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On generalized order-k modified Pell and Pell–Lucas numbers in terms of Fibonacci and Lucas numbers

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Abstract: This study shows that the generalized order-k Pell–Lucas and Modified Pell numbers can be expressed in terms of the well-known Fibonacci numbers. Certain n-square Hessenberg matrices with permanents equal to the Fibonacci numbers are defined. These Hessenberg matrices are then extended to super-diagonal $(0, 1, 2)$-matrices. In particular, the permanents of the super-diagonal matrices are shown to equal the components of the generalized order-k Pell–Lucas and Modified Pell numbers, and also their sums. In addition, two computer algorithms regarding our results are composed.
Keywords: Fibonacci number, Hessenberg matrix, Generalized modified Pell numbers, Super-diagonal matrix, Permanent.
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1 Introduction

The well-known Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ are defined by the recurrence relation

$$F_{n+1} = F_n + F_{n-1},\ n \geqslant 1 \tag{1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. The Lucas numbers $\{L_n\}_{n=0}^{\infty}$ are defined by the same recurrence relation but with different initial condition $(L_0 = 2$ and $L_1 = 1)$. The Fibonacci numbers have many properties and are exploited in many applications. Well-known systematic investigations of the Fibonacci and Lucas sequences are presented in [7, 8]. The Pell numbers $\{P_n\}_{n=0}^{\infty}$ form another sequence with the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2},\ n \geqslant 1 \tag{2}$$

starting from $P_0 = 0$ and $P_1 = 1$. The classical Pell–Lucas $\{Q_n\}_{n=0}^{\infty}$ and the Modified Pell $\{q_n\}_{n=0}^{\infty}$ numbers are defined by the same recurrence but with initial terms $Q_0 = Q_1 = 2$ and $q_0 = q_1 = 1$ respectively. It should be noted that the Pell–Lucas and Modified Pell numbers are related through $Q_n = 2q_n$ (see the reference in [5]). Therefore, the known properties of the Pell–Lucas numbers can also be written for the Modified Pell numbers. A researcher studying one sequence will inevitably become familiar with the other.

Many investigations of Fibonacci and Pell numbers by numerous researchers have been published. Horadam considered many properties of the usual and Modified Pell numbers [4, 5], and Ercolano derived the generating matrices of Pell sequences [3]. Daşdemir investigated certain properties of the Pell, Pell–Lucas and Modified Pell numbers by a matrix approach [1]. Employing various Hessenberg matrices, Kaygısız and Şahin presented certain determinantal and permanental representations of the generalized order-k Fibonacci numbers [6]. Daşdemir derived the recurrence relations corresponding to generalizations of the usual Pell–Lucas and Modified Pell numbers [2].

The present paper presents the generalized order-k Modified Pell and Pell–Lucas numbers in terms of Fibonacci numbers and obtains their permanental representations using Hessenberg and super-diagonal matrices. It also derives certain sum formulae from the super-diagonal matrices.

Before presenting our results, we remember that the generalized k-Modified Pell numbers are defined by Daşdemir [2] as follows:

$$q_n^i = 2q_{n-1}^i + q_{n-2}^i + \cdots + q_{n-k}^i \tag{3}$$

with initial conditions

$$q_n^i = \begin{cases} 1, & \text{if } n = 0 \\ -1, & \text{if } n + i = 1 \text{ and } i \neq 1 \quad \text{for } 1 - k \leq n \leq 0. \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Similarly, the generalized order-k Pell–Lucas numbers are defined by the same recurrence relation but with initial conditions

$$Q_n^i = \begin{cases} 2, & \text{if } n = 0 \\ -2, & \text{if } n + i = 1 \text{ and } i \neq 1 \quad \text{for } 1 - k \leq n \leq 0. \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

2 Main results

This section presents the main results of the paper, namely, that the generalized order-k Modified Pell and Pell–Lucas numbers can be represented in terms of the usual Fibonacci numbers. For this purpose, we first state the following lemma.

