Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 26, 2020, No. 3, 68–73
DOI: 10.7546/nntdm.2020.26.3.68-73

On quasimultiperfect numbers

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Received: 20 November 2019 **Revised:** 5 August 2020 **Accepted:** 5 August 2020

Abstract: For a positive integer n, let $\sigma(n)$ and $\omega(n)$ respectively denote the sum of the positive divisors of n and the number of distinct prime factors of n. A positive integer n is called a *quasimultiperfect* (QM) number if $\sigma(n) = kn + 1$ for some integer $k \geq 2$. In this paper we give some necessary conditions to be satisfied by the prime factors of QM number n with $\omega(n) = 3$ and $\omega(n) = 4$. Also we show that no QM n with $\omega(n) = 4$ can be a fourth power of an integer.
Keywords: Quasimultiperfect number, Quasitriperfect number.
2010 Mathematics Subject Classification: 11A25.

1 Introduction

For a positive integer n, let $\sigma(n)$ and $\omega(n)$ respectively denote the sum of the positive divisors of n and the number of distinct prime factors of n. Tang Min and Meng Li [4] called a positive integer n *quasimultiperfect* (QM) number if $\sigma(n) = kn+1$ for some integer $k \geq 2$. In particular, a positive integer n is said to be *quasiperfect* (QP) if $\sigma(n) = 2n + 1$ and *quasitriperfect* (QT) if $\sigma(n) = 3n + 1$. No QM number is known so far. P. Cattaneo [2] started the study of QP numbers which was continued in [1] and later by several researchers, the details of which can be seen in the book [7, p.38-39] and in recent papers [5] and [6].

If a QM number n exists, then it is shown in [4, Theorem 1] that $\omega(n) \geq 7$ or 3 according as n is odd or even. Also it is proved:

Lemma 1.1 ([4, Theorem 2]). *If n is an even QM with $\omega(n) = 3$, then n is QT and is of the form $n = 2^\alpha.3^\beta.p^2$, where α and β are even integers and p is an odd prime. Also $p = [F(\alpha, \beta)]$ in which $F(\alpha, \beta) = 2^{\alpha+1}.3^{\beta+1}/(2^{\alpha+1} + 3^{\beta+1} - 1)$ (Here $[x]$, as usual, denotes the greatest integer not exceeding the real number x).*

Lemma 1.2 ([3, Theorem 1.1]). *If n is QM with $\omega(n) = 4$, then n is QT and is of the form $n = 2^\alpha.3^\beta.p^\gamma.q^\delta$, where α, β, γ and δ are even integers and $p < q$ are odd primes.*

The purpose of this paper is to give some necessary conditions on the prime p in Lemma 1.1 and on the primes p and q in Lemma 1.2. Also we establish that no QM number with $\omega(n) = 4$ can be a fourth power of an integer In fact, we prove the following:

Theorem A. The odd prime p in Lemma 1.1 is such that

$$(i)\ p \equiv 29 \pmod{36} \text{ if } \alpha \equiv 0 \pmod 6$$
$$(ii)\ p \equiv 17 \pmod{36} \text{ if } \alpha \equiv 2 \pmod 6$$
$$\text{and } (iii)\ p \equiv 5 \pmod{36} \text{ if } \alpha \equiv 4 \pmod 6.$$

Theorem B. Suppose n is QM with $\omega(n) = 4$ and is of the form given in Lemma 1.2. Suppose $p \equiv a \pmod 8$ and $q \equiv b \pmod 8$. Then

$$(i)\ (a, b) \notin \{(3, 3), (3, 7), (7, 3), (7, 7)\}$$
$$(ii)\ (a, b) \in \{(1, 3), (5, 3), (1, 7), (5, 7)\} \text{ implies } \gamma \equiv 2 \pmod 4$$
$$(iii)\ (a, b) \in \{(3, 1), (3, 5), (7, 1), (7, 5)\} \text{ implies } \delta \equiv 2 \pmod 4$$
$$\text{and } (iv)\ (a, b) \in \{(1, 1), (1, 5), (5, 1), (5, 5)\} \text{ implies } \gamma \equiv 2 \pmod 4 \text{ or } \delta \equiv 2 \pmod 4.$$

Remark 1.3. In view of Theorem B, one of p^γ and q^δ in Lemma 1.2 is not a fourth power and therefore the number n in it cannot be a fourth power. That is, any QM n with $\omega(n) = 4$ is not a fourth power.

