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On generalized Fibonacci quadratics

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Abstract: In this paper, we consider generalized Fibonacci quadratics and give solutions of them under certain conditions. For example, for odd number k, under condition $n=U_k^2\left(V_kV_{k(4n+1)}-4\right)$, the equation

$$nx^2+\left(V_kn-2U_k^2D\right)x-\left(n+DU_k^2\left(V_k+2\right)\right)=0$$

has rational roots.
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1 Introduction

The second order sequence $\{W_n(a,b;p,q)\}$, or briefly $\{W_n\}$ is defined for $n>2$ by

$$W_n=pW_{n-1}+qW_{n-2},$$

in which $W_0=a, W_1=b$, where a,b are arbitrary integers and p,q are nonzero integers [1]. The Binet formula for $\{W_n\}$ is

$$W_n=A\alpha^n+B\beta^n,$$

where $A=\frac{b-a\beta}{\alpha-\beta}, B=\frac{a\alpha-b}{\alpha-\beta}$ and $\alpha,\beta=\left(p\pm\sqrt{p^2+4q}\right)/2$.

In [3, 4], E. Kılıç and P. Stanica derived the following recurrence relation for the sequence $\{W_{kn}\}$. For $n > 2$ and a fixed positive integer k,

$$W_{kn} = V_k W_{k(n-1)} - (-q)^k W_{k(n-2)},$$

where $V_k = \alpha^k + \beta^k$. Specifically define the generalized Fibonacci $\{U_n\}$ and Lucas $\{V_n\}$ sequences as $U_n = W_n(0, 1; p, 1)$, $V_n = W_n(2, p; p, 1)$, respectively. Thus;

$$\begin{aligned} U_{kn} &= V_k U_{k(n-1)} + (-1)^{k+1} U_{k(n-2)}, \\ V_{kn} &= V_k V_{k(n-1)} + (-1)^{k+1} V_{k(n-2)}. \end{aligned}$$

The Binet formulas are

$$U_{kn} = \frac{\alpha^{kn} - \beta^{kn}}{\alpha - \beta} \quad \text{and} \quad V_{kn} = \alpha^{kn} + \beta^{kn},$$

respectively. The equations

$$\begin{aligned} ax^2 + bx - c &= 0, \ \ ax^2 - bx - c = 0, \\ cx^2 + bx - a &= 0, \ \ cx^2 - bx - a = 0 \end{aligned}$$

have the same discriminant and then, the first one is considered. Rational roots of the quadratic equation $ax^2 + bx + c = 0$ are given under certain conditions.

In [7], the author gave that the solutions of the equation $F_n x^2 + F_{n+1} x - F_{n+2} = 0$ are -1 and F_{n+2}/F_n.

In [6], for $m \in \mathbb{Z}^+$, the author gave the rational solutions of the three equations under conditions $n = F_{2m+1} - 1, F_{2m+3}F_{2m}$ and $F_{2m+1}F_{2m}$, respectively:

$$\begin{aligned} nx^2 + (n+1)x - (n+2) &= 0, \\ nx^2 + (n+2)x - (n+1) &= 0, \\ nx^2 + (n-1)x - (n+1) &= 0. \end{aligned}$$

In [5], for $n, r \in \mathbb{Z}^+$, the authors obtained the solutions of equations

$$\begin{aligned} nx^2 + (n+r)x - (n+2r) &= 0, \\ nx^2 + (n+2r)x - (n+r) &= 0, \\ nx^2 + (n-r)x - (n+r) &= 0. \end{aligned}$$

In this paper, we consider generalized Fibonacci quadratics and give solutions of them under certain conditions. For example, for odd number k, under condition $n = U_k^2 \left(V_k V_{k(4n+1)} - 4\right)$, the equation

$$nx^2 + \left(V_k n - 2U_k^2 D\right) x - \left(n + DU_k^2 (V_k + 2)\right) = 0$$

has rational roots.