Lemma 2.1. *Let* $0 \leqslant t; s \leqslant k$. *Then*

$$q_{s+t+1}^k = F_{2(s+t+1)} - F_{2(s+t-k)} + \sum_{j=1}^{s+t-k} F_{2j} F_{2(s+t-k-j+1)}. \tag{6}$$

Proof. First consider the case of $s+t < k+1$. From the definition of the generalized order-k Modified Pell numbers given in [2], we directly obtain

$$q_1^k = F_2,\ q_2^k = F_4,\ q_3^k = F_6,\ \ldots,\ q_{k+1}^k = F_{2(k+1)}. \tag{7}$$

Now consider the case of $s+t \geqslant k+1$. When $s+t = k+1$, Eq. (7) gives

$$q_{k+2}^k = 2q_{k+1}^k + q_k^k + \cdots + q_2^k = 2F_{2(k+1)} + F_{2k} + \cdots + F_4. \tag{8}$$

Monograph [8] gives the famous summation formula comprising Fibonacci terms

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1. \tag{9}$$

Substituting Eq. (9) into the recurrence relation (8), we get

$$\begin{aligned} q_{k+2}^k &= F_{2(k+1)} + F_{2(k+1)} + F_{2k} + \cdots + F_4 + F_2 - F_2 \\ &= F_{2(k+1)} + \sum_{i=1}^{k+1} F_{2i} - F_2 \\ &= F_{2(k+2)} - F_2 - F_2 F_2 \end{aligned}$$

confirming that Lemma 2.1 is true for the present case. Assuming that Lemma 2.1 also holds in the case $k+2 \leqslant s+t \leqslant 2k-1$, we must validate the lemma for the case $s+t = 2k$. Under this assumption,

$$q_{2k}^k = F_{4k} - F_{2k-2} - \sum_{j=1}^{k-1} F_{2j} F_{2(k-j)}$$

can be written. Hence, it is seen that

$$\begin{aligned} q_{2k+1}^k &= 2q_{2k}^k + q_{2k-1}^k + \cdots + q_{k+2}^k + q_{k+1}^k \\ &= 2\left\{F_{4k} - F_{2(k-1)} - \sum_{j=1}^{k-1} F_{2j} F_{2(k-j)}\right\} + \left\{F_{2(2k-1)} - F_{2(k-2)} - \sum_{j=1}^{k-2} F_{2j} F_{2(k-1-j)}\right\} \\ &\qquad + \cdots + \left\{F_{2(k+2)} - F_2 - \sum_{j=1}^{1} F_{2j} F_{2(2-j)}\right\} + F_{2(k+1)} \\ &= F_{4k} - F_{2(k-1)} - \sum_{j=1}^{k-1} F_{2j} F_{2(k-j)} + \sum_{j=k+1}^{2k} F_{2j} - \sum_{j=1}^{k-1} F_{2j} - \sum_{i=1}^{k-1} \sum_{j=1}^{k-i} F_{2i} F_{2j} \\ &= F_{4k} - F_{2(k-1)} - \sum_{j=1}^{k-1} F_{2j} F_{2(k-j)} + \sum_{j=k+1}^{2k} F_{2j} - \sum_{j=1}^{k-1} F_{2j} - \sum_{i=1}^{k-1} F_{2i} \left(F_{2(k-i)+1} - 1\right) \end{aligned}$$

$$= F_{4k} - F_{2(k-1)} - \sum_{j=1}^{k-1} F_{2j}F_{2(k-j)} + \sum_{j=k+1}^{2k} F_{2j} - \sum_{i=1}^{k-1} F_{2i}\left(F_{2(k-i+1)} - F_{2(k-i)}\right)$$

$$= F_{4k} - F_{2(k-1)} + \sum_{j=k+1}^{2k} F_{2j} - \sum_{j=1}^{k-1} F_{2j}F_{2(k-j+1)}$$

$$= F_{4k} - F_{2(k-1)} + \sum_{j=1}^{2k} F_{2j} - \sum_{j=1}^{k} F_{2j} - \sum_{j=1}^{k-1} F_{2j}F_{2(k-j+1)}$$

$$= F_{4k+2} - F_{2k} - \sum_{j=1}^{k} F_{2j}F_{2(k-j+1)},$$

as desired. This completes the proof. □

Note that the generalized order-k Pell–Lucas numbers can be investigated by a similar approach. The author of [2] found that the generalized order-k Modified Pell and Pell–Lucas numbers are interrelated through

$${Q_n}^i = 2{q_n}^i, \tag{10}$$

where $1 \leqslant i \leqslant k$ and $n \geqslant 0$. Hence, the statement

$$Q^k_{s+t+1} = 2\left(F_{2(s+t+1)} - F_{2(s+t-k)} + \sum_{j=1}^{s+t-k} F_{2j}F_{2(s+t-k-j+1)}\right) \tag{11}$$

can be given. In addition, we can write

$$F_n + L_n = 2F_{n+1} \; for \; n \in \mathbb{Z}. \tag{12}$$

Consequently, Eq. (11) can be re-written as a combination of Fibonacci and Lucas numbers.

