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An alternative proof of Nyblom's results and a generalisation

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Abstract: Let n be a positive integer and k be a non-negative integer. We define

$$p(n,k) = \begin{cases} n(n+k), & \text{if } k \equiv 0 \pmod 2; \\ \frac{n(n+k)}{2}, & \text{if } k \equiv 1 \pmod 2, \end{cases}$$

and $D(n,k)$ to be the number of ways n can be expressed as a difference of two elements from the sequence $p(n,k)$. Nyblom found closed expressions for $D(n,0)$ and $D(n,1)$ in terms of some restricted number-of-divisors functions. Here we re-establish these two results of Nyblom in a relatively simple way. Along with the other interpretations for $D(n,k)$, an expression for $D(n,k)$ is presented in terms of restricted form of $D(n,0)$ and $D(n,1)$. Also we consider another function due to Nyblom, denoted $p_D(n)$, which counts the number of partitions of n with parts in arithmetic progression having common difference D. Nyblom and Evan found a simple expression for $p_2(n)$ and put $p_D(n)$ in terms of a divisor-counting functions when $D \geq 3$. Here we re-establish Nyblom's expression for $p_2(n)$, and find equinumerous expressions for $p_D(n)$ when $D \geq 3$. Finally, we present the following generalised version of $D(n,k)$: given a set of positive integers say, A, we denote by $D(n,A)$, the number of ways n can be written as a difference of two elements from the set A. And we express $D(n,A)$ in terms of partition enumerations when some restrictions are imposed upon the elements of A. We close with the hint that, boundedness of $D(n,A)$ together with the divergence of $\sum_{a\in A} \frac{1}{a}$ disproves Erdős arithmetic progression conjecture.

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1 Introduction

Nyblom [6] converted the problem of finding the number of representation of integers as a difference of two squares to a system of linear Diophantine equation and thereby got the following expression:

$$S(n) = \frac{1}{2}\left(\tau_e(n) + (-1)^{n+1}\tau_o(n) + \frac{1+(-1)^{\tau(n)+1}}{2}\right), \tag{1}$$

where $S(n)$ denote the number of ways n can be expressed as a difference of two square numbers, $\tau_e(n)$(resp. $\tau_o(n)$) denote the number of even (resp. odd) positive divisors of n and $\tau(n)$ denote the number of positive divisors of n. In Section 2, we establish this expression in a simple and different way by counting a particular kind of divisors of n; this counting has already been done in a paper by the author [2].

In one of the other paper of Nyblom [5] following expression was derived:

$$T(n) = \tau_o(n), \tag{2}$$

where $T(n)$ denote the number of ways n can be expressed as a difference of two triangular numbers. Even to derive this expression, Nyblom employed a technique similar to the one used in deriving (1). In Section 2, we establish (2) by counting another kind of divisors of n; essentials of this counting has been derived already in the same paper of the author [2].

Now, we define a sequence which is a common generalisation of square number sequence and triangular number sequence and find the expression of the above kind when elements of this sequence come into play.

Definition 1.1. Let n be a positive integer and k be a non-negative integer. We define

$$p(n,k) = \begin{cases} n(n+k), & \text{if } k \equiv 0 \pmod 2; \\ \dfrac{n(n+k)}{2}, & \text{if } k \equiv 1 \pmod 2, \end{cases}$$

and $D(n,k)$ to be the number of ways n can be expressed as a difference of two elements from the sequence $p(n,k)$.

From this definition, it follows that:

$$S(n) = \begin{cases} D(n,0) - 1, & \text{if } \delta(n) = 1; \\ D(n,0), & \text{if } \delta(n) = 0, \end{cases} \tag{3}$$

where δ denotes the characteristic function of square numbers. Moreover, we have

$$T(n) = D(n,1). \tag{4}$$

The sequences $p(n,k)$ for the initial cases $k = 2,3,4,5,6$ (with comments) can be found, respectively, in A005563, A000096, A028347, A055998 and A028560 of Online Encyclopedia of Integer Sequences. Following result concerning $p(n,k)$ is straightforward.

