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Notes on the Hermite-based poly-Euler polynomials with a *q*-parameter

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Abstract: We introduce and investigate the Hermite-based poly-Euler polynomials with a q-parameter. We give some basic properties and identities for these polynomials. Furthermore, we prove two explicit relations.
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1 Introduction

As usual, throughout this paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the complex numbers.

In the usual notations, let $B_n(x)$ and $E_n(x)$ denotes respectively, the classical Bernoulli polynomials and the classical Euler polynomials in x defined by the following generating functions, respectively;

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt},\ |t| < 2\pi \tag{1}$$

and

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt},\ |t| < \pi. \tag{2}$$

Also, let $x = 0$, $B_n(0) = B_n$ and $E_n(0) = E_n$, where B_n and E_n are respectively, the Bernoulli numbers and the Euler numbers.

The 2-variable Hermite–Kampé de Feriét polynomials are defined by (see [5, 16, 18])

$$\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = e^{xt+yt^2} \tag{3}$$

so that, obviously, we have the following relationships:

$$H_n(2x,-1) = H_n(x) \text{ and } H_n(x) = \left(\frac{i}{\sqrt{y}}\right)^n H_n(-2ix\sqrt{y}, y),\ \left(i = \sqrt{-1}\right) \tag{4}$$

with the classical Hermite polynomials $H_n(x)$, $n \in \mathbb{N}_0$.

Let $k \in \mathbb{Z}$, $k > 1$, the k-th polylogarithm function is defined by (see [1, 3, 7, 11, 16])

$$Li_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k},\ z \in \mathbb{C},\ z > 1 \tag{5}$$

when $k = 1$, $Li_1(z) = -\log(1-z)$. In the case $k \leq 0$, $Li_k(z)$ are the rational functions:

$$Li_0(z) = \frac{z}{1-z},\ Li_{-1}(z) = \frac{z}{(1-z)^2},\ Li_{-2}(z) = \frac{z^2+z}{(1-z)^3},\ \cdots.$$

Further information about polylogarithm function and polynomials (see [1–16]).

Hamahata in [7] defined the poly-Euler polynomials by

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(x)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{t\,(e^t+1)}e^{xt}, \tag{6}$$

for $k = 1$, we have $\mathcal{E}_n^{(k)}(x) = E_n(x)$.

Cenkci *et al.* in [3] defined the weighted Stirling numbers of the second kind as

$$\frac{e^{xt}\left(e^t-1\right)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k,x)\frac{t^n}{n!}. \tag{7}$$

Duran *et al.* in [5] defined the Hermite-based λ-Stirling polynomials of the second kind as

$$\frac{\left(\lambda e^t-1\right)^m}{m!}e^{xt+yt^j} = \sum_{n=0}^{\infty} S_2^{(\lambda,j)}(n,m,x,y)\frac{t^n}{n!}. \tag{8}$$

The special values of the (8) are given in [5].

Let $n,\ k \in \mathbb{Z}$, $n \geq 0$, $k > 0$ and $q \in \mathbb{R}\setminus\{0\}$. We define the Hermite-based poly-Euler polynomials with a q-parameter by the following generating functions:

$$\sum_{n=0}^{\infty} {}_H\mathcal{E}_{n,q}^{(k)}(x,y)\frac{t^n}{n!} = \frac{2qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{t\,(1+e^{qt})}e^{xt+yt^2}. \tag{9}$$

For $x = y = 0$, we get ${}_H\mathcal{E}_{n,q}^{(k)}(0,0) = {}_H\mathcal{E}_{n,q}^{(k)}$ which is called a new class of the Hermite-based poly-Euler numbers with a q-parameter. Some special cases of ${}_H\mathcal{E}_{n,q}^{(k)}(x,y)$ are following remarks.

Remark 1. For $y = 0$, we have ${}_H\mathcal{E}_{n,q}^{(k)}(x,0) = {}_H\mathcal{E}_{n,q}^{(k)}(x)$ called the Hermite-based poly-Euler polynomials with a q-parameter.

