Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 26, 2020, No. 2, 105–115
DOI: 10.7546/nntdm.2020.26.2.105-115

The linear combination of two polygonal numbers is a perfect square

Mei Jiang[1] and Yangcheng Li[2,*]

[1] School of Mathematics and Statistics, Changsha University of Science and Technology,
Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering,
Changsha, 410114, China
e-mail: jiangmeicsust@163.com

[2] School of Mathematics and Statistics, Changsha University of Science and Technology,
Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering,
Changsha, 410114, China
e-mail: liyangchengmlx@163.com

* *Corresponding author*

Received: 7 December 2019 **Revised:** 19 April 2020 **Accepted:** 21 April 2020

Abstract: By the theory of Pell equation and congruence, we study the problem about the linear combination of two polygonal numbers is a perfect square. Let $P_k(x)$ denote the x-th k-gonal number. We show that if $k \geq 5$, $2(k-2)n$ is not a perfect square, and there is a positive integer solution (Y', Z') of $Y^2 - 2(k-2)nZ^2 = (k-4)^2n^2 - 8(k-2)n$ satisfying

$$Y' + (k-4)n \equiv 0 \pmod{2(k-2)n},\ Z' \equiv 0 \pmod 2,$$

then the Diophantine equation $1+nP_k(y) = z^2$ has infinitely many positive integer solutions (y, z). Moreover, we give conditions about m, n such that the Diophantine equation $mP_k(x)+nP_k(y) = z^2$ has infinitely many positive integer solutions (x, y, z).

Keywords: Polygonal number, Diophantine equation, Pell equation, Positive integer solution.
2010 Mathematics Subject Classification: 11D09, 11D72.

1 Introduction

A polygonal number [4] is a positive number, corresponding to an arrangement of points on the plane, which forms a regular polygon.

The x-th k-gonal number [4, p. 5] is

$$P_k(x) = \frac{x\big((k-2)(x-1)+2\big)}{2},$$

where $x \geq 1, k \geq 3$. There are many papers about the polygonal numbers and many properties of them have been studied, we can refer to the first chapter of [5] and D3 of [8].

In 2005, Bencze [1] raised a problem: find all positive integers n for which

$$1 + \frac{9}{2}n(n+1) = 1 + 9P_3(n)$$

is a perfect square. In 2007, Le [11] gave a complete answer to Bencze's problem and showed that all such n are given by

$$n = \frac{1}{2}\big(\frac{1}{6}(a^{2k+1} + b^{2k+1}) - 1\big),$$

where $a = 3 + \sqrt{8}, b = 3 - \sqrt{8}$, and $k \in \mathbb{Z}^+$. In 2011, Guan [7] proved that all positive integers n for which $1+\frac{8s^2}{s^2-1}P_3(n)$ is a perfect square are given by

$$n = \frac{1}{2}\big(\frac{1}{2s}(a^{2k+1} + b^{2k+1}) - 1\big),$$

where $a = s + \sqrt{s^2-1}, b = s - \sqrt{s^2-1}$, and s is a positive odd integer with $s > 1$, $k \in \mathbb{Z}^+$. In 2013, Hu [9] used the theory of Pell equation to study the positive integer solutions of the Diophantine equation

$$1 + nP_3(y-1) = z^2,$$

where

$$n = \begin{cases} \dfrac{t^2 \pm 1}{2}, & t \equiv 1 \pmod 2, t \geq 3, \\ \dfrac{t^2 \pm 2}{2}, & t \equiv 0 \pmod 2, t \geq 2, \\ \dfrac{t(t-1)}{2}, & t \geq 2. \end{cases}$$

In 2019, Peng [12] showed that if $2n$ is not a perfect square, then the Diophantine equation $1 + nP_3(y-1) = z^2$ has infinitely many positive integer solutions, and if $n = \frac{d(t)}{2}$, then the Diophantine equation $1 + nP_3(y-1) = z^2$ has infinitely many positive integer solutions, where $d(t)$ are some special polynomials. Meanwhile, she studied the Diophantine equation

$$mP_3(x-1) + nP_3(y-1) = z^2,$$

where $m, n \in \mathbb{Z}^+$, and proved that when $\frac{m(m+1)}{2} = u^2, n = 1$, there exist infinitely many pairs (a, b) of integer numbers such that $mP_3(x-1)+nP_3(y-1) = z^2$ has integer parametric solutions $(t, at+b, u(ct+d))$, where t is a positive integer greater than 1.