2 Generalized Fibonacci quadratics

Throughout this paper, we denote $D = V_k^2 + 4$. We will consider some interesting quadratics including generalized Fibonacci numbers and give the solutions of them under conditions $\frac{U_k^2}{D}\left(4 - V_k V_{k(4n-1)}\right)$, $\frac{U_k^2}{D}\left(V_k V_{k(4n+3)} - 4\right)$ and $U_k^2 \left(V_k V_{k(4n+1)} - 4\right)$, respectively.

Lemma 2.1. *For odd number k, we have*

$$\begin{aligned} U_{kn}V_{km}+V_{kn}U_{km} &= 2U_{k(n+m)}, \\ U_{kn}V_{km}-V_{kn}U_{km} &= -2U_{k(n-m)}, \\ U_k^2V_{kn}V_{km}+DU_{kn}U_{km} &= 2U_k^2V_{k(n+m)}, \\ V_{k(n+m)}+V_{k(n-m)} &= \begin{cases} \dfrac{D}{U_k^2}U_{kn}U_{km} & \text{if } m \text{ is odd} \\ V_{kn}V_{km} & \text{if } m \text{ is even} \end{cases}. \end{aligned} \tag{1}$$

Proof. From the Binet formulas of sequences $\{U_{kn}\}$ and $\{V_{kn}\}$, we have the desired identities. □

In [2], the authors determined Pell equations involving the generalized Fibonacci and Lucas sequences by the following Lemmas:

Lemma 2.2. *For odd number k, the integer solutions of $Dx^2+4U_k^2=y^2U_k^2$ are precisely the pairs $(\pm U_{2kn}, \pm V_{2kn})$.*

Lemma 2.3. *For odd number k, the integer solutions of $Dx^2-4U_k^2=y^2U_k^2$ are precisely the pairs $\left(\pm U_{k(2n+1)}, \pm V_{k(2n+1)}\right)$.*

Lemma 2.4. *For odd number k, the integer solutions of $DU_k^2\left(x^2-4\right)=y^2$ are precisely the pairs $(\pm V_{2kn}, \pm DU_{2kn})$.*

Theorem 2.5. *For odd number k, rational solutions to*

$$nx^2+\left(nV_k+2U_k^2\right)x-\left(n-U_k^2\left(V_k+2\right)\right)=0 \tag{2}$$

exist if and only if $n=\frac{U_k^2}{D}\left(4-V_kV_{k(4n-1)}\right)$ and they are

$$-\frac{V_k}{2}-\frac{D}{4-V_kV_{k(4n-1)}}\pm\frac{DV_kU_{k(4n-1)}}{2U_k\left(4-V_kV_{k(4n-1)}\right)}.$$

Proof. The discriminant of (2) is

$$\Delta=V_k^2n^2+4\left(U_k^2-n\right)^2.$$

Rational solutions of (2) exist if and only if Δ is a perfect square. Hence,

$$\begin{aligned} n^2V_k^2+4\left(U_k^2-n\right)^2 &= V_k^2t^2 \\ n^2+4\left(\frac{U_k^2}{V_k}-\frac{n}{V_k}\right)^2 &= t^2. \end{aligned}$$

Thus, the Pythagorean triplet has $\left[n,\ \frac{2\left(-n+U_k^2\right)}{V_k},\ t\right]$, not necessarily primitive. If we present the triplet as

$$\left[g^2-h^2,\ 2gh,\ g^2+h^2\right],$$

then

$$n=g^2-h^2, \qquad \frac{U_k^2}{V_k}-\frac{n}{V_k}=gh, \qquad t=g^2+h^2,$$

and hence

$$g^2+V_khg-h^2-U_k^2=0.$$

From the discriminant of this equation, we have

$$Dh^2 + 4U_k^2 = U_k^2 s^2.$$

Then this equation has positive solutions $h = U_{2kn}$ and $s = V_{2kn}$ with $n \geq 1$ in Lemma 2.2. Hence