Remark 2. For $q = 1$, ${}_H\mathcal{E}_{n,q}^{(k)}(x,y)$ reduces to the Hermite-based poly-Euler polynomials.

Remark 3. For $q = 1$ and $y = 0$, ${}_H\mathcal{E}_{n,q}^{(k)}(x,y)$ reduces to poly-Euler polynomials which is defined Hamahata in [7].

Remark 4. When $q = k = 1$ and $y = 0$, we obtain the classical Euler polynomials.

Srivastava and Srivastava *et al.* in [20, 21] investigated some properties and proved some theorems for the Bernoulli, Euler and Genocchi polynomials. D. S. Kim *et al.* in [9–14] and T. Kim *et al.* in [15] introduced the poly-Bernoulli polynomials and gave some recurrences relations and identities. Cenkci *et al.* in [3] gave the poly-Bernoulli polynomials with a q-parameter. Kurt [16] gave the poly-Genocchi polynomials with a q-parameter. Duran *et al.* in [4–6] considered the (p,q)-Hermite polynomials and the (p,q)-Euler polynomials.

2 Main theorems

In this section, we give some basic identities and relations for the Hermite-based poly-Euler polynomial with a q-parameter. Further we give closed formula and explicit relation for these polynomials.

Theorem 2.1. *The Hermite-based poly-Euler polynomials with a q-parameter satisfy the following relation:*

$$ {}_H\mathcal{E}_{n,q}^{(k)}(x,y) = \sum_{m=0}^{n} \binom{n}{m} {}_H\mathcal{E}_{m,q}^{(k)}\ H_{n-m}(x,y), $$

$$ {}_H\mathcal{E}_{n,q}^{(k)}(x_1+x_2, y_1+y_2) = \sum_{m=0}^{n} \binom{n}{m} {}_H\mathcal{E}_{m,q}^{(k)}(x_1,y_1)\ {}_H\mathcal{E}_{n-m,q}^{(k)}(x_2,y_2) $$

and

$$ {}_H\mathcal{E}_{n,q}^{(k)}(x,y) = n! \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{{}_H\mathcal{E}_{n-2m,q}^{(k)}(x)}{(n-2m)!m!} y^m. $$

The proof of this theorem is easily obtained from (9).

Theorem 2.2. *The following relation holds true:*

$$ \begin{aligned} & n\ {}_H\mathcal{E}_{n,q}^{(k)}(x,y) + \sum_{v=0}^{n} \binom{n}{v} q^{n-v} v\ {}_H\mathcal{E}_{v-1,q}^{(k)}(x,y) \\ = \ & 2 \sum_{m=0}^{\infty} \frac{1}{q^m (m+1)^k} \sum_{r=0}^{m+1} \binom{m+1}{r} (-1)^r H_n(x-qr,y). \end{aligned} \tag{10} $$

Proof. By (3), (5) and (9), we write as

$$\sum_{n=0}^{\infty} n\ {}_{H}\mathcal{E}_{n-1,q}^{(k)}(x,y)\frac{t^n}{n!}\left(e^{qt}+1\right) = 2qLi_k\left(\frac{1-e^{-qt}}{q}\right)e^{xt+yt^2}.$$

The left-hand side of this equation is

$$\sum_{n=0}^{\infty}\left\{n\ {}_{H}\mathcal{E}_{n-1,q}^{(k)}(x,y)+\sum_{v=0}^{n}\binom{n}{v}q^{n-v}v\ {}_{H}\mathcal{E}_{v-1,q}^{(k)}(x,y)\right\}\frac{t^n}{n!}. \tag{11}$$

The right-hand side of this equation is

$$2q\sum_{m=0}^{\infty}\frac{\left(1-e^{-qt}\right)^{m+1}}{q^{m+1}}\frac{1}{(m+1)^k}\sum_{n=0}^{\infty}H_n\left(x,y\right)\frac{t^n}{n!}$$

$$\begin{aligned}
&= 2\sum_{m=0}^{\infty}\frac{1}{q^m\left(m+1\right)^k}\sum_{r=0}^{m+1}\binom{m+1}{r}\left(-1\right)^r e^{t(x-qr)+yt^2}\\
&= 2\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{\infty}\frac{1}{q^m\left(m+1\right)^k}\sum_{r=0}^{m+1}\binom{m+1}{r}\left(-1\right)^r H_n\left(x-qr,y\right)\right\}\frac{t^n}{n!}.
\end{aligned} \tag{12}$$