Moreover, she got two general results:

1) If $2(m+n)$ is not a perfect square, $r \in \mathbb{Z}$, and the Pell equation

$$X^2 - 2(m+n)Z^2 = \left(\frac{m+n}{2}\right)^2 - r^2mn$$

has a positive integer solution satisfying

$$X_0 - rn + \frac{m+n}{2} \equiv 0 \pmod{m+n},$$

then the Diophantine equation $mP_3(x-1) + nP_3(y-1) = z^2$ has infinitely many positive integer solutions.

2) Let u, v be integers with $u > \sqrt{2}v$, and u being a positive even integer. If $m = (u^2 - 2v^2)^2$, $n = 8u^2v^2$, then the Diophantine equation $mP_3(x-1) + nP_3(y-1) = z^2$ has infinitely many positive integer solutions.

For more related results, we refer to [2, 13–15].

2 Main results

In this paper, we continue the study of [12], and consider the positive integer solutions of the Diophantine equations

$$1 + nP_k(y) = z^2 \tag{2.1}$$

and

$$mP_k(x) + nP_k(y) = z^2, \tag{2.2}$$

where $k \geq 5$, and $k, m, n \in \mathbb{Z}^+$. When $k = 4$, there are general results (see [3, p. 345, Corollary 6.3.6]).

By the theory of Pell equation, we give a positive answer to Question 4.1 of [12] and have the following theorems.

Theorem 2.1. *If $k \geq 5$, $2(k-2)n$ is not a perfect square, and there is a positive integer solution (Y', Z') of $Y^2 - 2(k-2)nZ^2 = (k-4)^2n^2 - 8(k-2)n$ satisfying*

$$Y' + (k-4)n \equiv 0 \pmod{2(k-2)n}, \ Z' \equiv 0 \pmod{2},$$

then Eq. (2.1) *has infinitely many positive integer solutions (y, z).*

Theorem 2.2. *When $k \geq 5$ and $m = (r(k-2)-1)n$, if $\frac{(r(k-2)-1)nr}{2}$ is a perfect square, then there exist infinitely many pairs (a, b) of positive integers such that Eq.* (2.2) *has integer parametric solutions $(x, ax+b, u(cx+d))$, where r is a positive integers.*

Moreover, we get

Theorem 2.3. *If $k \geq 5$, $2(k-2)(m+n)$ is not a perfect square, $r \in \mathbb{Z}$, and the Pell equation*

$$X^2 - 2(k-2)(m+n)Z^2 = (k-4)^2(m+n)^2 - 4(k-2)^2mnr^2$$

has a positive integer solution (X_0, Z_0) satisfying

$$X_0 - 2(k-2)nr + (k-4)(m+n) \equiv 0 \pmod{2(k-2)(m+n)}, \ Z_0 \equiv 0 \pmod{2},$$

then Eq. (2.2) *has infinitely many positive integer solutions (x, y, z).*

In particular,

Theorem 2.4. *Let $k \geq 5$, $m = 2(u^2 - 4u - 4)^2$, $n = 2(u^2 + 4u - 4)^2$. If $2(k-2)$ is not a perfect square, and the Pell equation $X^2 - 8(k-2)(u^2+4)^2 Z^2 = 1$ has a positive integer solution (U_0, V_0) satisfying $U_0 - 1 \equiv 0 \pmod{2(k-2)}$, then Eq.* (2.2) *has infinitely many positive integer solutions (x, y, z).*

Remark 2.5. *When $k = 3$, these are the cases studied by Peng* [12].

3 Preliminaries

To prove the above results, we give the following well-known lemmas (for example, see [10]).

