

$$U_k < \frac{\alpha^k}{\alpha - \beta} < \alpha^k,$$

which is the required upper bound. Now, the inequality

$$\frac{U_k}{U_{k-1}} = \alpha\left(\frac{1 - (\beta/\alpha)^k}{1 - (\beta/\alpha)^{k-1}}\right) > \alpha$$

holds for each $k \geq 0$ and thus, $U_k > \alpha U_{k-1}$. Iterating this inequality recursively, we get

$$U_k > \alpha^{k-1} U_1 = \alpha^{k-1},$$

which is the desired lower bound. This completes the proof. □

Lemma 2.3. *For any positive integer j,*

(i) $25 + P_j^2 + P_{j+3}^2 = 15P_jP_{j+3}$ *if and only if* $j = 2$,

(ii) $841 + P_j^2 + P_{j+5}^2 = 87P_jP_{j+5}$ *if and only if* $j = 2$.

Proof. (i) When $j = 2$, we get $25 + P_2^2 + P_5^2 = 870 = 15P_2P_5$. Now, using the Binet formula of Pell numbers given in (1), we get

$$\begin{aligned}
&25 + P_j^2 + P_{j+3}^2 - 15P_jP_{j+3} \\
&= \left(\frac{\alpha^j - \beta^j}{2\sqrt{2}}\right)^2 + \left(\frac{\alpha^{j+3} - \beta^{j+3}}{2\sqrt{2}}\right)^2 - 15\left(\frac{\alpha^j - \beta^j}{2\sqrt{2}}\right)\left(\frac{\alpha^{j+3} - \beta^{j+3}}{2\sqrt{2}}\right) + 25 \\
&= \frac{1}{8}\Big((\alpha^{2j} + \beta^{2j} - 2(-1)^j) + (\alpha^{2j+6} + \beta^{2j+6} - 2(-1)^{j+3}) \\
&\qquad\qquad - 15(\alpha^{2j+3} + \beta^{2j+3} - \alpha^j\beta^{j+3} - \alpha^{j+3}\beta^j)\Big) + 25 \\
&= \frac{1}{8}(\alpha^{2j+3}(\alpha^3 + \frac{1}{\alpha^3}) + \beta^{2j+3}(\beta^3 + \frac{1}{\beta^3}) - 15(\alpha^{2j+3} + \beta^{2j+3}) + 15(-1)^j(\alpha^3 + \beta^3)) + 25 \\
&= \frac{1}{8}((\alpha^{2j+3} - \beta^{2j+3})(\alpha^3 - \beta^3) - 15(\alpha^{2j+3} + \beta^{2j+3}) + 15(-1)^j(\alpha^3 + \beta^3)) + 25 \\
&= P_{2j+3}P_3 - \frac{15}{4}q_{2j+3} + \frac{15}{4}(-1)^j q_3 + 25 \qquad \text{(using the Binet formula given in (2))} \\
&= \frac{1}{4}(20P_{2j+3} - 15q_{2j+3} + 105(-1)^j + 100)
\end{aligned}$$

$$= \frac{1}{4}(20P_{2j+3} - 15(P_{2j+3} + P_{2j+2}) + 105(-1)^j + 100) \qquad (\text{since } q_j = P_j + P_{j-1},\ j \in \mathbb{N})$$
$$= \frac{1}{4}(5P_{2j+3} - 15P_{2j+2} + 105(-1)^j + 100)$$
$$= \frac{5}{4}(P_{2j+3} - 3P_{2j+2} + 21(-1)^j + 20)$$
$$= \frac{-5}{4}(P_{2j+2} - P_{2j+1} - 21(-1)^j - 20).$$

If j is odd, then

$$25 + P_j^2 + P_{j+3}^2 - 15P_jP_{j+3} = \frac{-5}{4}(P_{2j+2} - P_{2j+1} + 1) < 0,$$

since $P_{2j+2} - P_{2j+1} + 1 > 0$ for all $j \geq 1$.

If j is even, then

$$25 + P_j^2 + P_{j+3}^2 - 15P_jP_{j+3} = \frac{-5}{4}(P_{2j+2} - P_{2j+1} - 41).$$

But, $P_{2j+2} - P_{2j+1} - 41 > 0$ for $j > 2$ and $P_{2j+2} - P_{2j+1} - 41 < 0$ for $j < 2$, which implies that $25 + P_j^2 + P_{j+3}^2 \neq 15P_jP_{j+3}$ for all positive integers $j \neq 2$.