Lemma 1.2. *Let n be a positive integer and let k be a non-negative integer. Then*

$$p(n,k) = \begin{cases} p(n+1,k-2)-(k-1), & \text{if } k \equiv 0 \pmod 2; \\ p(n+1,k-2)-\dfrac{k-1}{2}, & \text{if } k \equiv 1 \pmod 2. \end{cases} \tag{5}$$

This lemma is instrumental in deriving expressions (which are presented in Section 2) for $D(n,k)$ in a restricted form of $D(n,0)$ and $D(n,1)$.

Next we are concerned with another result of Nyblom and Evan. The following definition is essential for mentioning that result.

Definition 1.3. Let n and D be two positive integers. Then the function $p_D(n)$ is defined to be the number of partitions of n with parts in arithmetic progression having common difference D.

Note that, in the above definition, we consider the partition (n) of n as a partition in an arithmetic progression.

R. Cook and D. Sharp [1] found a necessary and sufficient condition for an integer n to be written as a sum of arithmetic progression. In this sequence, Nyblom and Evans [7] defined the function $p_D(n)$ and obtained that:

$$p_2(n) = \frac{1}{2}\Big(\tau(n) - 2 + \frac{(-1)^{\tau(n)+1}+1}{2}\Big) + 1, \tag{6}$$

and also they have obtained an interpretation for the term $p_D(n)$ in terms of a divisor-counting function. A. O. Munagi [4] pointed out that such an expression for $p_D(n)$ by Nyblom and Evans is a complicated one. In Section 2 we re-establish (6) and also we find alternative interpretation for $p_D(n)$ which is relatively simple to that of Nyblom and Evans.

Section 3 is concerned with a generalisation of the above problem, that is, we are concerned with the number of ways a positive integer n can be expressed as a difference of two elements of a given set of positive integers. Following definition forms a basis for Section 3.

Definition 1.4. Let n be a positive integer and let A be a set of positive integers. Then we define the function $D(n,A)$ to be the number of ways n can be expressed as a difference of two elements from the the set A.

In Section 3 we found that the function $D(n,A)$ is equinumerous with a kind of partitions involving the set A with some additional constraints. We close by mentioning a connection between boundedness of $D(n,A)$ and Erdős arithmetic progression conjecture.

Notations and definitions in this section bears the same meaning throughout the article when used.

2 Proof and an Extension of Nyblom's results

2.1 Alernative proof of Nyblom's results

In this section we give an alternative and simple proofs of Nyblom's expressions [5, 6].

Let n be a positive integer. Suppose that n can be written as a difference of two squares; both non-zero. Then we have the following equalities:

$$
\begin{aligned}
n &= y^2 - x^2 \\
&= (x+d)^2 - x^2 \\
&= 2dx + d^2.
\end{aligned}
$$

This gives the following congruence:

$$
\frac{n}{d} - d \equiv 0 \pmod 2 \tag{7}
$$

with $\frac{n}{d} - d \geq 1$.

Conversely, every solution to the congruence above gives a representation of n as a difference of two squares; both non-zero.

In [2] to conclude Theorem 5, the number of divisors d of n satisfying the congruence (7) was counted. Now in view of the above observations, we equate that counting with $D(n, 0)$.

$$
D(n,0) = \begin{cases}
\dfrac{\tau(n)}{2} & \text{if } n \equiv 1 \pmod 2 \text{ and } \delta(n) = 1; \\
\dfrac{\tau(n)-1}{2} & \text{if } n \equiv 1 \pmod 2 \text{ and } \delta(n) = 0; \\
\dfrac{(\beta-1)\tau(\frac{n}{2^\beta})-1}{2} & \text{if } n \equiv 0 \pmod 2 \text{ and } \delta(n) = 1; \\
\dfrac{(\beta-1)\tau(\frac{n}{2^\beta})}{2} & \text{if } n \equiv 0 \pmod 2 \text{ and } \delta(n) = 0,
\end{cases} \tag{8}
$$

where β is the highest power of 2 that divides n.

Now we observe that (8) is a disguised form of (1).