Lemma 3.1 ([10]). *Let D be a positive integer which is not a perfect square, then the Pell equation $x^2 - Dy^2 = 1$ has infinitely many positive integer solutions. If (U, V) is the least positive integer solution of the Pell equation $x^2 - Dy^2 = 1$, then all positive integer solutions are given by*

$$x_s + y_s\sqrt{D} = (U + V\sqrt{D})^s,$$

where s is an arbitrary integer.

Lemma 3.2 ([10]). *Let D be a positive integer which is not a perfect square, N be a nonzero integer, and (U, V) is the least positive integer solution of $x^2 - Dy^2 = 1$. If (x_0, y_0) is a positive integer solution of $x^2 - Dy^2 = N$, then an infinity of positive integer solutions are given by*

$$x_s + y_s\sqrt{D} = (x_0 + y_0\sqrt{D})(U + V\sqrt{D})^s,$$

where s is an arbitrary integer.

Lemma 3.3 ([6]). *Let D be a positive integer which is not a perfect square, m_1, m_2 are positive integers, and N be a nonzero integer. If the Pell equation $x^2 - Dy^2 = N$ has a positive integer solution satisfying*

$$u_0 \equiv a \pmod{m_1}, \quad v_0 \equiv b \pmod{m_2},$$

then it has infinitely many positive integer solutions satisfying

$$u \equiv a \pmod{m_1}, \quad v \equiv b \pmod{m_2}.$$

4 Proofs of the Theorems

Proof of Theorem 2.1. Multiplying Eq. (2.1) by $8(k-2)n$, we have

$$(n(2(k-2)y - (k-4)))^2 - 2(k-2)n(2z)^2 = (k-4)^2 n^2 - 8(k-2)n.$$

Setting $Y = n(2(k-2)y - (k-4)), Z = 2z$, we get the Pell equation

$$Y^2 - 2(k-2)nZ^2 = (k-4)^2 n^2 - 8(k-2)n. \tag{4.1}$$

By Lemma 3.1, if $k \geq 5$ and $2(k-2)n$ is not a perfect square, the Pell equation $Y^2 - 2(k-2)nZ^2 = 1$ always has an infinite number of positive integer solutions. And suppose (u, v) is the least positive integer solution of $Y^2 - 2(k-2)nZ^2 = 1$. It is easy to see that $(Y_0, Z_0) = ((k-4)n, 2)$ is a positive integer solution of Eq. (4.1). By Lemma 3.2, an infinity of positive integer solutions of Eq. (4.1) are given by

$$Y_s + Z_s\sqrt{2(k-2)n} = \left((k-4)n + 2\sqrt{2(k-2)n}\right)\left(u + v\sqrt{2(k-2)n}\right)^s, s \geq 0.$$

If there is a positive integer solution (Y', Z') of $Y^2 - 2(k-2)nZ^2 = (k-4)^2n^2 - 8(k-2)n$ satisfying

$$Y' + (k-4)n \equiv 0 \pmod{2(k-2)n},\ Z' \equiv 0 \pmod 2.$$

Lemma 3.3 guarantees that Eq. (4.1) has infinitely many positive integer solutions (Y, Z) with the above condition. Then there are infinitely many

$$y = \frac{Y + (k-4)n}{2(k-2)n} \in \mathbb{Z}^+,\ z = \frac{Z}{2} \in \mathbb{Z}^+.$$

Thus, if $k \geq 5$ and $2(k-2)n$ is not a perfect square, and there is a positive integer solution (Y', Z') of $Y^2 - 2(k-2)nZ^2 = (k-4)^2n^2 - 8(k-2)n$ satisfying

$$Y' + (k-4)n \equiv 0 \pmod{2(k-2)n},\ Z' \equiv 0 \pmod 2,$$

Eq. (2.1) has infinitely many positive integer solutions (y, z). □

Remark 4.1. *In Theorem 2.1,* (u, v) *is the least positive integer solution of* $Y^2 - 2(k-2)nZ^2 = 1$ *and* $(Y_0, Z_0) = ((k-4)n, 2)$ *is a positive integer solution of Eq.* (4.1)*, so we have*

$$\begin{cases} Y_s = 2uY_{s-1} - Y_{s-2}, & Y_0 = (k-4)n,\ Y_1 = ((k-4)u + 4(k-2)v)n, \\ Z_s = 2uZ_{s-1} - Z_{s-2}, & Z_0 = 2,\ Z_1 = (k-4)nv + 2u. \end{cases}$$