(ii) Proceeding as above it is easy to show that

$$841 + P_j^2 + P_{j+5}^2 - 87P_jP_{j+5} < 0$$

for all positive integers $j \neq 2$. Hence, the proof is complete. □

Similarly, we have the following lemmas.

Lemma 2.4. *If $s = 1, r = 5$ and $j \in \mathbb{N}$, then $25 + U_j^2 + U_{j+2}^2 = 15U_jU_{j+2}$ has no solution.*

Proof. Using the Binet formula (1) for the terms of $U(5, 1)$, we get

$$\begin{aligned}
&25 + U_j^2 + U_{j+2}^2 - 15U_jU_{j+2} \\
&= \left(\frac{\alpha^j - \beta^j}{\sqrt{29}}\right)^2 + \left(\frac{\alpha^{j+2} - \beta^{j+2}}{\sqrt{29}}\right)^2 - 15\left(\frac{\alpha^j - \beta^j}{\sqrt{29}}\right)\left(\frac{\alpha^{j+2} - \beta^{j+2}}{\sqrt{29}}\right) + 25 \\
&= \frac{1}{29}\Big((\alpha^{2j} + \beta^{2j} - 2(-1)^j) + (\alpha^{2j+4} + \beta^{2j+4} - 2(-1)^{j+2}) \\
&\qquad\qquad - 15(\alpha^{2j+2} + \beta^{2j+2} - \alpha^j\beta^{j+2} - \alpha^{j+2}\beta^j)\Big) + 25 \\
&= \frac{1}{29}((\alpha^{2j+2} + \beta^{2j+2})(\alpha^2 + \beta^2 - 15) + (-1)^j(15(\alpha^2 + \beta^2) - 4)) + 25 \\
&= \frac{1}{29}((\alpha^{2j+2} + \beta^{2j+2})(27 - 15) + (-1)^j(15(27) - 4)) + 25 \\
&= \frac{1}{29}(12V_{2j+2} + 401(-1)^j + 725),
\end{aligned}$$

which is positive for all $j \geq 1$. This completes the proof. □

Lemma 2.5. *If $s = 1$ and $r, j \in \mathbb{N}$, then $1 + U_j^2 + U_{j+1}^2 = 3U_jU_{j+1}$ if and only if $r, j \in \{1, 2\}$. Further, if $s = -1$ and $2 < r \in \mathbb{N}$, then $1 + U_j^2 + U_{j+1}^2 = 3U_jU_{j+1}$ has no solution.*

Proof. Let $s = 1$, then for $r \in \{1, 2\}, j = 1$ we have

$$1 + F_1^2 + F_2^2 = 3 = 3F_1F_2, \quad 1 + P_1^2 + P_2^2 = 6 = 3P_1P_2$$

and when $j = 2$ we have

$$1 + F_2^2 + F_3^2 = 6 = 3F_2F_3, \quad 1 + P_2^2 + P_3^2 = 30 = 3P_2P_3.$$

Conversely, when $s = 1$ and $j \in \{1, 2\}$,

$$1 + U_j^2 + U_{j+1}^2 = 3U_jU_{j+1}$$

gives $r^2 - 3r + 2 = 0$ and $r^4 - 3r^3 + 3r^2 - 3r + 2 = 0$ respectively and hence, $r \in \{1, 2\}$. Further, if $s = 1, r > 2, j > 2$, then

$$1 + U_j^2 + U_{j+1}^2 < 3U_jU_{j+1}$$

and if $s = -1, r > 2, j \geq 1$, then

$$1 + U_j^2 + U_{j+1}^2 > 3U_jU_{j+1}.$$

□

3 Proof of main results

3.1 Proof of Theorem 1.1

The terms of the generalized Lucas sequence satisfy the Cassini identity

$$U_{n+1}U_{n-1} - U_n^2 = (-1)^n s^{n-1}. \tag{6}$$

The right-hand side of (6) is a perfect square only if $s = 1$ and n is even. Thus, using the recurrence relation for $s = 1$ and considering the even parity of n, i.e., taking $n = 2k$, we get

$$r^2 + U_{2k-1}^2 + U_{2k+1}^2 = (r^2 + 2)U_{2k-1} + U_{2k+1}.$$

Therefore, if (r, U_{n-1}, U_{n+1}) is a Markov triple corresponding to the Markov equation (3), then $r^2 + 2 = 3r$ yielding $r = 1, 2$. This completes the proof.

