Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 26, 2020, No. 2, 85–104
DOI: 10.7546/nntdm.2020.26.2.85-104

Infinite series containing generalized harmonic functions

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Received: 4 December 2019 **Revised:** 10 April 2020 **Accepted:** 24 April 2020

Abstract: We use Abel's summation formula and the method of partial fraction decomposition to study infinite series involving generalized harmonic numbers of any positive integral order, with any positive integral power.

Keywords: Abel's summation formula, Generalized harmonic functions, Multiple Hurwitz zeta functions, Digamma function.
2010 Mathematics Subject Classification: 05A19, 11M06, 40A25.

1 Introduction

Let $\mathbb{N}$ be the set of all positive integers, and $\mathbb{N}_0$ be the set of all nonnegative integers. For $m, n \in \mathbb{N}$, $z \in \mathbb{C}$ with $z \neq -1, -2, \ldots, -n$, the generalized harmonic functions [5,7] are defined by $H_n^{(m)}(z) = \sum_{k=1}^{n} \frac{1}{(k+z)^m}$. The generalized harmonic function $H_n^{(m)}(z)$ is a generalization of the generalized harmonic number $H_n^{(m)}(0) = H_n^{(m)}$. Also we have

$$O_n^{(m)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^m} = 2^{-m} H_n^{(m)}(-1/2).$$

It is known that

$$H_n^{(1)} = H_n = \gamma + \psi(n+1), \tag{1.1}$$

where γ is Euler's constant and ψ is the digamma function given by

$$\psi(x) = -\gamma + \sum_{k=0}^{\infty} \left\{ \frac{1}{k+1} - \frac{1}{k+x} \right\}. \tag{1.2}$$

In physics, the computation of Feynman integrals in massive higher order perturbative calculations in renormalizable quantum field theories requires extensions of multiplying nested harmonic sums [1, 14]. Binomial, inverse binomial and harmonic number series have been studied in order to perform calculations of higher order corrections to scattering processes in particle physics [2, 9, 10, 13, 26]. Hence, there is a great motivation to study the representation of the following Euler series in closed forms:

$$\sum_{n=1}^{\infty} \frac{(H_n^{(p)})^m}{(n+a)(n+a+1)}, \quad \sum_{n=1}^{\infty} \frac{(H_n^{(p)})^m}{\binom{n+k}{k}}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(H_n^{(p)})^m}{n\binom{n+k}{k}}. \tag{1.3}$$

In the literature there exists a lot of results on these series; for example, Sofo [20] studied the case $m = 1$ and $p \in \mathbb{N}$, the cases with $m, p \leq 2$ were surveyed in [8, 18, 19, 21, 24, 25], and Xu, Zhang, Zhu [27] investigated the cases $m = 3, p = 1$ and $m = 2$, $p \in \mathbb{N}$, etc. The interested reader should refer to Hoffman's webpage [12] for more references.

Abel's summation formula is given by [23, Theorem 6.30]

$$\sum_{k=m}^{n} a_k b_k = A_{n,m} b_n - \sum_{k=m}^{n-1} A_{k,m}(b_{k+1} - b_k),$$

where $\{a_k\}$, $\{b_k\}$ are two sequences, and $A_{n,m} = \sum_{k=m}^{n} a_k$. This formula is often employed in studies dealing with the convergence of sequences. Abel's summation formula is modified by Chu [8] to derive many infinite series identities involving the harmonic numbers and their variants. We require Abel's summation formula in the following form: For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and Δ, respectively, by

$$\nabla \tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \Delta\, \tau_k = \tau_k - \tau_{k+1}.$$

Then we have

$$\sum_{k=\varepsilon}^{\infty} V_k \nabla U_k = [UV]_+ - U_{\varepsilon-1} V_\varepsilon + \sum_{k=\varepsilon}^{\infty} U_k \,\Delta\, V_k \tag{1.4}$$

provided that one of both series is convergent and the limit $[UV]_+ = \lim_{n\to\infty} U_n V_{n+1}$ exists.

Using Abel's summation formula, Wang [24] recently established infinite series involving generalized harmonic numbers of order 2 and order 3. For example, for any two positive real numbers a and b [24, Theorem 1],

$$\sum_{k\geq 1} \frac{a h_k^{(2)}(a,b)}{(ak+b)(ak-a+b)} = \sum_{k\geq 0} \frac{1}{(ak+b)^3},$$

where the harmonic numbers of order m is defined by $h_n^{(m)}(a,b)=\sum_{k=1}^n \frac{1}{(ak-a+b)^m}$. In fact, all results in [24] are special values of our formulas. In 2018, Wang & Chu [25] obtained infinite series involving quadratic and cubic harmonic numbers, for example [25, Eq. (6)],

$$\sum_{k\geq 1}\frac{a^2h_k^2(a,b)}{(ak+b)(ak+a+b)}=\frac{1}{ab}+\sum_{k\geq 0}\frac{1}{(ak+b)^2}.$$

Note that the generalized harmonic function $H_n^{(m)}(z)$ is related to $h_n^{(m)}(a,b)$ by

$$H_n^{(m)}(z)=a^m h_n^{(m)}(a,b),$$

with $z=(b-a)/a$. Following Wang and Chu, we find that if we apply the method of partial fraction decomposition additionally, we can then obtain infinite series involving generalized harmonic numbers of any positive integral order, with any positive integral power, for example (ref. Eq. (3.3))

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^2}{(n+z+a)(n+z+a+1)}&=\frac{-1}{a^{2s}}H_a(z)+\frac{2}{a^s}\sum_{k=1}^{a}\frac{H_k(z)}{k^s}+2\sum_{k=0}^{s-2}\frac{(-1)^k}{a^{k+1}}\zeta(s,s-k;z)\\&+\sum_{k=0}^{2s-2}\frac{(-1)^k}{a^{k+1}}\zeta(2s-k;z)+2\sum_{k=0}^{s-2}\frac{(-1)^{k+s+1}H_a^{(k+1)}}{a^s}\zeta(s-k;z),\end{aligned}$$

where the multiple Hurwitz zeta function of depth k and weight $s_1+\cdots+s_k$ is defined by [3, 5, 15, 17]

$$\zeta(s_1,\ldots,s_k;z)=\sum_{1\leq n_1<n_2<\cdots<n_k}\frac{1}{(n_1+z)^{s_1}(n_2+z)^{s_2}\cdots(n_k+z)^{s_k}}.$$

It is known that multiple zeta values (MZVs) $\zeta(s_1,\ldots,s_k)=\zeta(s_1,\ldots,s_k;0)$ and multiple t-values (MtVs) $t(s_1,\ldots,s_k)=2^{-(s_1+\cdots+s_k)}\zeta(s_1,\ldots,s_k;-1/2)$.

Here we highlight one new infinite series expression (ref. Example 5.3): for s,k any two positive integers,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{O_n^{(s)}}{n\binom{n+k}{k}}&=\sum_{r=0}^{k-1}\binom{k-1}{r}\frac{(-1)^{r+s+1}(H_r+2\log(2))}{(2r+1)^s}\\&\quad+2\sum_{\ell=0}^{s-2}(-1)^{\ell}t(s-\ell)\sum_{r=0}^{k-1}\binom{k-1}{r}\frac{(-1)^r}{(2r+1)^{\ell+1}}.\end{aligned}$$

This result can be compared with the following formula in [5, Example 5.2], [20, Eq.(2.3)].