When $(k-4)(u+1) \equiv 0 \pmod{2(k-2)}$ *and* $v \equiv 0 \pmod 2$*, it is easy to check that*

$$Y_s \equiv 0 \pmod n, Z_s \equiv 0 \pmod 2 \text{ and } Y_1 + (k-4)n \equiv 0 \pmod{2(k-2)n}.$$

Let $Y_s = nY'_s$*, then*

$$Y'_s = 2uY'_{s-1} - Y'_{s-2},\ Y'_0 = k-4,\ Y'_1 = (k-4)u + 4(k-2)v,$$

it is easy to prove that

$$Y_s \equiv \begin{cases} (k-4) & \pmod{2(k-2)}, \quad s \equiv 0 \pmod 2, \\ -(k-4) & \pmod{2(k-2)}, \quad s \equiv 1 \pmod 2. \end{cases}$$

Hence, when $s \equiv 1 \pmod 2$*, we have*

$$Y_s + (k-4)n \equiv 0 \pmod{2(k-2)n},\ Z_s \equiv 0 \pmod 2.$$

Example 4.2. *When $k=5,\ n=3$, then $2(k-2)n=18$ is not a perfect square.* $(u,v)=(17,4)$ *is the least positive integer solution of $Y^2-18Z^2=1$, so*

$$u+1\equiv 0 \pmod 6,\ \ v\equiv 0 \pmod 2.$$

$(Y_0,Z_0)=(3,2)$ *is the least positive integer solution of $Y^2-18Z^2=-63$, then*

$$\begin{cases} Y_s=34Y_{s-1}-Y_{s-2}, & Y_0=3,\ \ Y_1=195,\\ Z_s=34Z_{s-1}-Z_{s-2}, & Z_0=2,\ \ Z_1=46.\end{cases}$$

By Remark 4.1, when $s\equiv 1 \pmod 2$, we have

$$y_s=\frac{Y_s+3}{18}\in\mathbb{Z}^+,\ z_s=\frac{Z_s}{2}\in\mathbb{Z}^+.$$

Therefore, Eq. (2.1) *has infinitely many positive integer solutions* (y_s,z_s).

Proof of Theorem 2.2. If we let $m=tn$ and $y=ax+b$, then Eq. (2.2) reduces to

$$\frac{n(k-2)(a^2+t)}{2}x^2+\frac{n(2(k-2)ab-(k-4)(a+t))}{2}x+\frac{nb((k-2)b-(k-4))}{2}=z^2. \quad (4.2)$$

Consider

$$g(x)=\frac{n(k-2)(a^2+t)}{2}x^2+\frac{n(2(k-2)ab-(k-4)(a+t))}{2}x+\frac{nb((k-2)b-(k-4))}{2}$$

as a quadratic polynomial of x, if $g(x)=0$ has a root with multiplicity 2, the discriminant of $g(x)$ is zero, i.e.,

$$\frac{n^2}{4}((k-4)^2a^2-2t(k-4)(2(k-2)b-(k-4))a-t(4(k-2)^2b^2-4(k-2)(k-4)b-t(k-4)^2))=0.$$

It implies

$$a=\frac{2t(k-2)b-(k-4)t+2\sqrt{b(k-2)t(t+1)((k-2)b-(k-4))}}{k-4}. \quad (4.3)$$