From (11) and (12), we obtain (10). □

Theorem 2.3. *The following relation between the Hermite-based poly-Euler polynomials with a q-parameter and the Euler polynomials holds:*

$${}_{H}\mathcal{E}_{n,q}^{(k)}(x,y)=\frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}\ {}_{H}\mathcal{E}_{m,q}^{(k)}(0,y)q^{n-m}\left(E_{n-m}\left(\frac{x}{q}+1\right)+E_{n-m}\left(\frac{x}{q}\right)\right). \tag{13}$$

Proof. By (2) and (9), we write

$$\sum_{n=0}^{\infty}{}_{H}\mathcal{E}_{n,q}^{(k)}(x,y)\frac{t^n}{n!}=\frac{2qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{t\left(e^{qt}+1\right)}e^{yt^2}\frac{e^{qt}+1}{2}\frac{2}{e^{qt}+1}e^{\frac{x}{q}qt}$$

$$\begin{aligned}
&= \frac{1}{2}\left\{\sum_{m=0}^{\infty}{}_{H}\mathcal{E}_{m,q}^{(k)}(0,y)\frac{t^m}{m!}\sum_{l=0}^{\infty}E_l\left(\frac{x}{q}+1\right)q^l\frac{t^l}{l!}\right.\\
&\quad\left.+\sum_{m=0}^{\infty}{}_{H}\mathcal{E}_{m,q}^{(k)}(0,y)\frac{t^m}{m!}\sum_{l=0}^{\infty}E_l\left(\frac{x}{q}\right)q^l\frac{t^l}{l!}\right\}.
\end{aligned}$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{n!}$, we have (13). □

Theorem 2.4. *The following relation between the Hermite-based poly-Euler polynomials with a q-parameter and the Bernoulli polynomials holds:*

$${}_{H}\mathcal{E}_{n-1,q}^{(k)}(x,y)=\frac{1}{n}\sum_{m=0}^{n}\binom{n}{m}\ {}_{H}\mathcal{E}_{m,q}^{(k)}(0,y)\left\{-B_{n-m}\left(\frac{x}{q}\right)+B_{n-m}\left(1+\frac{x}{q}\right)\right\}q^{n-m-1}. \tag{14}$$

Proof. From (1) and (9), we write as:

$$\sum_{n=0}^{\infty} {}_H\mathcal{E}_{n,q}^{(k)}(x,y)\frac{t^n}{n!} = \frac{2qe^{yt^2}Li_k\left(\frac{1-e^{-qt}}{q}\right)}{t\left(e^{qt}+1\right)}\frac{1-e^{qt}}{q}\frac{q}{1-e^{qt}}e^{\frac{x}{q}qt}$$

$$= \frac{1}{qt}\left\{\frac{2qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{t\left(e^{qt}+1\right)}e^{yt^2}\frac{qte^{qt\left(\frac{x}{q}\right)}}{e^{qt}-1} - \frac{2qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{t\left(e^{qt}+1\right)}e^{yt^2}\frac{qte^{qt\left(\frac{x}{q}+1\right)}}{e^{qt}-1}\right\}$$

$$= \frac{1}{q}\left\{\sum_{m=0}^{\infty} -{}_H\,\mathcal{E}_{m,q}^{(k)}(0,y)\frac{t^m}{m!}\sum_{l=0}^{\infty}B_l\left(\frac{x}{q}\right)q^l\frac{t^l}{l!}\right.$$

$$\left. + \sum_{m=0}^{\infty} {}_H\mathcal{E}_{m,q}^{(k)}(0,y)\frac{t^m}{m!}\sum_{l=0}^{\infty}B_l\left(\frac{x}{q}+1\right)\frac{q^lt^l}{l!}\right\}.$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{n!}$, we have (14). □