The main goal of the paper is to derive the generalized order-k Modified Pell and Pell–Lucas numbers via certain special matrices, namely, certain n-square Hessenberg matrices and super-diagonal matrices. After defining these matrices, we can obtain many important properties of the generalized order-k Modified Pell and Pell–Lucas numbers.

First, we introduce a super-diagonal matrix $P(k,n) = [p_{ij}]_{n\times n}$, $k \leq n$, with entries $p_{11} = p_{i+1,i} = 1$ for $1 \leq i \leq n-1$, $p_{ii} = 2$ for $2 \leq i \leq n$, $p_{ij} = 1$ for $i+1 \leq j \leq i+k-1$ and 0 otherwise. Mathematically, $P(k,n)$ is given by

$$P(k,n) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & 2 & 1 & \cdots & 1 & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 2 & 1 & \cdots & 1 & 0 \\ \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 2 & 1 & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 2 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1 & 2 \end{bmatrix}. \tag{13}$$

Theorem 2.2. *For the super-diagonal* $(0,1,2)$*-matrix* $P(k,n)$ *given in* (13)*, we have*

$$perP(k,n) = q_n^k, \; n \geq 1. \tag{14}$$

Proof. In order to access the proof, we first consider the case of $1 \leq t \leq k$ and $k = n$. Clearly, the matrix $P(k,n)$ reduces to $P(t,t)$, in the form of a Hessenberg matrix, which is denoted by K_t. We can write

$$\begin{aligned}
&perK_1 = 1 = F_2, \; perK_2 = 3 = F_4, \; perK_3 = 8 = F_6, \\
&perK_4 = 21 = F_8, perK_5 = 55 = F_{10}, \; \dots \; , perK_n = F_{2n}.
\end{aligned}$$

Hence, from Lemma 2.1, expanding the permanent of the matrix $P(t,t)$ with respect to the last row, yields

$$\begin{aligned}
perP(t,t) &= 2perK_{t-1} + perK_{t-2} + \cdots + perK_1 + 1 \\
&= 2F_{2(t-1)} + F_{2(t-2)} + \cdots + F_2 + 1 \\
&= F_{2t-2} + F_{2t-1} - 1 + 1 \\
&= F_{2t} \\
&= q_t^k.
\end{aligned}$$

Now, consider the case of $k < n$ and $k+1 \leq t \leq n$. When $t = k+1$, the permanent of $perP(k,n)$ is computed as a Laplace expansion of the permanent with respect to the last row, we get

$$\begin{aligned}
perP(k,k+1) &= 2perP(k,k) + perP(k,k-1) + \cdots + perP(k,1) \\
&= 2q_k^k + q_{k-1}^k + \cdots + q_1^k \\
&= q_{k+1}^k
\end{aligned}$$

Assuming that Eq. (14) holds for $k+1 \leq t \leq n$, we must prove the validity of Theorem 2.2 in the case of $n = t+1$. We thus have

$$\begin{aligned}
perP(k,t+1) &= perP(k,t) + perP(k,t-1) + \cdots + perP(k,t-k+1) \\
&= 2q_t^k + q_{t-1}^k + \cdots + q_{t-k+1}^k \\
&= q_{t+1}^k,
\end{aligned}$$

which completes the proof. □

As another main result, we define a new super-diagonal $(0,1,2)$-matrix $R(k,n) = [r_{ij}]_{n \times n}$, with $r_{1i} = 2$ for $2 \leq i \leq k$, $r_{ii} = 2$ for $1 \leq i \leq n$, $r_{i+1,i} = 1$ for $1 \leq i \leq n-1$, $r_{ij} = 1$ for $i+1 \leq j \leq i+k-1$ and $i \neq 1$ and 0 otherwise. Clearly,

$$R(k,n) = \begin{bmatrix} 2 & 2 & 2 & \cdots & 2 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & 2 & 1 & \cdots & 1 & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 2 & 1 & \cdots & 1 & 0 \\ \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 2 & 1 & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 2 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1 & 2 \end{bmatrix}. \tag{15}$$

We hence obtain the following theorem.