Case i. When $n \equiv 1 \pmod 2$ and $\delta(n) = 0$, in accordance with (1), we can write

$$
\begin{aligned}
D(n,0) &= \frac{1}{2}\Big(\tau_e(n) + \tau_o(n) + \frac{1+1}{2}\Big) - 1 \\
&= \frac{1}{2}(\tau(n) + 1) - 1 \\
&= \frac{\tau(n)-1}{2}.
\end{aligned}
$$

Case ii. When $n \equiv 1 \pmod 2$ and $\delta(n) = 1$, in accordance with (1), we can write

$$
\begin{aligned}
D(n,0) &= \frac{1}{2}\Big(\tau_e(n) + \tau_o(n) + \frac{1-1}{2}\Big) \\
&= \frac{1}{2}\tau(n).
\end{aligned}
$$

Case iii. When $n \equiv 0 \pmod 2$ and $\delta(n) = 1$, by (1), we have

$$
\begin{aligned}
D(n,0) &= \frac{1}{2}(\tau_e(n) - \tau_o(n) + 1) - 1 \\
&= \frac{1}{2}\Big(\beta\tau\big(\frac{n}{2^\beta}\big) - \tau\big(\frac{n}{2^\beta}\big) + 1\Big) - 1 \\
&= \frac{1}{2}\Big((\beta-1)\tau\big(\frac{n}{2^\beta}\big) - 1\Big).
\end{aligned}
$$

Case iv. When $n \equiv 0 \pmod 2$ and $\delta(n) = 0$, as before, we have

$$\begin{aligned} D(n,0) &= \frac{1}{2}(\tau_e(n) - \tau_o(n)) \\ &= \frac{1}{2}\Big((\beta-1)\tau\big(\frac{n}{2^\beta}\big)\Big). \end{aligned}$$

Thus we have obtained a proof of (1).

Now we turn to the derivation of (2). Suppose that n can be written as a difference of two triangular numbers. Then we have

$$\begin{aligned} n &= \frac{y(y+1)}{2} - \frac{x(x+1)}{2} \\ &= \frac{(x+d)(x+d+1)}{2} - \frac{x(x+1)}{2}. \end{aligned}$$

This gives

$$\frac{2n}{d} - d = 2x + 1$$

for some non-negative integer x.

From this, we observe that $T(n)$ equals the number of divisors d of $2n$ which satisfies the following:

1. $d < \sqrt{2n}$;
2. Either d is odd or $d = 2^{\beta+1}k$ for some integer k, where β as before denotes the highest power of 2 that divides n.

Now again from the lines of proof of Theorem 5 in [2] we can write

$$T(n) = \begin{cases} \dfrac{\tau(2n)}{2} - \dfrac{(\beta+1-1)\tau\big(\frac{2n}{2^{\beta+1}}\big)}{2} & \text{if } \delta(2n) = 0; \\ \dfrac{\tau(2n)-1}{2} - \dfrac{(\beta+1-1)\tau\big(\frac{2n}{2^{\beta+1}}\big)-1}{2} & \text{if } \delta(2n) = 1. \end{cases}$$

Above equality is equivalent to

$$T(n) = \begin{cases} \dfrac{\tau_e(2n)+\tau_o(2n)-(\tau_e(2n)-\tau_o(2n))}{2} & \text{if } \delta(2n) = 0; \\ \dfrac{\tau_e(2n)+\tau_o(2n)-1}{2} - \dfrac{\tau_e(2n)-\tau_o(2n)-1}{2} & \text{if } \delta(2n) = 1. \end{cases}$$

Consequently, we get

$$\begin{aligned} T(n) =& \tau_o(2n) \\ =& \tau_o(n). \end{aligned}$$

Thus Nyblom's expression for $T(n)$ is re-established.

Now we turn to another result of Nyblom and Evans about $p_2(n)$ mentioned in (6). Our derivation is based upon the following observation: if n can be written as a sum of the terms in arithmetic progression with common difference D, then we have:

$$\begin{aligned} n &= a + (a + D) + \cdots + (a + (m-1)D) \\ &= ma + \frac{m(m-1)}{2} D. \end{aligned}$$

This gives

$$2n = 2ma + m^2 D - mD. \tag{9}$$

When $D = 2$, we have

$$\frac{n}{m} - m = a - 1.$$

Thus $p_2(n)$ equals the number of divisors of n satisfying the above equation. Since a is positive, $p_2(n)$ equals the number of divisors m of n satisfying the inequality $m \leq \sqrt{n}$. One can see that if n is a non-square then the mapping $m \to \frac{n}{m}$ is one-one and non-fixed. If n is a square number then the mapping $m \to \frac{n}{m}$ is again one-one and non-fixed when $m < \sqrt{n}$ and m fixes to itself under the map $m \to \frac{n}{m}$ if $m = \sqrt{n}$. These observations lead to the conclusion that

$$p_2(n) = \begin{cases} \frac{\tau(n)}{2} & \text{if } \delta(n) = 0; \\ \frac{\tau(n)-1}{2} + 1 & \text{if } \delta(n) = 1. \end{cases} \tag{10}$$

Thus the expression for $p_2(n)$ due to Nyblom and Evans [7] follows.