2 On QM numbers n with $\omega(n) = 3$

In this section n always denotes a QM number with $\omega(n) = 3$ so that, by Lemma 1.1., $3n + 1 = \sigma(n)$ and is of the form

$$n = 2^\alpha.3^\beta.p^2, \tag{2.1}$$

where α and β are even integers and p is an odd prime given by $p = [F(\alpha, \beta)]$. Therefore we have

$$\begin{aligned} 3.2^\alpha.3^\beta.p^2 + 1 &= \sigma(2^\alpha).\sigma(3^\beta).\sigma(p^2) \\ &= (2^{\alpha+1} - 1).\left(\frac{3^{\beta+1} - 1}{2}\right).\left(1 + p + p^2\right), \end{aligned}$$

which can be written as

$$2^{\alpha+1}.3^{\beta+1}.p^2 + 2 = (2^{\alpha+1} - 1).\left(3^{\beta+1} - 1\right).\left(1 + p + p^2\right). \tag{2.2}$$

First we prove

Lemma 2.1. $\alpha \geq 6$ and $\beta \geq 6$.

Proof. Note that $\alpha, \beta \in \{2, 4, 6, 8, ...\}$ and that $F(2,2) = 6.352, F(2,4) = 7.776$, $F(4,2) = 14.896$ and $F(4,4) = 28.3785$. Therefore $[F(2,2)] = 6, [F(2,4)] = 7$, $[F(4,2)] = 14$ and [F(4,4)]=28 showing that for $(\alpha, \beta) \in \{(2,2), (4,2), (4,4)\}$ the values of $[F(\alpha, \beta)]$ are composite so that the corresponding numbers n given in (2.1) are not QT. Also if $(\alpha, \beta) = (2, 4)$, then $p = 7$ so that in this case $n = 2^2.3^4.7^2 = 15876$ for which $\sigma(n) = \sigma(2^2).\sigma(3^4).\sigma(7^2)$ $= 7.121.57 = 48279$ showing $3n + 1 \neq \sigma(n)$. Therefore it is not QT. Thus for QT of the form (2.1) we have $(\alpha, \beta) \notin \{(2,2), (2,4), (4,2), (4,4)\}$, proving the lemma. □

Lemma 2.2. *The prime* p *in* (2.1) *is such that* $p \equiv 1 \pmod 4$.

Proof. For the integers α and β, it is clear that $2^{\alpha+1} - 1 \equiv -1 \pmod 8$ and $3^{\beta+1} - 1 \equiv 2 \pmod 8$ so that

$$\left(2^{\alpha+1} - 1\right) . \left(3^{\beta+1} - 1\right) \equiv -2 \pmod 8. \tag{2.3}$$

Writing the equation (2.2) to congruence modulo 8 and using (2.3) we get

$$2 \equiv -2 \left(1 + p + p^2\right) \pmod 8.$$

That is, the prime p should satisfy

$$2 + p + p^2 \equiv 0 \pmod 4. \tag{2.4}$$

Now p, being an odd prime, we have $p \equiv 1$ or $3 \pmod 4$ and in both cases $p^2 \equiv 1 \pmod 4$. Here (2.4) holds only if $p \equiv 1 \pmod 4$, proving the lemma. □

Lemma 2.3. *The prime* p *in* (2.1) *is such that* $p \equiv 2, 8$ or $5 \pmod 9$ *according to* $\alpha \equiv 0, 2$ or $4 \pmod 6$.

Proof. Write the equation (2.2) to congruence modulo 9 and use the fact that $3^{\beta+1} - 1 \equiv -1 \pmod 9$ for $\beta \geq 1$ to get

$$2 + G(\alpha, p) \equiv 0 \pmod 9, \tag{2.5}$$

where $G(\alpha, p) = (2^{\alpha+1} - 1)(1 + p + p^2)$.

Now α, being an even integer, $\alpha \equiv 0, 2$ or $4 \pmod 6$ so that $\alpha = 6k, 6k + 2$ or $6k + 4$ for some integer $k \geq 1$ (in view of Lemma 2.1). Hence

$$2^{\alpha+1} - 1 = \begin{cases} 2\left(2^6\right)^k - 1 & \text{if } \alpha \equiv 0 \pmod 6 \\ 2^3\left(2^6\right)^k - 1 & \text{if } \alpha \equiv 2 \pmod 6 \\ 2^5\left(2^6\right)^k - 1 & \text{if } \alpha \equiv 4 \pmod 6, \end{cases}$$

so that

$$2^{\alpha+1} - 1 \equiv \begin{cases} 1 \pmod 9 & \text{if } \alpha \equiv 0 \pmod 6 \\ 7 \pmod 9 & \text{if } \alpha \equiv 2 \pmod 6 \\ 4 \pmod 9 & \text{if } \alpha \equiv 4 \pmod 6, \end{cases} \tag{2.6}$$

since $2^6 \equiv 1 \pmod 9$.