$$g = \frac{-hV_k \pm U_k s}{2} = \frac{-U_{2kn}V_k \pm U_k V_{2kn}}{2},$$

and taking $m = 1$ and $2n$ instead of n in Lemma 2.1,

$$g = \frac{-U_{2kn}V_k + U_k V_{2kn}}{2} = U_{k(2n-1)},$$

$$g = \frac{-U_{2kn}V_k - U_k V_{2kn}}{2} = -U_{k(2n+1)}.$$

Since only the first solution gives positive, considering Binet formulas and recurrence relations of $\{U_{kn}\}$ and $\{V_{kn}\}$, we write

$$n = g^2 - h^2 = U_{k(2n-1)}^2 - U_{2kn}^2 = -\frac{U_k^2}{D}\left(V_k V_{k(4n-1)} - 4\right),$$

and using $m = 1$ in Lemma 2.1,

$$\begin{aligned} t &= g^2 + h^2 = U_{k(2n-1)}^2 + U_{2kn}^2 = \frac{U_k^2}{D}\left(V_{k(4n-2)} + V_{4kn}\right) \\ &= U_k U_{k(4n-1)}. \end{aligned}$$

Thus, using $x = \left(-V_k n - 2U_k^2 \pm t\right)/2n$, we obtain the solutions as claimed. □

For example, when $k = p = 1$ in Theorem 2.5, rational solutions to

$$nx^2 + (n+2)\,x - (n-3) = 0$$

exist if and only if $n = \frac{1}{5}\left(4 - L_{4n-1}\right)$ and they are

$$\frac{L_{4n-1} + 5F_{4n-1} - 14}{2\left(4 - L_{4n-1}\right)}, \ \frac{L_{4n-1} - 5F_{4n-1} - 14}{2\left(4 - L_{4n-1}\right)}.$$

We have the following theorem by using Lemma 2.3 and combinatoric identities.

Theorem 2.6. *For odd number k, rational solutions to*

$$nx^2 + \left(V_k n - 2U_k^2\right)x - \left(n + \left(V_k + 2\right)U_k^2\right) = 0$$

exist if and only if $n = \frac{U_k^2}{D}\left(V_k V_{k(4n+3)} - 4\right)$ *and they are*

$$-\frac{V_k}{2} + \frac{D}{V_k V_{k(4n+3)} - 4} \pm \frac{D V_k U_{k(4n+3)}}{2U_k\left(V_k V_{k(4n+3)} - 4\right)}.$$

For example, when $k = p = 1$ in Theorem 2.6, rational solutions to

$$nx^2 + (n-2)\,x - (n+3) = 0$$

exist if and only if $n = \frac{1}{5}(L_{4n-3} - 4)$ and they are

$$\frac{10F_{4n+3} - L_{4n+3} + 14}{2\left(L_{4n+3} - 4\right)}, \ \frac{-10F_{4n+3} - L_{4n+3} + 14}{2\left(L_{4n+3} - 4\right)}.$$

Theorem 2.7. *For odd number k, rational solutions to*

$$nx^2 + \left(nV_k - 2U_k^2 D\right) x - \left(n + DU_k^2 \left(V_k + 2\right)\right) = 0 \tag{3}$$

exist if and only if $n = U_k^2 \left(V_k V_{k(4n+1)} - 4\right)$ and they are

$$x_1 = \frac{2V_k + D - V_k V_{k(4n+2)}}{\left(V_k V_{k(4n+1)} - 4\right)} \text{ and } x_2 = \frac{2V_k + D + V_k V_{4kn}}{\left(V_k V_{k(4n+1)} - 4\right)}.$$