To find $a\in\mathbb{Z}^+$, we take $b(k-2)t(t+1)((k-2)b-(k-4))=v^2$, then

$$(2v)^2-t(t+1)(2(k-2)b-(k-4))^2=-(k-4)^2t(t+1).$$

Letting $X=2v, Y=2(k-2)b-(k-4)$, we obtain the Pell equation

$$X^2-t(t+1)Y^2=-(k-4)^2t(t+1). \quad (4.4)$$

It is easy to see that the pair $(X_0,Y_0)=(2(k-4)t(t+1),(k-4)(2t+1))$ is a positive integer solution of Eq. (4.4), and the pair $(U,V)=(2t+1,2)$ solves the Pell equation $X^2-t(t+1)Y^2=1$. So an infinity of positive integer solutions of Eq. (4.4) are given by

$$X_s+Y_s\sqrt{t(t+1)}=\left(2(k-4)t(t+1)+(k-4)(2t+1)\sqrt{t(t+1)}\right)\left(2t+1+2\sqrt{t(t+1)}\right)^s, s\geq 0.$$

Thus

$$\begin{cases} X_s = 2(2t+1)X_{s-1} - X_{s-2}, & X_0 = 2(k-4)t(t+1), \ X_1 = 4(k-4)t(t+1)(2t+1), \\ Y_s = 2(2t+1)Y_{s-1} - Y_{s-2}, & Y_0 = (k-4)(2t+1), \ Y_1 = (k-4)(8t^2+8t+1). \end{cases}$$

According to the above recurrence relations, we have

$$X_s \equiv 0 \pmod{(k-4)}, \ Y_s \equiv 0 \pmod{(k-4)}.$$

By Eq. (4.3), we get

$$a_s = \frac{X_s + 2(k-2)bt - (k-4)t}{k-4} = \frac{tY_s + X_s}{k-4}.$$

So a_s is a positive integer. From $Y_s = 2(k-2)b_s - (k-4)$, we obtain

$$b_s = \frac{Y_s + (k-4)}{2(k-2)}.$$

Further, we get

$$\begin{cases} a_s = 2(2t+1)a_{s-1} - a_{s-2}, & a_0 = (4t+3)t, \ a_1 = (16t^2+20t+5)t, \\ b_s = 2(2t+1)b_{s-1} - b_{s-2} - \dfrac{2t(k-4)}{k-2}, & b_0 = \dfrac{(k-4)(t+1)}{k-2}, \ b_1 = \dfrac{(k-4)(2t+1)^2}{k-2}. \end{cases}$$

In order for b_s to be a positive integer, we need $Y_s + (k-4) \equiv 0 \pmod{2(k-2)}$.

When $t \equiv -1 \pmod{(k-2)}$, we have $Y_0 + (k-4) \equiv 0 \pmod{2(k-2)}$, and the above recurrence relations imply that

$$Y_s \equiv \begin{cases} -(k-4) \pmod{2(k-2)}, & s \equiv 0 \pmod{2}, \\ (k-4) \pmod{2(k-2)}, & s \equiv 1 \pmod{2}. \end{cases}$$

Therefore, when $s \equiv 0 \pmod{2}$, we have

$$Y_s + (k-4) \equiv 0 \pmod{2(k-2)},$$

so b_s is a positive integer.

Taking $t = r(k-2) - 1$, Eq. (4.2) now becomes

$$\frac{(r(k-2)-1)nr}{2}(cx+d)^2 = z^2.$$

If $\frac{(r(k-2)-1)nr}{2}$ is a perfect square, there exist infinitely many pairs (a, b) of positive integers such that Eq. (2.2) has positive integer parametric solutions $(x, ax+b, u(cx+d))$, where r is a positive integers. □

Example 4.3. *When* $k = 5, \ r = 1, \ m = 2, \ n = 1, \ \frac{(r(k-2)-1)nr}{2} = 1$ *is a perfect square. Taking* $a_0 = 22, \ b_0 = 1$*, Eq.* (2.2) *has positive integer parametric solutions* $(x, 22x+1, 27x+1)$*, where* x *is a positive integer.*

Proof of Theorem 2.3. Letting $y = x + r, r \in \mathbb{Z}$, Eq. (2.2) equals

$$(2(k-2)(m+n)x + 2(k-2)nr - (k-4)(m+n))^2 - 2(k-2)(m+n)(2z)^2 \\ = (k-4)^2(m+n)^2 - 4(k-2)^2mnr^2.$$