Next we give an alternative interpretation for $p_{2D}(n)$ and $p_{2D+1}(n)$ in terms of a divisor-counting function which is the contention of the following result.

Theorem 2.1. *Let n and D be positive integers. Then we have*

1. *$p_{2D}(n)$ equals the number of divisors m of n such that $m < \frac{D + \sqrt{D^2 + 4Dn}}{2D}$.*
2. *$p_{2D-1}(n)$ equals the number of divisors m of $2n$ such that $m < \frac{(D - \frac{1}{2}) + \sqrt{(D - \frac{1}{2})^2 + 2(2D-1)n}}{2D-1}$.*

Proof. If common difference is $2D$ then from (9) it follows that

$$\frac{n}{m} - (m-1)D = a > 0. \tag{11}$$

This can be put as a quadratic inequality

$$m^2 D - mD - n < 0.$$

After factoring we get

$$\left(m - \frac{D - \sqrt{D^2 + 4Dn}}{2D}\right)\left(m - \frac{D + \sqrt{D^2 + 4Dn}}{2D}\right) < 0.$$

Since the first factor is always positive we have

$$\left(m - \frac{D + \sqrt{D^2 + 4Dn}}{2D}\right) < 0.$$

Also from (9) we conclude that m must be a divisor of n. Moreover every divisor m of n satisfying the above inequality contribute to the equality (11). Hence first part of the result follows. Similar approach will settle the second part. □

Definition 2.2. Let n be a positive integer. By unrestricted partition of n, we mean a non-increasing sequence of integers say $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$. We use $p_D^u(n)$ to denote the number of unrestricted partitions of n whose parts follows arithmetic progression with common difference D.

Theorem 2.3. *Let n be a positive integer. Then we have*

$$p_D^u(n) = \begin{cases} \tau(n) & \text{if } D \equiv 0 \pmod 2; \\ 2\tau_o(n) & \text{if } D \equiv 1 \pmod 2. \end{cases} \tag{12}$$

Proof. Assume that the common difference is $2D$, an even integer. Then from (9) it follows that

$$\frac{n}{m} - (m-1)D = a. \tag{13}$$

Since a can be any integer, each divisor m of n contribute an a. Consequently, corresponding to each divisor m of n there exists an unrestricted partition of the said type and vice versa. Hence the first part of the result follows.

Assume that the common difference is $2D + 1$, an odd integer. Then from (9) it follows that

$$\frac{2n}{m} - (2D+1)(m-1) = 2a. \tag{14}$$

Choosing divisors m of $2n$ such that either m is odd or m is of the form $m = 2^\beta(2r+1)$, where β is the highest power of 2 that divides $2n$ will alone contribute an even integer $2a$ at the right side of (14). As one can see, the number of such divisors is $\tau_o(2n) + \tau_o(2n)$, which is equal to $2\tau_o(n)$. Now the result follows. □

Remark 2.4. It is interesting to note that the function $p_D(n)$ is independent of the value of D, whereas it is dependent on the parity of D and the factorisation of n.

2.2 Other interpretations of the term $D(n, k)$

Now we confine to the expression $D(n, k)$ when k exceeds 1.

Case i. Assume that $k \equiv 1 \pmod 2$. If n can be written as a difference of two members from the sequence $p(n, k)$, then we have

$$\begin{aligned} n &= p(y, k) - p(x, k) \\ &= p(x+d, k) - p(x, k) \\ &= \frac{(x+d)(x+d+k)}{2} - \frac{x(x+k)}{2}. \end{aligned}$$

This implies that

$$\frac{2n}{d} - d = 2x + k. \tag{15}$$

From this observation, we have the following result.

Theorem 2.5. *Let $k \equiv 1 \pmod 2$ be a positive integer and let n be a positive integer. Then $D(n,k)$ counts the divisors d of $2n$ such that $\frac{2n}{d} - d$ is an odd integer not less than k.*

Corollary 2.6. Let $k \equiv 1 \pmod 2$ be a positive integer and let n be a positive integer such that: for each divisor d of $2n$, $\frac{2n}{d} - d$ is an odd integer not less than k. Then $D(n,k) = D(n,1)$.