For an odd prime p we have $p \equiv 1, 2, 4, 5, 7$ or $8 \pmod 9$ and in these respective cases $p^2 \equiv 1, 4, 7, 7, 4$ or $1 \pmod 9$. Therefore,

$$1 + p + p^2 \equiv \begin{cases} 3 \pmod 9 & \text{if } p \equiv 1 \pmod 9 \\ 7 \pmod 9 & \text{if } p \equiv 2 \pmod 9 \\ 3 \pmod 9 & \text{if } p \equiv 4 \pmod 9 \\ 4 \pmod 9 & \text{if } p \equiv 5 \pmod 9 \\ 3 \pmod 9 & \text{if } p \equiv 7 \pmod 9 \\ 1 \pmod 9 & \text{if } p \equiv 8 \pmod 9). \end{cases} \tag{2.7}$$

For different cases of $\alpha \equiv 0, 2$ or $4 \pmod 6$ and for different cases of $p \equiv 1, 2, 4, 5, 7$ or $8 \pmod 9$, the values of k such that $G(\alpha, p) \equiv k \pmod 9$ are given in Table 1 below, using (2.6) and (2.7):

	$p \equiv 1 \pmod 9$	$p \equiv 2 \pmod 9$	$p \equiv 4 \pmod 9$	$p \equiv 5 \pmod 9$	$p \equiv 7 \pmod 9$	$p \equiv 8 \pmod 9$
$\alpha \equiv 0 \pmod 6$	3	7	3	4	3	1
$\alpha \equiv 2 \pmod 6$	3	4	3	1	3	7
$\alpha \equiv 4 \pmod 6$	3	1	3	7	3	4

Table 1. The values of k such that $G(\alpha, p) \equiv k \pmod 9$.

It is clear from the Table 1 that (2.5) holds only in the cases (i) $\alpha \equiv 0 \pmod 6$, $p \equiv 2 \pmod 9$ (ii) $\alpha \equiv 2 \pmod 6$, $p \equiv 8 \pmod 9$ and (iii) $\alpha \equiv 4 \pmod 6$, $p \equiv 5 \pmod 9$, proving the lemma. □

Proof of Theorem A. (i) Suppose $\alpha \equiv 0 \pmod 6$, so that by Lemma 2.3, we have $p \equiv 2 \pmod 9$. Also by Lemma 2.2, $p \equiv 1 \pmod 4$. Hence by the Chinese remainder theorem, we have $p \equiv 29 \pmod{36}$.

Parts (ii) and (iii) of Theorem A can be proved similarly, using Lemmas 2.2 and 2.3. □

3 On QM numbers n with $\omega(n) = 4$

Throughout this section n stands for a QM number with $\omega(n) = 4$ so that, by Lemma 1.2, $\sigma(n) = 3n + 1$ and n is of the form

$$n = 2^\alpha.3^\beta.p^\gamma.q^\delta, \tag{3.1}$$

where α, β, γ and δ are even integers; and $p < q$ are odd primes. Therefore

$$3.2^\alpha.3^\beta.p^\gamma.q^\delta + 1 = \left(2^{\alpha+1} - 1\right)\left(\frac{3^{\beta+1} - 1}{2}\right).\sigma\left(p^\gamma\right).\sigma\left(q^\delta\right)$$

which can be written as

$$2^{\alpha+1}.3^{\beta+1}.p^\gamma.q^\delta + 2 = \left(2^{\alpha+1} - 1\right)\left(3^{\beta+1} - 1\right).\sigma\left(p^\gamma\right).\sigma\left(q^\delta\right). \tag{3.2}$$

Proof of Theorem B. Writing the equation (3.2) to congruence modulo 8 and using (2.3) we get that p, q, γ and δ of (3.1) must satisfy that $2 \equiv -2.\sigma\left(p^{\gamma}\right)\sigma\left(q^{\delta}\right) \pmod 8$ or equivalently

$$\sigma\left(p^{\gamma}\right)\sigma\left(q^{\delta}\right)+1 \equiv 0 \pmod 4. \tag{3.3}$$