Theorem 2.5. *The following relations hold true:*

$$n\,{}_H\mathcal{E}_{n-1,q}^{(k)}(x,y) = 2\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^{r+s}H_n\left(x+qs-qr,y\right). \tag{15}$$

Proof. By (9),

$$\sum_{n=0}^{\infty} n\,{}_H\mathcal{E}_{n-1,q}^{(k)}(x,y)\frac{t^n}{n!} = \frac{2qe^{xt+yt^2}}{t\left(e^{qt}+1\right)}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\frac{\left(1-e^{-qt}\right)^{m+1}}{q^{m+1}}$$

$$= 2\sum_{s=0}^{\infty}(-1)^s e^{qts}e^{xt+yt^2}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^r e^{-qrt}$$

$$= 2\sum_{n=0}^{\infty}\left\{\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^{r+s}H_n\left(x+qs-qr,y\right)\right\}\frac{t^n}{n!}.$$

From here, we have (15). □

Corollary 2.5.1. *We have the following relation from* (10) *and* (15)*:*

$$\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^{r+s}H_n\left(x+qs-qr,y\right)$$

$$+\sum_{v=0}^{n}\binom{n}{v}q^{n-v}v\,{}_H\mathcal{E}_{n-1,q}^{(k)}(x,y)$$

$$= \sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^r H_n\left(x-qr,y\right).$$

Theorem 2.6. *The following relationship between the Hermite-based poly-Euler polynomials with a q-parameter and the Stirling numbers of the second kind holds:*

$$n\,{}_H\mathcal{E}_{n-1,q}^{(k)}(x,y) + \sum_{m=0}^{n}\binom{n}{m}q^{n-m}m\,{}_H\mathcal{E}_{m-1,q}^{(k)}(x,y)$$

$$= 2\sum_{m=0}^{\infty}\frac{m!\,(-1)^{m+1+n}}{(m+1)^k}\sum_{r=0}^{n}\binom{n}{r}q^{r-m}H_{n-r}\left(x,y\right)\left(S_2\left(r,m,1\right)-S_2\left(r,m\right)\right). \tag{16}$$

Proof. By (7) and (9), we write as:

$$\begin{aligned}
&\sum_{n=0}^{\infty} n\ {}_H\mathcal{E}_{n-1,q}^{(k)}(x,y)\frac{t^n}{n!} + e^{qt}\sum_{n=0}^{\infty} n\ {}_H\mathcal{E}_{n-1,q}^{(k)}(x,y)\frac{t^n}{n!} \\
= \;& 2qLi_k\left(\frac{1-e^{-qt}}{q}\right)e^{xt+yt^2}.
\end{aligned}$$

The left-hand side of this equation is

$$\sum_{n=0}^{\infty}\left\{n\ {}_H\mathcal{E}_{n-1,q}^{(k)}(x,y) + \sum_{m=0}^{n}\binom{n}{m}q^{n-m}m\ {}_H\mathcal{E}_{m-1,q}^{(k)}(x,y)\right\}\frac{t^n}{n!}. \tag{17}$$

The right-hand side of this equation

$$\begin{aligned}
&2q\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+1)^k}\frac{(e^{-qt}-1)^{m+1}}{q^{m+1}}\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!} \\
= \;& 2q\sum_{m=0}^{\infty}\frac{m!}{q^{m+1}}\frac{(-1)^{m+1}}{(m+1)^k}\frac{(e^{-qt}-1)^m e^{-qt}}{m!}\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!} \\
&-2q\sum_{m=0}^{\infty}\frac{m!}{q^{m+1}}\frac{(-1)^{m+1}}{(m+1)^k}\frac{(e^{-qt}-1)^m}{m!}\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!} \\
= \;& 2q\sum_{m=0}^{\infty}\frac{m!\,(-1)^{m+1}}{q^{m+1}(m+1)^k}\sum_{r=0}^{\infty}S_2(r,m,1)\frac{(-qt)^r}{r!}\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!} \\
&-2q\sum_{m=0}^{\infty}\frac{m!\,(-1)^{m+1}}{q^{m+1}(m+1)^k}\sum_{r=0}^{\infty}S_2(r,m)\frac{(-qt)^r}{r!}\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}.
\end{aligned}$$