Theorem 2.3. *Let $R(k,n)$ take the form of Eq.* (15). *Then, for $n \geq 1$, we have*

$$perR(k,n) = Q_n^k. \tag{16}$$

Proof. It is easily be seen that $r_{1i} = 2p_{1i}$ for $1 \leqslant i \leqslant n$. The following result is then evident from matrix theory:

$$perR(k,n) = 2perP(k,n) = 2q_n^k = Q_t^k, \tag{17}$$

which completes the proof. □

3 Computer algorithm

We have presented our fundamental results above. It is well-known that one who studies on a generalized order-k sequence has three fundamental factors for determining the values of the sequence, i.e. i, j, and k. However, considering the foregoing results concludes that determining the permanents of the corresponding super-diagonal matrices $P(k,n)$ and $R(k,n)$ requires both cumbersome and very tedious computations; further, it can be very difficult to find the terms of the generalized order-k Modified Pell and Pell–Lucas numbers for larger values of k and n. To eliminate this obstacle, we, therefore, presented two computer algorithms, which are used for controlling and computing both the permanents of the mentioned super-diagonal matrices and the terms of the corresponding generalized sequence, based on the utilization of Mathematica© 11.2.

In this study, we only considered the case $i = k$ for the results. Consequently, we have two variables, namely k and n. Nevertheless, we designed the algorithms regarding our generalized sequences for the case wherein i, j, and k can arbitrarily be chosen so that these programs can then be used for other problems. Fig. 1 shows a computer algorithm related to the result of Theorem 2.2, while Fig. 2 exhibits a computer algorithm for the equation of Theorem 2.3.

```
(* Algorithm for the generalized order-k  Modified Pell number *)

k = Input["Please enter the order of the sequence."];

i = Input["Please enter the value of i."];

n = Input["Please enter the number of terms of the sequence."];

GMPN = Table[0, {n + k}];
Do[GMPN[[k - i + 1]] = -1;
  GMPN[[k]] = 1;
  GMPN[[tt]] = 2 * GMPN[[tt - 1]] + Sum[GMPN[[tt - j]], {j, 2, k}], {tt, k + 1, n + k}];

MP = Drop[GMPN, k - 1];

P = Table[0, {n}, {n}];
tt = 0;
Do[tt = tt + 1; pp = 0; Do[pp = pp + 1; P[[1, 1]] = 1; P[[j, j]] = 2;
   If[pp + tt ≤ n, P[[pp, pp + tt]] = 1]; If[j > 1, P[[j, j - 1]] = 1]
   , {j, 1, n}], {i, 1, k - 1}];

Print["q(", k ",", n, ")", "=", MP[[n + 1]]]

Print["P(", k ",", n, ")", "=", P // MatrixForm]

Print["perP(", k ",", n, ")", "=", Permanent[P]]
```

Fig. 1. Computer algorithm for the generalized order-k Modified Pell numbers

When the programs are started, we obtain the results regarding the value of the corresponding sequence, the respective matrix itself, and the permanent of that, respectively, with respect to the chosen values of k and n. For convenience and readability, in the algorithms, we denoted by $q\left(k,n\right) = {q_n}^k$ the generalized order-k Modified Pell numbers and by $Q\left(k,n\right) = {Q_n}^k$ the generalized order-k Pell–Lucas numbers.

```
(* Algorithm for the generalized order-k  Pell-Lucas number *)

k = Input["Please enter the order of the sequence."];

i = Input["Please enter the value of i."];

n = Input["Please enter the number of terms of the sequence."];

GPL = Table[0, {n + k}];
Do[GPL[[k - i + 1]] = -2;
  GPL[[k]] = 2;
  GPL[[tt]] = 2 * GPL[[tt - 1]] + Sum[GPL[[tt - i]], {i, 2, k}], {tt, k + 1, n + k}];

PL = Drop[GPL, k - 1];

R = Table[0, {n}, {n}];
tt = 0;
Do[tt = tt + 1; pp = 1; Do[pp = pp + 1; If[j > 1, R[[j, j - 1]] = 1];
   If[pp + tt <= n, R[[pp, pp + tt]] = 1]; R[[1, i + 1]] = 2;
   R[[j, j]] = 2, {j, 1, n}], {i, 1, k - 1}];

Print["Q(", k ",", n, ")", "=", PL[[n + 1]]]

Print["R(", k ",", n, ")", "=", R // MatrixForm]

Print["perR(", k ",", n, ")", "=", Permanent[R]]
```

Fig. 2. Computer algorithm for the generalized order-k Pell–Lucas numbers

4 Conclusions

This study established that the generalized order-k Pell–Lucas and Modified Pell numbers can be written in terms of the usual Fibonacci numbers. Certain families of the Hessenberg and super-diagonal matrices were presented. The terms of the generalized order-k Modified Pell and Pell–Lucas numbers, and also their sums, were obtained through the permanents of the given super-diagonal matrices.

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