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^{(s)}}{n\binom{n+k}{k}}&=\zeta(s+1)+\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\\&\quad\times\left[\sum_{j=1}^{r-1}\frac{(-1)^{s+1}}{j^s}H_j+\sum_{\ell=2}^{s}(-1)^s-\ell H_{r-1}^{(s+1-\ell)}\zeta(\ell)\right].\end{aligned}$$

2 Preliminaries

Let $\mathbb{Q}[[x_1, x_2, \ldots]]$ be the set of formal power series of bounded degree. An element $u \in \mathbb{Q}[[x_1, x_2, \ldots]]$ such that the coefficient in u of any monomial $x_{a_1}^{\alpha_1} x_{a_2}^{\alpha_2} \cdots x_{a_k}^{\alpha_k}$ with $a_1 < a_2 < \cdots < a_k$ is the same as that of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ is called quasi-symmetric [11, 16, 22].

Let QSym be the subring of all the quasi-symmetric functions. For any composition (ordered partition) $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n, we call the number of parts k as the length of I; and the sum of parts $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ as the weight of $\boldsymbol{\alpha}$.

The "smallest" quasi-symmetric function containing the term $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ is denoted by $M_{(\alpha_1, \alpha_2, \ldots, \alpha_k)}$. It is convenient to denote $M_0 = 1$. These elements form a basis for QSym.

For positive integers n, s, we set $x_i = 1/(i + z)^s$, for $1 \leq i \leq n$; $x_i = 0$, for $i > n$. Then

$$M_{(\alpha_1, \alpha_2, \ldots, \alpha_k)} = \zeta_n(s\alpha_1, s\alpha_2, \ldots, s\alpha_k; z)$$

forms a basis for this particular QSym, which we denote by $\mathfrak{Q}$. By the well-known identity [5, 6, 16]

$$\left(M_{(1)}\right)^k = \sum_{\lambda \vdash k} \binom{k}{\lambda} M_\lambda,$$

where $\lambda \vdash k$ means that λ is a partition of k, or from an elementary result in the harmonic algebra (Hoffman's harmonic algebra) [6, Eq. (5)], we know that

$$H_n^{(s)}(z)^k = \sum_{r=1}^{k} \sum_{\substack{|\boldsymbol{\alpha}|=k \\ \alpha_i \geq 1}} \binom{k}{\alpha_1, \alpha_2, \ldots, \alpha_r} \zeta_n(s\alpha_1, s\alpha_2, \ldots, s\alpha_r; z), \tag{2.1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ and $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_r$.

Lemma 2.1. *[4, Proposition 3.2] For $m, n \in \mathbb{N}_0$, and $x \in \mathbb{C} \backslash \{0, -1, \ldots, -n\}$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(x+k)^m} = B(n+1, x) P_{m-1}(H_{n+1}^{(1)}(1, x), H_{n+1}^{(2)}(1, x), \ldots, H_{n+1}^{(m-1)}(1, x)),$$

where $B(x, y)$ is Euler beta function, $P_m(x_1, x_2, \ldots, x_m)$ is the modified Bell polynomials defined by

$$\exp\left(\sum_{k=1}^{\infty} \frac{x_k}{k} z^k\right) = \sum_{m=0}^{\infty} P_m(x_1, x_2, \ldots, x_m) z^m,$$

and the finite Hurwitz–Lerch function $H_n^{(s)}(x, y)$ is defined by

$$H_n^{(s)}(x, y) = \sum_{k=0}^{n-1} \frac{x^k}{(k+y)^s}.$$

Since

$$\begin{aligned}\sum_{n=0}^{\infty} z^n P_n(ax_1, a^2 x_2, \ldots, a^n x_n) &= \exp\left(\sum_{k=1}^{\infty} \frac{a^k x_k}{k} z^k\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \frac{x_k}{k} (az)^k\right) = \sum_{n=0}^{\infty} z^n a^n P_n(x_1, x_2, \ldots, x_n),\end{aligned}$$

the modified Bell polynomials P_m have the following property:

$$P_n(ax_1, a^2x_2, \ldots, a^nx_n) = a^nP_n(x_1, x_2, \ldots, x_n). \tag{2.2}$$

For a complex number z, positive integers k, ℓ, and $(s_1, \ldots, s_k) \in \mathbb{N}_0^k$, with $z + \frac{j}{\ell} \neq 0$, for $1 \leq j \leq k$, we denote the symbol $\eta_{s_1,\ldots,s_k}(z; \ell)$ as follows:

$$\eta_{s_1,\ldots,s_k}(z; \ell) = \frac{1}{(z + \frac{1}{\ell})^{s_1}(z + \frac{2}{\ell})^{s_2} \cdots (z + \frac{k}{\ell})^{s_k}}.$$

If we do not change the variables z and ℓ of $\eta_{s_1,\ldots,s_k}(z; \ell)$, then we would abbreviate it as $\eta_{s_1,\ldots,s_k}$. For convenience, we denote the k repetitions of a in the subscript of the η function by a^k, for example, $\eta_{1,0,0,0,2,2} = \eta_{1,0^3,2^2}$.

Lemma 2.2. *[5, Lemma 2.2] Let $s_1, \ldots, s_k$ be a nonnegative integer sequence with $s_i, s_j \geq 1$ for $1 \leq i < j \leq k$. If $s_1 + \cdots + s_k \geq 3$, then*

$$\eta_{s_1,\ldots,s_i,\ldots,s_j,\ldots,s_k}(z; \ell) = \frac{\ell}{j - i}\left(\eta_{s_1,\ldots,s_i,\ldots,s_j-1,\ldots,s_k}(z; \ell) - \eta_{s_1,\ldots,s_i-1,\ldots,s_j,\ldots,s_k}(z; \ell)\right).$$

In this paper we will use the following four basic partial fraction decomposition formulas.

Lemma 2.3. *For integers $a \geq 0$ and $s \geq 1$, we have*

$$\eta_{s,0^a,1}(z; \ell) = \frac{(-\ell)^{s-1}}{(a+1)^{s-1}}\eta_{1,0^a,1}(z; \ell) - \sum_{k=0}^{s-2} \frac{(-\ell)^{k+1}}{(a+1)^{k+1}}\eta_{s-k}(z; \ell), \tag{2.3}$$

$$\eta_{1,0^a,1}(z; \ell) = \frac{1}{a+1}\sum_{k=0}^{a} \eta_{0^k,1^2}(z; \ell), \tag{2.4}$$

$$\eta_{1,0^a,s}(z; \ell) = \frac{\ell^{s-1}}{(a+1)^{s-1}}\eta_{1,0^a,1}(z; \ell) - \sum_{k=0}^{s-2} \frac{\ell^{k+1}}{(a+1)^{k+1}}\eta_{0^{a+1},s-k}(z; \ell), \tag{2.5}$$

$$\eta_{1^{a+2}}(z; \ell) = \frac{\ell^a}{(a+1)!}\sum_{r=0}^{a}(-1)^r\binom{a}{r}\eta_{0^r,1^2}(z; \ell). \tag{2.6}$$

Proof. We use induction on the positive integer s to prove Eq. (2.3) and Eq. (2.5). Lemma 2.2 gives

$$\eta_{s,0^a,1} = \frac{\ell}{a+1}\left(\eta_s - \eta_{s-1,0^a,1}\right), \quad \text{and} \quad \eta_{1,0^a,s} = \frac{\ell}{a+1}\left(\eta_{1,0^a,s-1} - \eta_{0^{a+1},s}\right),$$

from which the results follow when we invoke the induction hypothesis.