Taking $X = 2(k-2)(m+n)x + 2(k-2)nr - (k-4)(m+n),\ Z = 2z$, we get

$$X^2 - 2(k-2)(m+n)Z^2 = (k-4)^2(m+n)^2 - 4(k-2)^2mnr^2. \tag{4.5}$$

By Lemma 3.1, if $2(k-2)(m+n)$ is not a perfect square, the Pell equation

$$X^2 - 2(k-2)(m+n)Z^2 = 1$$

has infinitely many positive integer solutions. By Lemma 3.2, if Eq. (4.5) has a positive integer solution, it has infinitely many positive integer solutions. Assume that Eq. (4.5) has a positive integer solution (X_0, Z_0) satisfying

$$X_0 - 2(k-2)nr + (k-4)(m+n) \equiv 0 \pmod{2(k-2)(m+n)},\ Z_0 \equiv 0 \pmod 2.$$

By Lemma 3.3, Eq. (4.5) has infinitely many positive integer solutions (X, Z) satisfying the above condition, which leads to infinitely many $x, z \in \mathbb{Z}^+$. Then there are infinitely many $y = x + r \in \mathbb{Z}^+$. Hence, Eq. (2.2) has infinitely many positive integer solutions (x, y, z). □

Example 4.4. *When $k = 5,\ r = 34,\ m = 2,\ n = 1$, Eq. (4.5) becomes*

$$X^2 - 18Z^2 = -83223. \tag{4.6}$$

It has a positive integer solution $(X_0, Z_0) = (237, 88)$ satisfying

$$X_0 - 201 \equiv 0 \pmod{18},\ Z_0 \equiv 0 \pmod 2.$$

Note that $(u, v) = (17, 4)$ is the least positive integer solution of $X^2 - 18Z^2 = 1$. By Lemma 3.3, Eq. (4.6) has infinitely many positive integer solutions (X, Z) satisfying the above condition, which leads to infinitely many $x, z \in \mathbb{Z}^+$. Then there are infinitely many $y = x + 34 \in \mathbb{Z}^+$. Hence, Eq. (2.2) has infinitely many positive integer solutions (x, y, z).

Proof of Theorem 2.4. By Theorem 2.3, we need to find a positive integer solution (X_0, Z_0) satisfying

$$X_0 - 2(k-2)nr + (k-4)(m+n) \equiv 0 \pmod{2(k-2)(m+n)},\ Z_0 \equiv 0 \pmod 2.$$

Suppose that $X_0 = k(m+n)$ and $r = -t(m+n)$, then Z_0 satisfies

$$Z_0^2 = 2(m+n)((k-2)mnt^2 + 2).$$

From $X_0 = 2(k-2)(m+n)x_0 + 2(k-2)nr - (k-4)(m+n)$, we have $x_0 = 1 + nt$. Since we require Z_0 to be a positive integer, $2(m+n)((k-2)mnt^2 + 2)$ should be a perfect square. In order to get a concrete expression of m, n, we let

$$m = 2\alpha^2,\ \ n = \frac{\beta^2}{2},\ \ m + n = \gamma^2,$$

where $\alpha, \beta, \gamma \in \mathbb{Z}^+$. Then we get a quadratic equation

$$2\alpha^2 + \frac{\beta^2}{2} = \gamma^2,$$

which has a positive integer solution

$$\alpha = |u^2 - 4u - 4|, \;\; \beta = 2(u^2 + 4u - 4), \;\; \gamma = 2u^2 + 8,$$

where $u \in \mathbb{Z}^+$. Hence,

$$m = 2(u^2 - 4u - 4)^2, \;\; n = 2(u^2 + 4u - 4)^2.$$

Now Eq. (4.5) becomes

$$Z_0^2 = 16(u^2 + 4)^2 w^2,$$

where

$$w^2 = 1 + 2(k - 2)(u^2 - 4u - 4)^2(u^2 + 4u - 4)^2 t^2.$$

By Lemma 3.1, if $2(k - 2)$ is not a perfect square, the Pell equation

$$w^2 - 2(k - 2)(u^2 - 4u - 4)^2(u^2 + 4u - 4)^2 t^2 = 1 \tag{4.7}$$

has infinitely many positive integer solutions. And suppose (w_0, t_0) is a positive integer solution of Eq. (4.7). Hence,