By using Cauchy product, we have

$$=\sum_{n=0}^{\infty}\left\{2\sum_{m=0}^{\infty}\frac{m!\,(-1)^{m+1+n}}{(m+1)^k}\sum_{r=0}^{n}\binom{n}{r}q^{r-m}H_{n-r}(x,y)\left(S_2(r,m,1)-S_2(r,m)\right)\right\}\frac{t^n}{n!}. \tag{18}$$

From (17) and (18), we get (16). □

Theorem 2.7. *The following relation holds true:*

$${}_H\mathcal{E}_{n+m,q}^{(k)}(x,y) = \sum_{p=0}^{n}\sum_{r=0}^{m}\binom{n}{p}\binom{m}{r}(x-v)^{p+r}\ {}_H\mathcal{E}_{n+m-p-r,q}^{(k)}(x,y). \tag{19}$$

Proof. By (9),

$$\sum_{n=0}^{\infty}{}_H\mathcal{E}_{n,q}^{(k)}(x,y)\frac{t^n}{n!} = \frac{2qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{t\left(1+e^{qt}\right)}e^{xt+yt^2}. \tag{20}$$

We replace t by $t+u$ in (20)

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}{}_H\mathcal{E}_{n+m,q}^{(k)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!} = \frac{2qLi_k\left(\frac{1-e^{-q(t+u)}}{q}\right)}{(t+u)\left(1+e^{q(t+u)}\right)}e^{x(t+u)+y(t+u)^2}.$$

From this equation, we write as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {}_{H}\mathcal{E}_{n+m,q}^{(k)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!}e^{-x(t+u)} = \frac{2qLi_k\left(\frac{1-e^{-q(t+u)}}{q}\right)}{(t+u)\left(1+e^{q(t+u)}\right)}e^{y(t+u)^2}. \tag{21}$$

In the last equation, we replace x by v, we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {}_{H}\mathcal{E}_{n+m,q}^{(k)}(v,y)\frac{t^n}{n!}\frac{u^m}{m!}e^{-v(t+u)} = \frac{2qLi_k\left(\frac{1-e^{-q(t+u)}}{q}\right)}{(t+u)\left(1+e^{q(t+u)}\right)}e^{y(t+u)^2}. \tag{22}$$

By (21) and (22), we write

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {}_{H}\mathcal{E}_{n+m,q}^{(k)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!} = e^{(x-v)(t+u)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {}_{H}\mathcal{E}_{n+m,q}^{(k)}(v,y)\frac{t^n}{n!}\frac{u^m}{m!}. \tag{23}$$

Now, by applying the following known series identity [22, p.52, Eq. 1.6(2)]

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!} \tag{24}$$

in the right-hand side of (23), we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {}_{H}\mathcal{E}_{n+m,q}^{(k)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!} = \sum_{p=0}^{\infty}\sum_{r=0}^{\infty}(x-v)^{p+r}\frac{t^p}{p!}\frac{u^r}{r!}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} {}_{H}\mathcal{E}_{n+m,q}^{(k)}(v,y)\frac{t^n}{n!}\frac{u^m}{m!}. \tag{25}$$

Finally, upon first replacing n by $n-p$ and m by $m-r$ by using the Cauchy product in the left-hand side of the above equation (25) and comparing the coefficients of $\frac{t^n}{n!}$ and $\frac{u^m}{m!}$ on both sides of the resulting equation, we have (19). □

Theorem 2.8 (Closed Formula)**.** *The following relation holds true:*

$$\begin{aligned} n\ {}_{H}\mathcal{E}_{n-1,q}^{(-k)}(x,y) &= 2\sum_{l=0}^{\infty}(-1)^l\sum_{m=0}^{\min(n,k)}(m!)^2 S_2^{q^{-1}}(k,m,1)\left\{S_2^{(1,2)}\left(n,m;\frac{x}{q}+1+l,\frac{y}{q^2}\right)q^n\right. \\ &\quad \left.-S_2^{(1,2)}\left(n,m;\frac{x}{q}+l,\frac{y}{q^2}\right)q^n\right\}. \end{aligned} \tag{26}$$