We use Lemma 2.2 to obtain two expressions for $\eta_{1,0^a,1^2}(z; \ell)$:

$$\ell(\eta_{1,0^a,1} - \eta_{1,0^{a+1},1}) = \eta_{1,0^a,1^2}(z; \ell) = \frac{\ell}{a+2}(\eta_{1,0^a,1} - \eta_{0^a,1^2}).$$

Therefore we have

$$\eta_{1,0^{a+1},1} = \frac{a+1}{a+2}\eta_{1,0^a,1} + \frac{1}{a+2}\eta_{0^a,1^2}.$$

We use this identity and use induction on the positive integer a, then we can obtain the result of Eq. (2.4).

Eq. (2.6) is clearly true for $a = 0$. Assume that

$$\eta_{1^{a+1}}(z;\ell) = \frac{\ell^{a-1}}{a!}\sum_{r=0}^{a-1}(-1)^r\binom{a-1}{r}\eta_{0^r,1^2}(z;\ell)$$

is true for $a \geq 1$. We use Lemma 2.2 to obtain

$$\eta_{1^{a+2}}(z;\ell) = \frac{\ell}{a+1}(\eta_{1^{a+1}}(z;\ell) - \eta_{0,1^{a+1}}(z;\ell)).$$

Since $\eta_{0,1^{a+1}}(z;\ell) = \eta_{1^{a+1}}(z+\frac{1}{\ell};\ell)$, we can apply the inductive hypothesis to these two terms of the right-hand side of the above identity.

$$\begin{aligned}
\eta_{1^{a+2}}(z;\ell) &= \frac{\ell}{a+1}\left(\frac{\ell^{a-1}}{a!}\sum_{r=0}^{a-1}(-1)^r\binom{a-1}{r}\eta_{0^r,1^2}(z;\ell)\right) \\
&\quad - \frac{\ell}{a+1}\left(\frac{\ell^{a-1}}{a!}\sum_{r=0}^{a-1}(-1)^r\binom{a-1}{r}\eta_{0^r,1^2}(z+\frac{1}{\ell};\ell)\right) \\
&= \frac{\ell^a}{(a+1)!}\sum_{r=0}^{a-1}(-1)^r\binom{a-1}{r}\eta_{0^r,1^2}(z;\ell) \\
&\quad - \frac{\ell^a}{(a+1)!}\sum_{r=0}^{a-1}(-1)^r\binom{a-1}{r}\eta_{0^{r+1},1^2}(z;\ell) \\
&= \frac{\ell^a}{(a+1)!}\left(\sum_{r=0}^{a-1}(-1)^r\binom{a-1}{r}\eta_{0^r,1^2}(z;\ell) - \sum_{r=1}^{a}(-1)^{r-1}\binom{a-1}{r-1}\eta_{0^r,1^2}(z;\ell)\right).
\end{aligned}$$

Using the fact $\binom{a-1}{r} + \binom{a-1}{r-1} = \binom{a}{r}$ we get the desired result. By the mathematical induction we get Eq. (2.6). □

Using Eq.(2.6) the last two Euler series in Eq. (1.3) can be represented as

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(H_n^{(p)})^m}{\binom{n+k}{k}} &= k\sum_{r=0}^{k-2}(-1)^r\binom{k-2}{r}\sum_{n=1}^{\infty}\frac{(H_n^{(p)})^m}{(n+r+1)(n+r+2)}, &&\text{if } k \geq 2; \\
\sum_{n=1}^{\infty}\frac{(H_n^{(p)})^m}{n\binom{n+k}{k}} &= \sum_{r=0}^{k-1}(-1)^r\binom{k-1}{r}\sum_{n=1}^{\infty}\frac{(H_n^{(p)})^m}{(n+r+1)(n+r+2)}, &&\text{if } k \geq 1.
\end{aligned}$$

Therefore we investigate the following series in this paper

$$Y_a^{(s)}(m;z) = \sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^m}{(n+z+a)(n+z+a+1)},$$

where s, m are positive integers and $a \in \mathbb{Z}$, $z \in (-1, 0]$ in order to give them some simple representations in terms of multiple Hurwitz zeta functions and generalized harmonic functions.

3 Infinite series of $Y_a^{(s)}(m;z)$ with $a \in \mathbb{N}_0$

Theorem 3.1. *Let s, m, a be three positive integers and $z \in (-1, 0]$. Then we have*

$$
\begin{aligned}
Y_a^{(s)}(m;z) &= \sum_{n=1}^{\infty} \frac{\left(H_n^{(s)}(z)\right)^m}{(n+z+a)(n+z+a+1)} \\
&= \frac{(-1)^{ms-1}}{a^{ms}} H_a(z) - \sum_{k=0}^{ms-2} \frac{(-1)^{k+1}}{a^{k+1}} \zeta(ms-k;z) \\
&\quad + \sum_{k=1}^{m-1} \binom{m}{k} \sum_{\ell=0}^{(m-k)s-2} \frac{(-1)^{\ell}}{a^{\ell+1}} \sum_{r=1}^{k} \sum_{\substack{|\boldsymbol{\alpha}|=k \\ \alpha_i \geq 1}} \binom{k}{\alpha_1, \ldots, \alpha_r} \zeta(s\alpha_1, \ldots, s\alpha_r, (m-k)s-\ell; z) \\
&\quad + \sum_{k=1}^{m-1} \binom{m}{k} \frac{(-1)^{(m-k)s-1}}{a^{(m-k)s}} \sum_{r=1}^{a} Y_r^{(s)}(k;z).
\end{aligned}
$$

Proof. Consider two sequences V_n and U_n, given by

$$
V_n = \left(H_n^{(s)}(z)\right)^m, \quad \text{and} \quad U_n = \frac{-1}{n+z+a+1}. \tag{3.1}
$$