$$X_0 = 4k(u^2 + 4)^2, \; Z_0 = 4(u^2 + 4)w_0, r = -4t_0(u^2 + 4)^2.$$

Note that $2(k - 2)(m + n) = 2(k - 2)\gamma^2$ is not a perfect square, by Lemma 3.1, the Pell equation $X^2 - 8(k - 2)(u^2 + 4)^2 Z^2 = 1$ has infinitely many positive integer solutions. Let (U_0, V_0) be the least positive integer solution of $X^2 - 8(k - 2)(u^2 + 4)^2 Z^2 = 1$. And the Pell equation

$$\begin{aligned} X^2 - 8(k - 2)(u^2 + 4)^2 Z^2 =& 16(k - 4)^2(u^2 + 4)^4 \\ & - 256 t_0^2 (k - 2)^2 (u^2 + 4)^4 (u^2 - 4u - 4)^2 (u^2 + 4u - 4)^2 \end{aligned} \tag{4.8}$$

has a positive integer solution $(X_0, Z_0) = (4k(u^2 + 4)^2, 4(u^2 + 4)w_0)$. It is easy to prove that

$$\begin{aligned} & X_0 + 4(u^2 + 4)^2(4(k - 2)t_0(u^2 + 4u - 4)^2 + (k - 4)) \equiv 0 \pmod{8(k - 2)(u^2 + 4)^2}, \\ & Z_0 \equiv 0 \pmod{2}. \end{aligned}$$

By Lemma 2.2, an infinity of positive integer solutions of Eq. (4.8) are given by

$$\begin{aligned} X_s + Z_s\sqrt{8(k - 2)(u^2 + 4)^2} =& \big(4k(u^2 + 4)^2 + 4(u^2 + 4)w_0\sqrt{8(k - 2)(u^2 + 4)^2}\big) \\ & \times (U_0 + V_0\sqrt{8(k - 2)(u^2 + 4)^2})^s, s \geq 0. \end{aligned}$$

Thus,

$$\begin{cases} X_s = 2U_0 X_{s-1} - X_{s-2}, & X_0 = 4k(u^2 + 4)^2, \\ & X_1 = 4(u^2 + 4)^2(8w_0(k - 2)(u^2 + 4)V_0 + kU_0), \\ Z_s = 2U_0 Z_{s-1} - Z_{s-2}, & Z_0 = 4(u^2 + 4)w_0, \\ & Z_1 = 4(u^2 + 4)(k(u^2 + 4)V_0 + w_0 U_0). \end{cases}$$

Then

$$\begin{cases} x_s = 2U_0 x_{s-1} - x_{s-2} - \frac{U_0-1}{2(k-2)} \cdot (8(k-2)t_0(u^2+4u-4)^2 + 2(k-4)), \\ y_s = x_s - 4t_0(u^2+4)^2, \\ z_s = 2U_0 z_{s-1} - z_{s-2}, \end{cases} \tag{4.9}$$

where

$$\begin{aligned} x_0 =& 1 + 2t_0(u^2+4u-4)^2,\ x_1 = 2t_0(u^2+4u-4)^2 + 4w_0(u^2+4)V_0 + 1 + \frac{k(U_0-1)}{2(k-2)}, \\ y_0 =& 1 - 2t_0(u^2-4u-4)^2,\ y_1 = -2t_0(u^2-4u-4)^2 + 4w_0(u^2+4)V_0 + 1 + \frac{k(U_0-1)}{2(k-2)}, \\ z_0 =& 2(u^2+4)w_0,\ z_1 = 2(u^2+4)(k(u^2+4)V_0 + w_0U_0). \end{aligned}$$

For $k \geq 5$, $u \in \mathbb{Z}^+$, by Eq. (4.7), we get $w_0 > 2|u^2-4u-4|(u^2+4u-4)t_0$, it is easy to check that $y_1 > 1$.