<u>**Case ii.**</u> Assume that $k \equiv 0 \pmod 2$. If n can be written as a difference of two members from the sequence $p(n,k)$, then we have

$$\begin{aligned} n &= p(y,k) - p(x,k) \\ &= p(x+d,k) - p(x,k) \\ &= (x+d)(x+d+k) - x(x+k). \end{aligned}$$

This implies that

$$\frac{n}{d} - d = 2x + k. \tag{16}$$

From this observation, we have the following result.

Theorem 2.7. *Let $k \equiv 0 \pmod 2$ be a positive integer and let n be a positive integer. Then $D(n,k)$ counts the divisors d of n such that $\frac{n}{d} - d$ is an even integer not less than k.*

Corollary 2.8. Let $k \equiv 0 \pmod 2$ be a positive integer and let n be a positive integer such that: for each divisor d of n, $\frac{n}{d} - d$ is an even integer not less than k. Then $D(n,k) = D(n,0)$.

To mention some of the forthcoming results, some basic terminologies from partition theory are required.

Definition 2.9. Let n be a positive integer. By a partition of n, we mean a sequence of non-increasing positive integers say $\pi = (a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$. Each a_i is called a part of π and the number of times part a_i occurs is referred as the frequency of a_i. Each element in the set of parts of π is called a size of π.

We observe that (15) can be put in the form: $2n = (f_1 + f_2)f_1 + f_1(f_2 + k)$ for some positive integers f_1 and f_2. This leads to the following result.

Theorem 2.10. *Let n be a positive integer and let $k \equiv 1 \pmod 2$ be a positive integer. Then $D(n,k)$ equals the number of partitions of $2n$ with sizes $f_1 + f_2$ and f_1 with their respective frequencies f_1 and $f_2 + k$.*

Similar observation on (16) gives the following result.

Theorem 2.11. *Let n be a positive integer and let $k \equiv 0 \pmod 2$ be a positive integer. Then $D(n,k)$ equals the number of partitions of n with sizes $f_1 + f_2$ and f_1 with their respective frequencies f_1 and $f_2 + k$.*

We observe that: $p(m, 2k-1) = k + (k+1) + \cdots + (k+m-1)$. Therefore, if n can be written as a difference of two elements from $p(m, 2k-1)$, then this difference gives a partition of n with consecutive integers as parts with least part not less than k and vice versa. This gives the following result.

Theorem 2.12. *Let n be a positive integer. Then $D(n, 2k-1)$ equals the number of partitions of n with consecutive integers as parts with least part not less than k.*

Since $p(m, 2k) = (2k+1) + (2k+3) + \cdots + (2k+2m-1)$, as in the previous case, we have the following result.

Theorem 2.13. *Let n be a positive integer. Then $D(n, 2k)$ equals the number of partitions of n with consecutive odd integers as parts with least part not less than $2k+1$.*

Remark 2.14. From the above results we see that the following enumerations are equivalent and are equal with $D(n, k)$ when k is an odd integer:

1. Number of divisors d of $2n$ such that $\frac{2n}{d} - d$ is an odd integer not less than k.
2. Number of partitions of $2n$ with sizes $f_1 + f_2$ and f_1 with their respective frequencies f_1 and $f_2 + k$.
3. Number of partitions of n with consecutive integers as parts with least part not less than $\frac{k+1}{2}$.

Similarly following enumerations are equivalent and are equal with $D(n, k)$ when k is even:

1. Number of divisors d of n such that $\frac{n}{d} - d$ is an even integer not less than k.
2. Number of partitions of n with sizes $f_1 + f_2$ and f_1 with their respective frequencies f_1 and $f_2 + k$.
3. Number of partitions of n with consecutive odd integers as parts with least part not less than $k+1$.

To realise the essence of Lemma 1.2 we need the following definition.

Definition 2.15. Let n and r be two positive integers and k be a non-negative integer. Denote by $D(n, k, r)$, the number of ways n can be expressed as a difference of two elements from the sequence $\{p(n, k)\}_{n \geq r+1}$.

As a consequence of Lemma 1.2, the function $D(n, k)$ can be written in terms of $D(n, k, r)$ which is the contention of the following result.