3.2 Proof of Theorem 1.2 and 1.3

Let U_k be the k-th term of the generalized Lucas sequence with $s \in \{\pm 1\}$. Further, let $\Delta = \sqrt{r^2 + 4s}$ and $2\epsilon + s = 1$. Now, assume that (U_i, U_j, U_n) satisfy (3) with $U_i \leq U_j \leq U_n$.

When $r = s = 1$, the Markov triple (U_i, U_j, U_n) has only solution $(F_i, F_{2k-1}, F_{2k+1})$ for $i \in \{1, 2\}$ (see [3]). Without loss of generality we assume that $r > 1$ and since $U_1 = 1$, we have $1 \leq i \leq j \leq n$. Thus, (U_i, U_j, U_n) satisfy

$$U_n - 3U_iU_j = -\frac{U_i^2 + U_j^2}{U_n}. \tag{7}$$

Using (1) to the left-hand side of (7), we get

$$\frac{\alpha^n}{\Delta} - \frac{3\alpha^{i+j}}{\Delta^2} = -\frac{U_i^2 + U_j^2}{U_n} + \frac{\beta^n}{\Delta} - \frac{3}{\Delta^2}(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j}). \tag{8}$$

In view of Lemma 2.2, we have

$$\frac{U_i^2 + U_j^2}{U_n} \leq \frac{2U_j^2}{U_n} \leq 2\alpha^{2j-n+\epsilon} \leq 2\alpha^{j+\epsilon},$$

$$\left|\frac{\beta^n}{\Delta}\right| \leq \frac{\alpha^{-j}}{\Delta} < \frac{\alpha^j}{\Delta^2},$$

and

$$\left|\frac{3}{\Delta^2}(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j})\right| \leq \frac{3}{\Delta^2}(2\alpha^j + 1) \leq \frac{9\alpha^j}{\Delta^2}.$$

Using these values in (8), we get

$$\left|\frac{\alpha^n}{\Delta} - \frac{3\alpha^{i+j}}{\Delta^2}\right| \leq 2\alpha^{j+\epsilon} + \frac{10\alpha^j}{\Delta^2} \leq \alpha^{j+\epsilon}\left(2 + \frac{10}{\Delta^2}\right).$$

Dividing $\frac{\alpha^{i+j}}{\Delta}$ on both sides of the above inequality, we get

$$\left|\alpha^{n-i-j} - \frac{3}{\Delta}\right| < \frac{2\Delta + 10/\Delta}{\alpha^{i-\epsilon}}.$$

Since $\frac{3}{\Delta} < \alpha$, we have

$$\min_{k\in\mathbb{Z}}\left|\alpha^k - \frac{3}{\Delta}\right| > 0.01999 \quad (\text{when } s = 1) \tag{9}$$

and

$$\min_{k\in\mathbb{W}}\left|\alpha^k - \frac{3}{\Delta}\right| > 0.13397 \quad (\text{when } s = -1). \tag{10}$$

In view of (9) and (10), we have

$$\alpha^i < \frac{2\Delta^2 + 10}{0.01999\Delta} \quad \text{and} \quad \alpha^{i-1} < \frac{2\Delta^2 + 10}{0.13397\Delta}$$

respectively. Hence, the possible values of i and the corresponding values of r are given by

$$\{1 \leq i \leq 3,\ 2 \leq r \leq 10\} \text{ and } (i,r) \in \{(4,2),(5,2),(6,2),(4,3),(5,3),(4,4)\} \text{ when } s = 1$$

and

$$\{1 \leq i \leq 3,\ 3 \leq r \leq 15\} \text{ and } (i,r) \in \{(4,3),(5,3),(4,4)\} \text{ when } s = -1.$$

Searching for the Markov numbers with the above possibilities of i, r and s yield

$$P_2 = 2, P_3 = 5, P_5 = 29, U_2(5,1) = 5, U_2(5,-1) = 5, U_2(13,-1) = 13$$

and hence, we have the following result.

Lemma 3.1. *If* $(x,y,z) = (U_i, U_j, U_n)$ *satisfies* (3) *with* $2 \leq i \leq j \leq n$ *and* $r > 1, s = \pm 1$*, then* $(i,r,s) = (2,2,1),(3,2,1),(5,2,1),(2,5,1),(2,5,-1),(2,13,-1)$.

Lemma 3.2. *If $s = 1$ and $(x, y, z) = (U_i, U_j, U_n)$ satisfies* (3) *with $2 \leq i \leq j \leq n$, then $r = 2$, j is odd, $j = n - 2$ and $U_i = 2$.*

Proof. In view of Lemma 3.1, $(r, s) = (2, 1)$ when $i \in \{2, 3, 5\}$ for which the generalized Lucas sequence is nothing but the Pell sequence.