Then it is easy to check that

$$
\Delta V_n = -\sum_{\ell=0}^{m-1} \binom{m}{\ell} \frac{\left(H_n^{(s)}(z)\right)^{\ell}}{(n+1+z)^{(m-\ell)s}}, \quad \text{and} \quad \nabla U_n = \frac{1}{(n+z+a)(n+z+a+1)}
$$

as well as the relations

$$
[UV]_+ = 0, \quad \text{and} \quad U_0 V_1 = \frac{-1}{(1+z)^{sm}(z+a+1)}.
$$

Applying Abel's summation formula Eq. (1.4) with $\varepsilon = 1$, we can reformulate

$$
\begin{aligned}
Y_a^{(s)}(m;z) &= \sum_{n=1}^{\infty} \frac{\left(H_n^{(s)}(z)\right)^m}{(n+z+a)(n+z+a+1)} \\
&= \sum_{n=1}^{\infty} V_n \nabla U_n \;=\; \frac{1}{(1+z)^{sm}(z+a+1)} + \sum_{n=1}^{\infty} U_n \Delta V_n \\
&= \sum_{n=1}^{\infty} \frac{1}{(n+z)^{ms}(n+z+a)} + \sum_{k=1}^{m-1} \binom{m}{k} \sum_{n=1}^{\infty} \frac{\left(H_n^{(s)}(z)\right)^k}{(n+1+z)^{(m-k)s}(n+z+a+1)}.
\end{aligned}
$$

Using Eq. (2.3), the first term of the last equation can be rewritten as

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{(n+z)^{ms}(n+z+a)} &= \sum_{n=1}^{\infty} \eta_{ms,0^{a-1},1}(n+z-1;1) \\
&= \sum_{n=1}^{\infty} \left[\frac{(-1)^{ms-1}}{a^{ms-1}} \eta_{1,0^{a-1},1}(n+z-1;1) - \sum_{k=0}^{ms-2} \frac{(-1)^{k+1}}{a^{k+1}} \eta_{ms-k}(n+z-1;1) \right] \\
&= \frac{(-1)^{ms-1}}{a^{ms-1}} \sum_{n=1}^{\infty} \frac{1}{(n+z)(n+z+a)} - \sum_{k=0}^{ms-2} \frac{(-1)^{k+1}}{a^{k+1}} \zeta(ms-k;z).
\end{aligned}
$$

Using a similar trick, the second term can be written as

$$\sum_{k=1}^{m-1}\binom{m}{k}\frac{(-1)^{(m-k)s-1}}{a^{(m-k)s-1}}\sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^k}{(n+1+z)(n+z+a+1)}$$
$$+\sum_{k=1}^{m-1}\binom{m}{k}\sum_{\ell=0}^{(m-k)s-2}\frac{(-1)^{\ell}}{a^{\ell+1}}\sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^k}{(n+1+z)^{(m-k)s-\ell}}.$$

The series $\sum_{n=1}^{\infty}\frac{1}{(n+z)(n+z+a)}$ can be simplifed as $\frac{1}{a}H_a(z)$. By Eq. (2.1), the last term in the above equation can be rewritten as

$$\begin{aligned}&\sum_{k=1}^{m-1}\binom{m}{k}\sum_{\ell=0}^{(m-k)s-2}\frac{(-1)^{\ell}}{a^{\ell+1}}\sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^k}{(n+1+z)^{(m-k)s-\ell}}\\ =\;&\sum_{k=1}^{m-1}\binom{m}{k}\sum_{\ell=0}^{(m-k)s-2}\frac{(-1)^{\ell}}{a^{\ell+1}}\sum_{r=1}^{k}\sum_{\substack{|\boldsymbol{\alpha}|=k\\ \alpha_i\geq 1}}\binom{k}{\alpha_1,\ldots,\alpha_r}\zeta(s\alpha_1,\ldots,s\alpha_r,(m-k)s-\ell;z).\end{aligned}$$

We use Eq. (2.4) to reformula the term $\sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^k}{(n+1+z)(n+z+a+1)}$ into the following

$$\sum_{r=0}^{a-1}\frac{1}{a}\sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^k}{(n+r+z+1)(n+r+z+2)}=\sum_{r=1}^{a}\frac{1}{a}Y_r^{(s)}(k;z).$$

Therefore, we finally obtain

$$\begin{aligned}&Y_a^{(s)}(m;z)\\ =\;&\frac{(-1)^{ms-1}}{a^{ms}}H_a(z)-\sum_{k=0}^{ms-2}\frac{(-1)^{k+1}}{a^{k+1}}\zeta(ms-k;z)\\ &+\sum_{k=1}^{m-1}\binom{m}{k}\sum_{\ell=0}^{(m-k)s-2}\frac{(-1)^{\ell}}{a^{\ell+1}}\sum_{r=1}^{k}\sum_{\substack{|\boldsymbol{\alpha}|=k\\ \alpha_i\geq 1}}\binom{k}{\alpha_1,\ldots,\alpha_r}\zeta(s\alpha_1,\ldots,s\alpha_r,(m-k)s-\ell;z)\\ &+\sum_{k=1}^{m-1}\binom{m}{k}\frac{(-1)^{(m-k)s-1}}{a^{(m-k)s}}\sum_{r=1}^{a}Y_r^{(s)}(k;z).\end{aligned}$$

□

For any fixed positive integer a, the formula in Theorem 3.1 is a recursive formula for $Y_a^{(s)}(m;z)$. The initial value $Y_a^{(s)}(1;z)$ is also obtainable from this recursive formula. Therefore we use this recursive relation of $Y_a^{(s)}(m;z)$, we can find all the explicit formula for $Y_a^{(s)}(m;z)$. We state the explicit formulas of $Y_a^{(s)}(m;z)$ for the first two values $m=1,2$ as follows.

Corollary 3.2. *Let s, a be two positive integers, and $z \in (-1, 0]$. Then we have*

$$\sum_{n=1}^{\infty} \frac{H_n^{(s)}(z)}{(n+z+a)(n+z+a+1)} = \frac{(-1)^{s-1}}{a^s} H_a(z) - \sum_{k=0}^{s-2} \frac{(-1)^{k+1}}{a^{k+1}} \zeta(s-k; z), \tag{3.2}$$

$$\sum_{n=1}^{\infty} \frac{\left(H_n^{(s)}(z)\right)^2}{(n+z+a)(n+z+a+1)} = \frac{-1}{a^{2s}} H_a(z) + \frac{2}{a^s} \sum_{k=1}^{a} \frac{H_k(z)}{k^s} + 2 \sum_{k=0}^{s-2} \frac{(-1)^k}{a^{k+1}} \zeta(s, s-k; z)$$
$$+ \sum_{k=0}^{2s-2} \frac{(-1)^k}{a^{k+1}} \zeta(2s-k; z) + 2 \sum_{k=0}^{s-2} \frac{(-1)^{k+s+1} H_a^{(k+1)}}{a^s} \zeta(s-k; z). \tag{3.3}$$

Substituting $a = 1$, $z = (d-c)/c$, and $s = 2$ into Eq. (3.2) we have [24, Theorem 4]

$$\sum_{n=1}^{\infty} \frac{c h_n^{(2)}(c,d)}{(cn+d)(cn+c+d)} = \frac{1}{c} \sum_{n=0}^{\infty} \frac{1}{(cn+d)^2} - \frac{1}{c^2 d},$$

where c, d are positive integers. If we substitute $a = 2$, $z = (d-c)/c$, and $s = 2$ into Eq. (3.2), then the resulted identity is the main identity appeared in [24, Theorem 7]. Furthermore, if we set $s = 3$ and $z = (d-c)/c$ in Eq. (3.2), then we have [24, Theorem 28] with $a = 1$ and [24, Theorem 32] with $a = 2$.