Theorem 2.16. *Let n be a positive integer and k be a non-negative integer. Then we have*

$$D(n,k) = \begin{cases} D\big(n, 1, \frac{k-1}{2}\big) & \text{if } k \equiv 1 \pmod 2; \\ D\big(n, 0, \frac{k}{2}\big) & \text{if } k \equiv 0 \pmod 2. \end{cases} \tag{17}$$

Proof. Assume $k \equiv 1 \pmod 2$. In view of Lemma 1.2 we can write

$$\begin{aligned} D(n,k) &= D(n, k-2, 1) \\ &= D(n, k-4, 2) \end{aligned}$$

and so on. Then after the $\frac{k-1}{2}$ times of repeated application of Lemma 1.2 as above we get

$$D(n,k) = D\big(n, 1, \frac{k-1}{2}\big),$$

which is the expected end. Similar application of Lemma 1.2 serves good in odd case. $\square$

3 Generalised version of Nyblom's results

3.1 An interpretation for $D(n, A)$

Recall from Definition 1.4 that $D(n, A)$ is the number of ways n can be written as a difference of two elements from the set A. We presumably take $\gcd(A) = 1$. For otherwise, integer n which are non-multiples of $\gcd(A)$ cannot be expressed as a difference of elements from A. Following theorem gives an interpretation for $D(n, A)$ in terms of an integer partition enumeration.

Theorem 3.1. *Let n be a positive integer and let $P = \{a_1, a_2, \dots\}$ be a set of positive integers with $\gcd(P) = 1$ and $a_1 < a_2 < \cdots$. Define $s_n = a_1 + a_2 + \cdots + a_n$ for every $n \geq 1$ and $A = \{s_1, s_2, \dots\}$. Then $D(n, A)$ equals the number of partitions of n with parts as consecutive elements of P.*

Proof. If n can be written as difference of two elements from A then we have

$$\begin{aligned} n &= s_k - s_r \\ &= a_{r+1} + a_{r+2} + \cdots + a_k \end{aligned}$$

for some $k > r$. Thus this difference gives a representation of n as a sum of consecutive members of P.

On the other side, if

$$n = a_{r+1} + a_{r+2} + \cdots + a_k$$

with $k > r$, then we can write

$$n = s_k - s_r.$$

Hence, the representation as a sum of consecutive elements of P gives a representation of n as a difference of elements of A. This correspondence establishes the result. □

Corollary 3.2. Let n be a positive integer. Then we have

1. The number of ways n can be written as a difference of squares of triangular numbers equals the number of ways n can be written as the sum of consecutive cubes.
2. No prime can be expressed as a sum of consecutive cubes.
3. The number of ways n can be expressed as a difference of two Fibonacci numbers equals the number partitions of n with parts as consecutive Fibonacci numbers.

Proof. Statements 1. and 2. follows from the identity

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Statement 3. follows from the identity:

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1.$$ □

3.2 Boundedness of $D(n, A)$ and Erdős Arithmetic Progression Conjecture

Erdős conjectured that, if $\sum_{a \in A} \frac{1}{a}$ diverges then A contains arithmetic progression of arbitrary length. Now if we assume that the conjecture is true then the boundedness of $D(n, A)$ implies the convergence of $\sum_{a \in A} \frac{1}{a}$. For if $\sum_{a \in A} \frac{1}{a}$ is diverging, then since we assume the truthness of Erdős conjecture, for any given positive integer k we have

$$\{a, a + d, a + 2d, \cdots, a + kd\} \subset A$$

for some positive integers a and d. Consequently, $D(d, A) \geq k$. That is for any given positive integer k we can find a positive integer d such that $D(d, A) \geq k$. Thus $D(n, A)$ is unbounded.

Next we observe that the converse of the above statement need not be true; for we have the converging series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ but $D(n, \{1^2, 2^2, \cdots\})$ is not bounded. Based upon this discussion we state our closing result.

Theorem 3.3. *If there exist a set of positive integer, say A, such that $\sum_{a \in A} \frac{1}{a}$ diverges and $D(n, A)$ is bounded, then the Erdős arithmetic progression conjecture fails.*

Remark 3.4. If one finds a set of positive integers satisfying the hypothesis of the above theorem, then the Erdős conjecture will be laid to rest.

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