Assuming $i = 5$, we have $P_5 = 29$ and hence,

$$29^2 + P_j^2 + P_n^2 = 87P_jP_n$$

or

$$841 + P_j^2 = P_n(87P_j - P_n) > 0,$$

which gives $P_n < 87P_j$. By virtue of Lemma 2.1, $87P_j < P_j + P_n$ and so

$$86P_j < P_n < 87P_j.$$

The inequalities

$$\begin{aligned} P_{j+4} &= 2P_{j+3} + P_{j+2} = 5P_{j+2} + 2P_{j+1} = 12P_{j+1} + 5P_j \\ &= 29P_j + 12P_{j-1} < 41P_j < 86P_j < P_n \end{aligned}$$

and

$$\begin{aligned} P_{j+6} &= 2P_{j+5} + P_{j+4} = 5P_{j+4} + 2P_{j+3} = 12P_{j+3} + 5P_{j+2} \\ &= 29P_{j+2} + 12P_{j+1} = 70P_{j+1} + 29P_j > 87P_j > P_n \end{aligned}$$

implies that the only possibility is $n = j + 5$ for which the Markov equation is

$$29^2 + P_j^2 + P_{j+5}^2 = 87P_jP_{j+5}.$$

Use of Lemma 2.3 gives $j = 2$, which is a contradiction to our assumption $i \leq j$.

Now, assume that $i = 3$. Hence, $P_3 = 5$ and

$$25 + P_j^2 = P_n(15P_j - P_n),$$

which gives $P_n < 15P_j$. By virtue of Lemma 2.1, $15P_j < P_j + P_n$ and so

$$14P_j < P_n < 15P_j.$$

The inequalities

$$P_{j+2} = 2P_{j+1} + P_j = 5P_j + 2P_{j-1} < 7P_j$$

and

$$P_{j+4} = 2P_{j+3} + P_{j+2} = 12P_{j+1} + 5P_j > 17P_j$$

implies that the only possibility is $n = j + 3$ for which the Markov equation is

$$5^2 + P_j^2 + P_{j+3}^2 = 15P_jP_{j+3}.$$

Use of Lemma 2.3 gives $j = 2$, which is a contradiction to our assumption $i \leq j$.

Similarly, for $i=2$, $P_2=2$ and

$$4+P_j^2+P_n^2=6P_jP_n,$$

which gives $5P_j<P_n<6P_j$ by using Lemma 2.1. The inequality

$$P_{j+1}<3P_j<5P_j<P_n<6P_j<7P_j<P_{j+3}$$

implies that $n=j+2$. Thus,

$$4+P_{n-2}^2+P_n^2=6P_{n-2}P_n. \tag{11}$$

Let n be even. Then using the property $P_{2n}=2B_n$, we rewrite (11) as

$$1+B_{(n-2)/2}^2+B_{n/2}^2=6B_{(n-2)/2}B_{n/2}. \tag{12}$$

But, in view of Lemma 2.5, there is no positive integer solution to (12). Therefore, n must be odd and $j=n-2$. Moreover, since the Pell numbers satisfy the Cassini identity

$$P_{n+1}P_{n-1}-P_n^2=(-1)^n,$$

it is obvoius to have the Markov triples $(2,P_{2n-1},P_{2n+1})$ for every positive integer n.

Now, we are left with the case $(i,r,s)=(2,5,1)$. For this, we have $U_5=5$ and hence,

$$5^2+U_j^2+U_n^2=15P_jP_n$$

or

$$25+U_j^2=U_n(15U_j-U_n)>0,$$

which gives $U_n<15U_j$. By virtue of Lemma 2.1, $15U_j<U_j+U_n$ and so

$$14U_j<U_n<15U_j.$$

The inequalities

$$U_{j+1}=5U_j+U_{j-1}<6U_j<14U_j<U_n$$

and

$$U_{j+3}=5U_{j+2}+U_{j+1}=26U_{j+1}+5U_j>29U_j>15U_j>U_n$$

implies that the only possibility is $n=j+2$. Thus, we have the Markov equation

$$5^2+U_j^2+U_{j+2}^2=15U_jU_{j+2},$$

which has no solution by Lemma 2.4. This completes the proof. □

Now we will prove Theorem 1.2 and 1.3.