Moreover, if we substitue $s = 1 = a$ and $z = (d-c)/c$ into Eq. (3.3), then we have (see [25, Eq. (6)])

$$\sum_{n=1}^{\infty} \frac{c^2 h_n^2(c,d)}{(cn+d)(cn+c+d)} = \frac{1}{cd} + \sum_{n=0}^{\infty} \frac{1}{(cn+d)^2}. \tag{3.4}$$

Applying $s = 1$, $a = 2$, and $z = (d-c)/c$ in Eq. (3.3) we have the following new identity

$$\sum_{n=1}^{\infty} \frac{c^2 h_n^2(c,d)}{(cn+c+d)(cn+2c+d)} = \frac{5c+6d}{4cd(c+d)} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{(cn+d)^2}.$$

Using Eq. (2.6) we know that $\eta_{111} = \frac{1}{2}(\eta_{11} - \eta_{011})$. Therefore

$$\sum_{n=1}^{\infty} \frac{4c^3 h_n^2(c,d)}{(cn+d)(cn+c+d)(cn+2c+d)}$$
$$= \frac{1}{2} \left(\sum_{n=1}^{\infty} \frac{4c^2 h_n^2(c,d)}{(cn+d)(cn+c+d)} - \sum_{n=1}^{\infty} \frac{4c^2 h_n^2(c,d)}{(cn+c+d)(cn+2c+d)} \right)$$
$$= -\frac{c+2d}{2cd(c+d)} + \sum_{n=0}^{\infty} \frac{1}{(cn+d)^2}.$$

The above identity is the same as [25, Eq. (7)].

Following the same method of the above theorem we derive the closed form of $Y_0^{(s)}(m; z)$ as the following.

Theorem 3.3. *For positive integers s, m and $z \in (-1, 0]$, we have*

$$\sum_{n=1}^{\infty} \frac{\left(H_n^{(s)}(z)\right)^m}{(n+z)(n+z+1)} = \zeta(ms+1; z) + \sum_{k=1}^{m-1} \binom{m}{k} \sum_{r=1}^{k} \sum_{\substack{|\boldsymbol{\alpha}|=k \\ \alpha_i \geq 1}} \binom{k}{\alpha_1, \ldots, \alpha_r} \zeta(s\alpha_1, \ldots, s\alpha_r, (m-k)s+1; z). \tag{3.5}$$

Proof. Using the same sequences V_k and U_k as Eq. (3.1), we again apply Abel's summation formula Eq. (1.4) with $\varepsilon = 1$. Then $Y_0^{(s)}(m; z)$ becomes

$$\sum_{n=1}^{\infty} \frac{1}{(n+z)^{ms+1}} + \sum_{k=1}^{m-1} \binom{m}{k} \sum_{n=1}^{\infty} \frac{\left(H_n^{(s)}(z)\right)^k}{(n+1+z)^{(m-k)s+1}}.$$

From the known result Eq. (2.1), we get the desired identity. □

If we put $m = 1$ and $z = (d-c)/c$ in Eq. (3.5), then we have

$$\sum_{n=1}^{\infty} \frac{ch_n^{(s)}(c, d)}{(cn - c + d)(cn + d)} = \sum_{n=0}^{\infty} \frac{1}{(cn+d)^{s+1}}. \tag{3.6}$$

In particular, [8, Proposition 1] is the identity with $s = 1$, [24, Theorem 1] is the identity with $s = 2$, and [24, Theorem 25] is the identity with $s = 3$.

4 Infinite series of $Y_{-a}^{(s)}(m; z)$ with $a \in \mathbb{N}$

Theorem 4.1. *Let s, m, a be three positive integers and $z \in (-1, 0)$. Then we have*

$$\begin{aligned}
Y_{-a}^{(s)}(m; z) &= \sum_{n=1}^{\infty} \frac{\left(H_n^{(s)}(z)\right)^m}{(n+z-a)(n+z-a+1)} \\
&= \frac{H_a(z-a)}{a^{ms}} - \sum_{k=0}^{ms-2} \frac{1}{a^{k+1}} \zeta(ms-k; z) \\
&\quad - \sum_{k=1}^{m-1} \binom{m}{k} \sum_{\ell=0}^{(m-k)s-2} \frac{1}{a^{\ell+1}} \sum_{r=1}^{k} \sum_{\substack{|\boldsymbol{\alpha}|=k \\ \alpha_i \geq 1}} \binom{k}{\alpha_1, \ldots, \alpha_r} \zeta(s\alpha_1, \ldots, s\alpha_r, (m-k)s-\ell; z) \\
&\quad + \sum_{k=1}^{m-1} \binom{m}{k} \frac{1}{a^{(m-k)s}} \sum_{r=0}^{a-1} Y_{-r}^{(s)}(k; z).
\end{aligned}$$

Proof. We let the two sequences V_n and U_n be as follows:

$$V_n = \left(H_n^{(s)}(z)\right)^m, \quad \text{and} \quad U_n = \frac{-1}{n+z-a+1}.$$

Then it is easy to check that

$$\Delta V_n = -\sum_{\ell=0}^{m-1}\binom{m}{\ell}\frac{\left(H_n^{(s)}(z)\right)^{\ell}}{(n+1+z)^{(m-\ell)s}}, \quad \text{and} \quad \nabla U_n = \frac{1}{(n+z-a)(n+z-a+1)}$$

as well as the relations

$$[UV]_+ = 0, \quad \text{and} \quad U_0V_1 = \frac{-1}{(1+z)^{sm}(z-a+1)}.$$

Applying Abel's summation formula Eq. (1.4) with $\varepsilon = 1$, we can reformulate

$$\begin{aligned}
Y_{-a}^{(s)}(m;z) &= \sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^{m}}{(n+z-a)(n+z-a+1)} \\
&= \sum_{n=1}^{\infty} V_n \nabla U_n = \frac{1}{(1+z)^{sm}(z-a+1)} + \sum_{n=1}^{\infty} U_n \Delta V_n \\
&= \sum_{n=1}^{\infty}\frac{1}{(n+z)^{ms}(n+z-a)} + \sum_{k=1}^{m-1}\binom{m}{k}\sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^{k}}{(n+1+z)^{(m-k)s}(n+z-a+1)}.
\end{aligned}$$

Using Eq. (2.5), the above last equation becomes

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{a^{ms-1}}\frac{1}{(n+z)(n+z-a)} - \sum_{k=0}^{ms-2}\frac{1}{a^{k+1}}\zeta(ms-k;z) \\
&+ \sum_{k=1}^{m-1}\binom{m}{k}\frac{1}{a^{(m-k)s-1}}\sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^{k}}{(n+1+z)(n+z-a+1)} \\
&- \sum_{k=1}^{m-1}\binom{m}{k}\sum_{\ell=0}^{(m-k)s-2}\frac{1}{a^{\ell+1}}\sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^{k}}{(n+1+z)^{(m-k)s-\ell}}.
\end{aligned}$$