Proof. The discriminant of (3) is

$$\Delta_1 = V_k^2 n^2 + 4\left(n + U_k^2 D\right)^2.$$

Rational solutions of (3) exist if and only if Δ_1 is a perfect square, $\Delta_1 = V_k^2 t^2$. Hence,

$$n^2 + 4\left(\frac{U_k^2 D}{V_k} + \frac{n}{V_k}\right)^2 = t^2.$$

Thus, $\left[n,\ \frac{2\left(n+U_k^2 D\right)}{V_k},\ t\right]$ form Pythagorean triplet. Considering the triplet as

$$\left[g^2 - h^2,\ 2gh,\ g^2 + h^2\right],$$

we have

$$n = g^2 - h^2, \qquad \frac{U_k^2}{V_k} D + \frac{n}{V_k} = gh, \qquad t = g^2 + h^2,$$

and hence

$$g^2 - V_k hg - h^2 + U_k^2 D = 0.$$

From the discriminant of this equation, taking $h = U_k h_1$, we get

$$U_k^2 D\left(h_1^2 - 4\right) = s^2$$

which has positive solutions $h_1 = V_{2kn}$ and $s = DU_{2kn}$ in Lemma 2.4. Hence

$$\begin{aligned} h &= U_k V_{2kn}, \\ g &= \frac{hV_k \pm s}{2} = \frac{V_{2kn} U_k V_k \pm DU_{2kn}}{2} \\ &= \frac{V_k\left(V_{2kn} U_k \pm V_k U_{2kn}\right)}{2} + 2U_{2kn}. \end{aligned}$$

Taking $m = 1$ and $2n$, $2n-1$ instead of n in Lemma 2.1, respectively, and Binet formula of $\{U_{kn}\}$,

$$\begin{aligned} g &= V_k U_{k(2n+1)} + 2U_{2kn} = U_k V_{k(2n+1)}, \\ g &= V_k U_{k(2n-1)} - 2U_{2kn} = -V_k U_{k(2n-1)}. \end{aligned}$$

Only the first solution gives positive n. Using Lemma 2.1 and Binet formula, recurrence relation of $\{V_{kn}\}$, we write

$$n = g^2 - h^2 = U_k^2 V_{k(2n+1)}^2 - U_k^2 V_{2kn}^2 = U_k^2\left(V_k V_{k(4n+1)} - 4\right).$$

and by (1)

$$\begin{aligned} t &= g^2 + h^2 = U_k^2 V_{k(2n+1)}^2 + U_k^2 V_{2kn}^2 \\ &= U_k^2 \left(V_{k(2n+1)}^2 + V_{2kn}^2 \right) = U_k^2 \left(V_{k(4n+2)} + V_{4kn} \right) = DU_k U_{k(4n+1)}. \end{aligned}$$

Thus, from $x = \left(-nV_k + 2U_k^2 D \pm t\right)/2n$, we obtain the solutions as claimed. □

For example, when $k = p = 1$ in Theorem 2.7, rational solutions to

$$nx^2 + (n - 10)\,x - (n + 15) = 0$$

exist if and only if $n = L_{(4n+1)} - 4$ and they are

$$x_1 = \frac{7 - L_{4n+2}}{L_{4n+1} - 4} \text{ and } x_2 = \frac{7 + L_{4n}}{L_{4n+1} - 4}.$$

We have the following theorem by using Lemma 2.4 and combinatoric identities.

Theorem 2.8. *For odd number k, rational solutions to*

$$nx^2 + \left(V_k n + 2U_k^2 D\right) x - \left(n - U_k^2 D\left(V_k - 2\right)\right) = 0$$

exist if and only if $n = U_k^2 \left(V_k V_{k(4n+1)} - 4\right)$ and they are

$$x_1 = \frac{2V_k - D - V_k V_{k(4n+2)}}{V_k V_{k(4n+1)} - 4} \text{ and } x_2 = \frac{2V_k - D + V_k V_{4kn}}{V_k V_{k(4n+1)} - 4}.$$

For example, rational solutions to

$$nx^2 + (n + 10)\,x - (n + 5) = 0$$

exist if and only if $n = L_{4n+1} - 4$ and they are

$$x_1 = \frac{-3 - L_{4n+2}}{L_{4n+1} - 4} \text{ and } x_2 = \frac{-3 + L_{4n}}{L_{4n+1} - 4}.$$

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