Proof. By replacing k by $(-k)$ in (9). We get

$$\sum_{n=0}^{\infty} n\ {}_{H}\mathcal{E}_{n-1,q}^{(-k)}(x,y)\frac{t^n}{n!} = \frac{2qLi_{-k}\left(\frac{1-e^{-qt}}{q}\right)}{(1+e^{qt})}e^{xt+yt^2}$$

$$\begin{aligned} \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} n\ {}_{H}\mathcal{E}_{n-1,q}^{(-k)}(x,y)\frac{t^n}{n!}\frac{u^k}{k!} &= \frac{2q}{e^{qt}+1}\sum_{m=0}^{\infty}\left(\frac{1-e^{-qt}}{q}\right)^{m+1}(m+1)^k e^{xt+yt^2}\frac{u^k}{k!} \\ &= \frac{2q}{e^{qt}+1}e^{xt+yt^2}\left(\frac{1-e^{-qt}}{q}\right)e^u\sum_{m=0}^{\infty}\left(\left(\frac{1-e^{-qt}}{q}\right)e^u\right)^m \\ &= \frac{2e^{xt+yt^2}}{e^{qt}+1}\left(1-e^{-qt}\right)e^u\frac{e^{qt}}{1-(e^{qt}-1)(q^{-1}e^u-1)} \\ &= 2\sum_{l=0}^{\infty}(-1)^l e^{qlt}\left(e^{qt}-1\right)\sum_{m=0}^{\infty}\left(e^{qt}-1\right)^m e^u\left(q^{-1}e^u-1\right)^m e^{xt+yt^2} \end{aligned}$$

$$
\begin{aligned}
= \ & 2\sum_{l=0}^{\infty}(-1)^{l}\left\{\sum_{m=0}^{\infty}e^{(x+q+ql)t+yt^{2}}\left(e^{qt}-1\right)^{m}e^{u}\left(q^{-1}e^{u}-1\right)^{m}\right.\\
&\left.-\sum_{m=0}^{\infty}e^{(x+ql)t+yt^{2}}\left(e^{qt}-1\right)^{m}e^{u}\left(q^{-1}e^{u}-1\right)^{m}\right\}. \qquad (27)
\end{aligned}
$$

For $\lambda = 1$ and $j = 2$ in (8), we get

$$
\sum_{n=0}^{\infty}S_{2}^{(1,2)}\left(n,m;x,y\right)\frac{t^{n}}{n!}=\frac{\left(e^{t}-1\right)^{m}}{m!}e^{xt+yt^{2}}. \qquad (28)
$$

We put the equation (8) and (28) in (27). We have

$$
\begin{aligned}
= \ & 2\sum_{l=0}^{\infty}(-1)^{l}\left\{\sum_{m=0}^{\infty}\left[m!\sum_{n=0}^{\infty}S_{2}^{(1,2)}\left(n,m;\frac{x}{q}+1+l,\frac{y}{q^{2}}\right)q^{n}\frac{t^{n}}{n!}\right]\left[m!\sum_{k=0}^{\infty}S_{2}^{q^{-1}}\left(k,m1\right)\frac{u^{k}}{k!}\right]\right.\\
&\left.-\sum_{m=0}^{\infty}\left[m!\sum_{n=0}^{\infty}S_{2}^{(1,2)}\left(n,m;\frac{x}{q}+l,\frac{y}{q^{2}}\right)q^{n}\frac{t^{n}}{n!}\right]\left[m!\sum_{k=0}^{\infty}S_{2}^{q^{-1}}\left(k,m1\right)\frac{u^{k}}{k!}\right]\right\}.
\end{aligned}
$$

From the last equation, comparing the coefficients of $\frac{t^n}{n!}$ and $\frac{u^k}{k!}$, we have (26). □

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