If $U_0 - 1 \equiv 0 \pmod{2(k-2)}$, for any $s \geq 1$, we deduce that x_s, y_s, z_s are positive integers greater than 1. Thus, Eq. (2.2) has infinitely many positive integer solutions (x_s, y_s, z_s). □

Example 4.5. *When $k = 5,\ u = 1$, we get $m = 98,\ n = 2$, and Eq. (4.8) becomes*

$$X^2 - 600Z^2 = -5531903990000.$$

It has a positive integer solution $(X_0, Z_0) = (500, 96020)$ satisfying

$$X_0 + 336100 \equiv 0 \pmod{600},\ Z_0 \equiv 0 \pmod{2}.$$

Note that $(U_0, V_0) = (49, 2)$ is the least positive integer solution of $Y^2 - 600Z^2 = 1$, and $U_0 - 1 \equiv 0 \pmod{6}$. By (4.9), we have

$$\begin{cases} x_s = 98x_{s-1} - x_{s-2} - 53776, & x_0 = 561,\ \ x_1 = 192641, \\ y_s = x_s - 28000, & y_0 = -27439,\ \ y_1 = 164641, \\ z_s = 98z_{s-1} - z_{s-2}, & z_0 = 48010,\ \ z_1 = 2352990. \end{cases}$$

Thus, for any $s \geq 1$, Eq. (2.2) has infinitely many positive integer solutions (x_s, y_s, z_s).

References

[1] Bencze, M. (2012). Proposed Problem 7508, *Octogon Mathematical Magazine*, 13 (1B), 678.

[2] Chen, J. P. (2012). The squares with the form $p^2 + \frac{4n(n+1)s^2k^2}{s^2-1}$, *Natural Science Journal of China West Normal University*, 33 (2), 196–198, 217.

[3] Cohen, H. (2007). *Number Theory, Vol. I: Tools and Diophantine Equations, Graduate Texts in Mathematics*, Springer.

[4] Deza E., & Deza M. M. (2012). *Figurate Numbers*, Word Scientific.

[5] Dickson, L. E. (1934). *History of the Theory of Numbers, Vol. II: Diophantine Analysis*, Dover Publications.

[6] Eggan, L. C., Eggan, P. C., & Selfridge, J. L. (1982). Polygonal products of polygonal numbers and the Pell equation, *Fibonacci Quart.*, 20 (1), 24–28.

[7] Guan, X. G. (2011). The squares with the form $1+ \frac{4n(n+1)s^2}{s^2-1}$, *Natural Science Journal of Ningxia Teachers University*, 32 (3), 97–107.

[8] Guy, R. K. (2007). *Unsolved Problems in Number Theory*, Springer-Verlag.

[9] Hu, M. J. (2013). The positive integer solutions of the Diophantine equation $1 + n\binom{x}{2} = y^2$, *Journal of Zhejiang International Studies University*, 4, 70–76.

[10] Ke, S., & Sun, Q. (1980). *About Indeterminate Equation*, Harbin Institute of Technology Press.

[11] Le, M. H. (2007). The squares with the form $1+ \frac{9}{2}n(n+1)$, *Natural Science Journal of Hainan University*, 25 (1), 13–14.

[12] Peng, J. Y. (2019). The linear combination of two triangular numbers is a perfect square, *Notes on Number Theory and Discrete Mathematics*, 25 (3), 1–12.

[13] Ran, Y. X., Yan, S. J., Ran, Y. P., & Yang, X. Y. (2008). The squares with the form $\frac{1+25n(n+1)}{2}$, *Journal of Tianshui Normal University*, 28 (5), 9–15.

[14] Sun, Z. H. (2009). On the number of representations of n by $a\frac{x(x-1)}{2} + b\frac{y(y-1)}{2}$, *Journal of Number Theory*, 129 (5), 971–989.

[15] Wu, H. M. (2011). The square numbers with the form $1 + p^2\frac{n(n+1)}{2}$, *Journal of Zhanjiang Normal College*, 32 (3), 20–22.