Now $p \equiv 1, 3, 5$ or $7 \pmod 8$ so that $p^2 \equiv 1 \pmod 8$ in each case. Let $\gamma = 2a$ for some $a \geq 1$. Then

$$\begin{aligned}
\sigma\left(p^{\gamma}\right) &= \sigma\left(p^{2a}\right) = 1+p+p^2+...+p^{2a}\\
&= (1+p)\left(1+p^2+...+p^{2(a-1)}\right)+p^{2a}\\
&\equiv (1+p)a+1 \pmod 8\\
&\equiv \begin{cases} 2a+1 \pmod 8 & \text{if } p \equiv 1 \pmod 8\\ 4a+1 \pmod 8 & \text{if } p \equiv 3 \pmod 8\\ 6a+1 \pmod 8 & \text{if } p \equiv 5 \pmod 8\\ 1 \quad \pmod 8 & \text{if } p \equiv 7 \pmod 8 \end{cases}
\end{aligned}$$

which shows

$$\sigma\left(p^{\gamma}\right) \equiv \begin{cases} \gamma+1 \pmod 8 & \text{if } p \equiv 1 \pmod 8\\ 2\gamma+1 \pmod 8 & \text{if } p \equiv 3 \pmod 8\\ 3\gamma+1 \pmod 8 & \text{if } p \equiv 5 \pmod 8\\ 1 \quad \pmod 8 & \text{if } p \equiv 7 \pmod 8 \end{cases} \tag{3.4}$$

Similarly

$$\sigma\left(q^{\delta}\right) \equiv \begin{cases} \delta+1 \pmod 8 & \text{if } q \equiv 1 \pmod 8\\ 2\delta+1 \pmod 8 & \text{if } q \equiv 3 \pmod 8\\ 3\delta+1 \pmod 8 & \text{if } q \equiv 5 \pmod 8\\ 1 \quad \pmod 8 & \text{if } q \equiv 7 \pmod 8 \end{cases} \tag{3.5}$$

Table 2 gives the values of k such that $\sigma\left(p^{\gamma}\right)\sigma\left(q^{\delta}\right) \equiv k \pmod 8$ for different cases of $p \equiv 1, 3, 5$ or $7 \pmod 8$ and for different cases of $q \equiv 1, 3, 5$ or $7 \pmod 8$.

	$p \equiv 1 \pmod 8$	$p \equiv 3 \pmod 8$	$p \equiv 5 \pmod 8$	$p \equiv 7 \pmod 8$
$q \equiv 1 \pmod 8$	$(\gamma+1)(\delta+1)$	$(2\gamma+1)(\delta+1)$	$(3\gamma+1)(\delta+1)$	$\delta+1$
$q \equiv 3 \pmod 8$	$(\gamma+1)(2\delta+1)$	$(2\gamma+1)(2\delta+1)$	$(3\gamma+1)(2\delta+1)$	$2\delta+1$
$q \equiv 5 \pmod 8$	$(\gamma+1)(3\delta+1)$	$(2\gamma+1)(3\delta+1)$	$(3\gamma+1)(3\delta+1)$	$3\delta+1$
$q \equiv 7 \pmod 8$	$(\gamma+1)$	$(2\gamma+1)$	$(3\gamma+1)$	1

Table 2. The values of k such that $\sigma\left(p^{\gamma}\right)\sigma\left(q^{\delta}\right) \equiv k \pmod 8$.

In view of (3.3), the values of k must satisfy the condition

$$k+1 \equiv 0 \pmod 4. \tag{3.6}$$

Now Theorem B follows from (3.6) and the Table 2. For instance,

(i) if $(a, b) = (3, 3)$, then $(2\gamma + 1)(2\delta + 1) + 1 \equiv 0 \pmod 4$ and this is impossible since γ and δ are even integers, giving $(a, b) \neq (3, 3)$.

(ii) if $(a, b) = (5, 3)$, then $(3\gamma + 1)(2\delta + 1) + 1 \equiv 0 \pmod 4$ holds only for $(\gamma + 1) + 1 \equiv 0 \pmod 4$ giving $\gamma \equiv 2 \pmod 4$. That is, $(a, b) = (5, 3) \Rightarrow \gamma \equiv 2 \pmod 4$.

(iii) if $(a, b) = (3, 1)$, then $(2\gamma + 1)(\delta + 1) + 1 \equiv 0 \pmod 4 \Rightarrow \delta + 2 \equiv 0 \pmod 4$ or $\delta \equiv 2 \pmod 4$. That is, $(a, b) = (3, 1) \Rightarrow \delta \equiv 2 \pmod 4$.

(iv) if $(a, b) = (1, 5)$, then $(\gamma + 1)(3\delta + 1) + 1 \equiv 0 \pmod 4 \Rightarrow \gamma + \delta + 2 \equiv 0 \pmod 4$ showing either $\gamma \equiv 2 \pmod 4$ or $\delta \equiv 2 \pmod 4$, but not both. That is, $(a, b) = (1, 5) \Rightarrow \gamma \equiv 2 \pmod 4$ or $\delta \equiv 2 \pmod 4$.

The other cases can be proved similarly. □

Acknowledgements

The authors are grateful to the referees for their valuable suggestions in improving the quality of the presentation.

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