The series $\sum_{n=1}^{\infty}\frac{1}{(n+z)(n+z-a)}$ can be simplifed as $\frac{1}{a}H_a(z-a)$. Applying Eq. (2.1), we can rewrite the last term in the above equation as

$$\begin{aligned}
&\sum_{k=1}^{m-1}\binom{m}{k}\sum_{\ell=0}^{(m-k)s-2}\frac{1}{a^{\ell+1}}\sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^{k}}{(n+1+z)^{(m-k)s-\ell}} \\
= &\sum_{k=1}^{m-1}\binom{m}{k}\sum_{\ell=0}^{(m-k)s-2}\frac{1}{a^{\ell+1}}\sum_{r=1}^{k}\sum_{\substack{|\boldsymbol{\alpha}|=k \\ \alpha_i\ge 1}}\binom{k}{\alpha_1,\ldots,\alpha_r}\zeta_n(s\alpha_1,\ldots,s\alpha_r,(m-k)s-\ell;z).
\end{aligned}$$

We use Eq. (2.4) to reformula the term $\sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^{k}}{(n+1+z)(n+z-a+1)}$ into the following

$$\sum_{r=0}^{a-1}\frac{1}{a}\sum_{n=1}^{\infty}\frac{\left(H_n^{(s)}(z)\right)^{k}}{(n+r+z+1-a)(n+r+z+2-a)} = \sum_{r=0}^{a-1}\frac{1}{a}Y_{-r}^{(s)}(k;z).$$

Therefore we finally get the desired form. □

For any fixed positive integer a, the formula in Theorem 4.1 is a recursive formula for $Y_{-a}^{(s)}(m;\, z)$. Note that we use Theorem 3.3 to give the value of $Y_0^{(s)}(1; z) = \zeta(s+1;\, z)$. In what follows, we list the explicit formulas of $Y_{-a}^{(s)}(m;\, z)$ for the first two values $m = 1$ and $m = 2$.

Corollary 4.2. *Let s, a be two positive integers, and $z \in (-1, 0)$. Then we have*

$$\sum_{n=1}^{\infty} \frac{H_n^{(s)}(z)}{(n+z-a)(n+z-a+1)} = \frac{H_a(z-a)}{a^s} - \sum_{k=0}^{s-2} \frac{\zeta(s-k; z)}{a^{k+1}}, \tag{4.1}$$

$$\sum_{n=1}^{\infty} \frac{\left(H_n^{(s)}(z)\right)^2}{(n+z-a)(n+z-a+1)} = \frac{H_a(z-a)}{a^{2s}} + \frac{2}{a^s}\zeta(s+1; z) + \frac{2}{a^s}\sum_{k=1}^{a-1} \frac{H_k(z-k)}{k^s}$$
$$- \sum_{k=0}^{2s-2} \frac{\zeta(2s-k; z)}{a^{k+1}} - \sum_{k=0}^{s-2} \frac{2}{a^{k+1}}\zeta(s, s-k; z) - \frac{2}{a^s}\sum_{k=0}^{s-2} H_{a-1}^{(k+1)}\zeta(s-k; z). \tag{4.2}$$

Substituting $a = 1$, $z = (d-c)/c$ into Eq. (4.1) we have

$$\sum_{n=1}^{\infty} \frac{ch_n^{(s)}(c,d)}{(cn+d-2c)(cn+d-c)} = \frac{1}{c^s(d-c)} - \sum_{k=0}^{s-2} \frac{1}{c^{k+1}} \sum_{n=0}^{\infty} \frac{1}{(cn+d)^{s-k}}.$$

It is clear that if we set $s = 2$, then the above identity becomes the identity in [24, Theorem 10]. If we set $s = 3$, then it gives the identity in [24, Theorem 34].

Moreover, if we substitute $s = a = 1$ and $z = (d-c)/c$ in Eq. (4.2) we have

$$\sum_{n=1}^{\infty} \frac{h_n^2(c,d)}{(cn+d-c)(cn+d-2c)} = \frac{1}{c^3(d-c)} + \sum_{n=0}^{\infty} \frac{1}{c^2(cn+d)^2}. \tag{4.3}$$

We substract Eq. (3.4) from Eq. (4.3), we have

$$\sum_{n=1}^{\infty} \frac{h_n^2(c,d)}{(cn+d-c)(cn+d-2c)} - \sum_{n=1}^{\infty} \frac{h_n^2(c,d)}{(cn+d)(cn+c+d)}$$
$$= \frac{1}{c^3(d-c)} + \sum_{n=0}^{\infty} \frac{1}{c^2(cn+d)^2} - \sum_{n=0}^{\infty} \frac{1}{c^2(cn+d)^2} - \frac{1}{c^3 d} = \frac{1}{c^2 d(d-c)}.$$

We rewrite the lefthand side of the above identity, then we have

$$\sum_{n=1}^{\infty} \frac{2c(2cn-c+2d)h_n^2(c,d)}{(cn-2c+d)(cn-c+d)(cn+d)(cn+c+d)} = \frac{1}{c^2 d(d-c)}.$$

This gives a correct form of [8, Theorem 3], where the original statement is imprecise. Also the result of [8, Corollary 4] should be corrected as the following.

$$\sum_{n=1}^{\infty} \frac{nO_n^2}{(2n-3)(2n-1)(2n+1)(2n+3)} = \frac{-1}{64}.$$

When we allow the variable $z = 0$, the number n should begin from $a+1$ to avoid the denominator in the infinite series $Y_{-a}^{(s)}(m; z)$ being zero. Therefore we consider the infinite series $\sum_{n=a+1}^{\infty} \frac{\left(H_n^{(s)}\right)^m}{(n-a)(n+1-a)}$ instead of $Y_{-a}^{(s)}(m; z)$.

Theorem 4.3. *For any positive integers a, m, s, we have*

$$\sum_{n=a+1}^{\infty} \frac{\left(H_n^{(s)}\right)^m}{(n-a)(n-a+1)}$$
$$= \left(H_{a+1}^{(s)}\right)^m + \frac{H_{a+1}-1}{a^{ms}} - \sum_{k=0}^{ms-2} \frac{\zeta(ms-k) - H_{a+1}^{(ms-k)}}{a^{k+1}}$$
$$- \sum_{k=1}^{m-1} \binom{m}{k} \sum_{\ell=0}^{(m-k)s-2} \frac{1}{a^{\ell+1}} \sum_{r=1}^{k} \sum_{\substack{|\boldsymbol{\alpha}|=k \\ \alpha_i \geq 1}} \binom{k}{\alpha_1, \ldots, \alpha_r}$$
$$\times \left[\zeta(s\alpha_1, \ldots, s\alpha_r, (m-k)s-\ell) - \zeta_{a+1}(s\alpha_1, \ldots, s\alpha_r, (m-k)s-\ell)\right]$$
$$+ \sum_{k=1}^{m-1} \binom{m}{k} \frac{1}{a^{(m-k)s}} \sum_{r=1}^{a} \sum_{n=a+1}^{\infty} \frac{\left(H_n^{(s)}\right)^k}{(n+r-a)(n+r-a+1)}.$$