Proof of Theorem 1.2. Since the case $i=1,2$ for the Fibonacci sequence ($r=s=1$) has already been studied in [3], without loss of generality we assume that $r>1$ and $s=1$. Further, we have already discussed the case $i\geq 2$ and $r>1$ in Lemma 3.2. Thus, we are left with the case $i=1$, $r>1$ and $s=1$.

Hence,

$$1 + U_j^2 + U_n^2 = 3U_jU_n$$

or

$$1 + U_j^2 = U_n(3U_j - U_n),$$

which gives $U_n < 3U_j$. From Lemma 2.1, $3U_j < U_j + U_n$ and so

$$2U_j < U_n < 3U_j.$$

The inequality

$$U_j < 2U_j < U_n < 3U_j < U_{j+2}$$

implies that the only possibility is $n = j + 1$ for which the Markov equation is

$$1 + U_j^2 + U_{j+1}^2 = 3U_jU_{j+1}. \tag{13}$$

In view of Lemma 2.5, the only solution to (13) are $(1, 1, 1), (1, 1, 2), (1, 2, 5)$ of which the first one is trivial and the other two can be viewed as the Markov triple $(F_2, F_{2k-1}, F_{2k+1})$ and $(P_2, P_{2k-1}, P_{2k+1})$ for $k = 1$ as stated in Theorem 1.2. Now, when $r = 1$ the case $i = 2$ yields the same result as the case $i = 1$ because $F_1 = F_2 = 1$. Thus, we are left with the case $i \geq 2$ and $r > 1$, which has already been discussed in Lemma 3.2. This completes the proof.

Proof of Theorem 1.3. In view of Lemma 3.1, $(r, s) = (5, -1), (13, -1)$ when $i = 2$ and $i = 1$ for all $r > 2$ and $s = -1$.

Assuming $i = 2$, we have $U_2(5, -1) = 5$ and $U_2(13, -1) = 13$. Correspondingly

$$25 + U_j^2(5, -1) + U_n^2(5, -1) = 15U_j(5, -1)U_n(5, -1)$$

and

$$169 + U_j^2(13, -1) + U_n^2(13, -1) = 39U_j(13, -1)U_n(13, -1).$$

Proceeding as in Lemma 3.2 and using Lemma 2.1, we obtain

$$14U_j(5, -1) < U_n(5, -1) < 15U_j(5, -1) \text{ and } 38U_j(13, -1) < U_n(13, -1) < 39U_j(13, -1).$$

Further, the inequalities

$$U_{j+1}(5, -1) < 14U_j(5, -1) < U_n(5, -1) < 15U_j(5, -1) < U_{j+2}(5, -1)$$

and

$$U_{j+1}(13, -1) < 38U_j(13, -1) < U_n(13, -1) < 39U_j(13, -1)) < U_{j+2}(13, -1)$$

confirms the non existence of n satisfying (3).

Similarly, for the case $i = 1$, we have

$$1 + U_j^2 + U_n^2 = 3U_jU_n \quad \text{and} \quad 2U_j < U_n < 3U_j.$$

The inequality

$$U_j < 2U_j < U_n < 3U_j < U_{j+2}$$

gives the possibility $n = j + 1$ for which the Markov equation is

$$1 + U_{n-1}^2 + U_n^2 = 3U_{n-1}U_n \tag{14}$$

and Lemma 2.5 confirms the non existence of n satisfying (14) and hence, there is no solution of the form $(x, y, z) = (U_i, U_j, U_n)$ to (3) for all $r > 2, s = -1$. This completes the proof. □

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References

[1] Behera, A., & Panda, G. K. (1999). On the square roots of triangular numbers, *The Fibonacci Quarterly*, 37 (2), 98–105.

[2] Bolat, C., & Köse, H. (2010). On the Properties of k-Fibonacci Numbers, *Int. J. Contemp. Math. Sciences*, 5 (22), 1097–1105.

[3] Luca, F., & Srinivasan, A. (2018). Markov equation with Fibonacci components, *The Fibonacci Quarterly*, 56 (2), 126–129.

[4] Markoff, A. (1880). Sur les formes quadratiques binaires indéfinies, *Math. Ann.*, 17 (3), 379–399.

[5] Panda, G. K., & Ray, P. K. (2011). Some Links of Balancing and Cobalancing Numbers with Pell and Associated Pell Numbers, *Bull. Inst. Math., Acad. Sin.*, 6 (1), 41–72.

[6] Sloane, N. J. A. *On-Line Encyclopedia of Integer Sequences*, Available online at: `https://oeis.org/`.