Proof. We let the two sequences V_n and U_n as

$$V_n = \left(H_n^{(s)}\right)^m, \quad \text{and} \quad U_n = \frac{-1}{n-a+1}.$$

Then it is easy to check that

$$\vartriangle V_n = -\sum_{\ell=0}^{m-1} \binom{m}{\ell} \frac{\left(H_n^{(s)}\right)^\ell}{(n+1)^{(m-\ell)s}}, \quad \text{and} \quad \nabla U_n = \frac{1}{(n-a)(n-a+1)}$$

as well as the relations

$$[UV]_+ = 0, \quad \text{and} \quad U_a V_{a+1} = -\left(H_{a+1}^{(s)}\right)^m.$$

Applying Abel's summation formula Eq. (1.4) with $\varepsilon = a+1$, we can reformulate

$$\sum_{n=a+1}^{\infty} \frac{\left(H_n^{(s)}\right)^m}{(n-a)(n-a+1)}$$
$$= \sum_{n=a+1}^{\infty} V_n \nabla U_n = \left(H_{a+1}^{(s)}\right)^m + \sum_{n=a+1}^{\infty} U_n \vartriangle V_n$$
$$= \left(H_{a+1}^{(s)}\right)^m + \sum_{n=a+1}^{\infty} \frac{1}{(n+1-a)(n+1)^{ms}} + \sum_{k=1}^{m-1} \binom{m}{k} \sum_{n=a+1}^{\infty} \frac{\left(H_n^{(s)}\right)^k}{(n+1-a)(n+1)^{(m-k)s}}.$$

Using Eq. (2.5), the above last equation becomes

$$\left(H_{a+1}^{(s)}\right)^m + \frac{1}{a^{ms-1}} \sum_{n=a+1}^{\infty} \frac{1}{(n+1-a)(n+1)} - \sum_{k=0}^{ms-2} \frac{1}{a^{k+1}} \sum_{n=a+1}^{\infty} \frac{1}{(n+1)^{ms-k}}$$
$$+ \sum_{k=1}^{m-1} \binom{m}{k} \frac{1}{a^{(m-k)s-1}} \sum_{n=a+1}^{\infty} \frac{\left(H_n^{(s)}\right)^k}{(n+1-a)(n+1)}$$
$$- \sum_{k=1}^{m-1} \binom{m}{k} \sum_{\ell=0}^{(m-k)s-2} \frac{1}{a^{\ell+1}} \sum_{n=a+1}^{\infty} \frac{\left(H_n^{(s)}\right)^k}{(n+1)^{(m-k)s-\ell}}.$$

The second term and the third term can be simplified as follows.

$$\sum_{n=a+1}^{\infty}\frac{1}{(n+1-a)(n+1)}=\frac{H_{a+1}-1}{a},\quad \text{and}\quad \sum_{n=a+1}^{\infty}\frac{1}{(n+1)^{ms-k}}=\zeta(ms-k)-H_{a+1}^{(ms-k)}.$$

We use Eq. (2.4) to reformula the term $\sum_{n=a+1}^{\infty}\frac{\left(H_n^{(s)}\right)^k}{(n+1-a)(n+1)}$ in the fourth term as

$$\sum_{r=1}^{a}\frac{1}{a}\sum_{n=a+1}^{\infty}\frac{\left(H_n^{(s)}\right)^k}{(n+r-a)(n+r-a+1)}.$$

Moreover we apply Eq. (2.1) to the fifth term, we have

$$\sum_{n=a+1}^{\infty}\frac{\left(H_n^{(s)}\right)^k}{(n+1)^{(m-k)s-\ell}}=\sum_{r=1}^{k}\sum_{\substack{|\boldsymbol{\alpha}|=k\\ \alpha_i\geq 1}}\binom{k}{\alpha_1,\ldots,\alpha_r}\sum_{n=a+1}^{\infty}\frac{\zeta_n(s\alpha_1,\ldots,s\alpha_r)}{(n+1)^{(m-k)s-\ell}}.$$

The rightmost summation in the right-hand side of the above identity is

$$\zeta(s\alpha_1,\ldots,s\alpha_r,(m-k)s-\ell)-\zeta_{a+1}(s\alpha_1,\ldots,s\alpha_r,(m-k)s-\ell).$$

Thus we conclude our result. □

The following identity is the identity in Theorem 4.3 with $a=m=1$,

$$\sum_{n=2}^{\infty}\frac{H_n^{(s)}}{(n-1)n}=1+s-\sum_{k=2}^{s}\zeta(k). \tag{4.4}$$

It gives the result in [24, Corollary 12] with $s=2$ and the result in [24, Corollary 36] with $s=3$. Furthermore, the result in [24, Corollary 24] can be derived by Eq. (3.6) (with special values $c=d=1$, $s=2$) and Eq. (4.4) (with $s=2$):

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{2H_n^{(2)}}{(n-1)n(n+1)}&=\sum_{n=2}^{\infty}\frac{H_n^{(2)}}{(n-1)n}-\sum_{n=2}^{\infty}\frac{H_n^{(2)}}{n(n+1)}\\&=3-\zeta(2)+\frac{1}{2}-\zeta(3)\;=\;\frac{7}{2}-\frac{\pi^2}{6}-\zeta(3).\end{aligned}$$

5 Infinite series of $Y_a^{(s)}(1;z)$ with $a\in\mathbb{Q}^+$

In order to give formulas of $\sum_{n=1}^{\infty}\frac{O_n^{(s)}}{n\binom{n+k}{k}}$ and $\sum_{n=1}^{\infty}\frac{O_n^{(s)}}{\binom{n+k}{k}}$, we present one simple identitiy concerning $Y_a^{(s)}(1;z)$ with $a\in\mathbb{Q}^+$ in Theorem 5.1.

Theorem 5.1. *Let a, s, p, q be four integers with $s \geq 1$, $ap+q \geq 1$, $0 \leq q < p$, and $z \in (-1, 0]$. Then we have*

$$\sum_{n=1}^{\infty} \frac{H_n^{(s)}(z)}{(n+z+a+\frac{q}{p})(n+z+a+1+\frac{q}{p})}$$
$$= \frac{(-p)^s}{(ap+q)^s}\left(\psi(1+z) - \psi(1+z+a+\frac{q}{p})\right) - \sum_{k=0}^{s-2} \frac{(-p)^{k+1}}{(ap+q)^{k+1}} \zeta(s-k; z).$$

Proof. We let the two sequences V_n and U_n as

$$V_n = H_n^{(s)}(z), \quad \text{and} \quad U_n = \frac{-1}{n+z+a+1+\frac{q}{p}}.$$

Then it is easy to check that

$$\triangle V_n = \frac{-1}{(n+z+1)^s}, \quad \text{and} \quad \nabla U_n = \frac{1}{(n+z+a+\frac{q}{p})(n+z+a+1+\frac{q}{p})}$$

as well as the relations

$$[UV]_+ = 0, \quad \text{and} \quad U_0 V_1 = \frac{-1}{(1+z)^s(z+a+1+\frac{q}{p})}.$$

Applying Abel's summation formula Eq. (1.4) with $\varepsilon = 1$, we can reformulate

$$\sum_{n=1}^{\infty} \frac{H_n^{(s)}(z)}{(n+z+a+\frac{q}{p})(n+z+a+1+\frac{q}{p})}$$
$$= \sum_{n=1}^{\infty} V_n \nabla U_n = \frac{1}{(1+z)^s(z+a+1+\frac{q}{p})} + \sum_{n=1}^{\infty} U_n \triangle V_n$$
$$= \sum_{n=1}^{\infty} \frac{1}{(n+z)^s(n+z+a+\frac{q}{p})}.$$

Using Eq. (2.3), the above last equation becomes

$$\frac{(-p)^{s-1}}{(ap+q)^{s-1}} \sum_{n=1}^{\infty} \frac{1}{(n+z)(n+z+a+\frac{q}{p})} - \sum_{k=0}^{s-2} \frac{(-p)^{k+1}}{(ap+q)^{k+1}} \zeta(s-k; z).$$

Using Eq. (1.2) we have

$$\sum_{n=1}^{\infty} \frac{1}{(n+z)(n+z+a+\frac{q}{p})} = \frac{-p}{ap+q}\left(\psi(1+z) - \psi(1+z+a+\frac{q}{p})\right),$$

therefore we conclude our result. □

Applying $a = 0$, $p = 2$, $q = 1$ in Theorem 5.1 and using the fact

$$\psi\left(\frac{d}{c}\right) - \psi\left(\frac{d}{c} + \frac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2c}{nc + 2d},$$

we have the following identity.

$$\sum_{n=1}^{\infty} \frac{2c^{s+2}h_n^{(s)}(c,d)}{(2cn-c+2d)(2cn+c+2d)} = \sum_{n=0}^{\infty} \frac{(-1)^{s+n+1}2^s c}{nc+2d} + \sum_{k=0}^{s-2}\sum_{n=0}^{\infty} \frac{(-1)^k 2^k c^{s-k}}{(cn+d)^{s-k}}. \tag{5.1}$$

Setting $s=2$ yields identical result in [24, Theorem 13].

Example 5.2. *For nonnegative integers a, s with $s \geq 1$, we have*

$$\sum_{n=1}^{\infty} \frac{O_n^{(s)}}{(n+a)(n+a+1)} = \frac{(-1)^{s+1}(H_a + 2\log(2))}{(2a+1)^s} - 2\sum_{k=0}^{s-2} \frac{(-1)^{k+1}}{(2a+1)^{k+1}} t(s-k).$$

Proof. We set $z=-1/2$, $p=2$, and $q=1$ in Theorem 5.1, then we have

$$\sum_{n=1}^{\infty} \frac{2^s O_n^{(s)}}{(n+a)(n+a+1)} = \frac{(-2)^s}{(2a+1)^s}\left(\psi(1/2) - \psi(a+1)\right) - \sum_{k=0}^{s-2} \frac{(-2)^{s+1}}{(2a+1)^{k+1}} t(s-k).$$

Since $\psi(1/2) = -\gamma - 2\log(2)$ and by Eq. (1.1) $\psi(a+1) = H_a - \gamma$, we conclude the result. □

Example 5.3. *Let s, k be two positive integers. Then*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{O_n^{(s)}}{n\binom{n+k}{k}} &= \sum_{r=0}^{k-1} \binom{k-1}{r} \frac{(-1)^{r+s+1}(H_r + 2\log(2))}{(2r+1)^s} \\ &\quad + 2\sum_{\ell=0}^{s-2} (-1)^\ell t(s-\ell) \sum_{r=0}^{k-1} \binom{k-1}{r} \frac{(-1)^r}{(2r+1)^{\ell+1}}. \end{aligned} \tag{5.2}$$

Let s, k be two positive integers with $k \geq 2$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{O_n^{(s)}}{\binom{n+k}{k}} &= k\sum_{r=0}^{k-2} \binom{k-2}{r} \frac{(-1)^{r+s+1}(H_{r+1} + 2\log(2))}{(2r+3)^s} \\ &\quad + 2k\sum_{\ell=0}^{s-2} (-1)^\ell t(s-\ell) \sum_{r=0}^{k-2} \binom{k-2}{r} \frac{(-1)^r}{(2r+3)^{\ell+1}}. \end{aligned} \tag{5.3}$$

Proof.

$$\sum_{n=1}^{\infty} \frac{O_n^{(s)}}{n\binom{n+k}{k}} = \sum_{n=1}^{\infty} \frac{k!\,O_n^{(s)}}{n\cdot(n+1)\cdots(n+k)} = k!\sum_{n=1}^{\infty} O_n^{(s)} \eta_{1^{k+1}}(n-1;1),$$

$$\sum_{n=1}^{\infty} \frac{O_n^{(s)}}{\binom{n+k}{k}} = \sum_{n=1}^{\infty} \frac{k!\,O_n^{(s)}}{(n+1)\cdots(n+k)} = k!\sum_{n=1}^{\infty} O_n^{(s)} \eta_{1^k}(n;1).$$

Using Eq. (2.6) with $\ell = 1$,

$$\eta_{1^{a+2}} = \frac{1}{(a+1)!}\sum_{r=0}^{a} (-1)^r \binom{a}{r} \eta_{0^r,1^2},$$

and Example 5.2, we get the results. □

Applying Lemma 2.1 with $x = 1/2$, we have

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+\frac{1}{2})^m} = B(n+1, \frac{1}{2}) P_{m-1}(H_{n+1}(1, \frac{1}{2}), \dots, H_{n+1}(1, \frac{1}{2})).$$

Since

$$H_n^{(s)}(1, 1/2) = 2^s O_{n+1}^{(s)}, \quad \text{and} \quad B(n+1, 1/2) = \frac{2^{2n+1}}{(2n+1)\binom{2n}{n}},$$

we rewrtite the above identity as

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k 2^m}{(2k+1)^m} = \frac{2^{2n+1}}{(2n+1)\binom{2n}{n}} P_{m-1}(2O_n^{(1)}, \dots, 2^{m-1} O_n^{(m-1)}).$$

Using Eq. (2.2) we can simplify the identity as

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(2k+1)^m} = \frac{2^{2n}}{(2n+1)\binom{2n}{n}} P_{m-1}(O_n^{(1)}, \dots, O_n^{(m-1)}).$$

Therefore the summation in Eq. (5.2) can write as

$$\sum_{r=0}^{k-1} \binom{k-1}{r} \frac{(-1)^r}{(2r+1)^{\ell+1}} = \frac{2^{2k-2}}{(2k-1)\binom{2k-2}{k-1}} P_\ell(O_k^{(1)}, \dots, O_k^{(\ell)}).$$

Similarly the summation in Eq. (5.3) can rewrite as

$$\sum_{r=0}^{k-1} \binom{k-2}{r} \frac{(-1)^r}{(2r+3)^{\ell+1}} = \frac{2^{2k-3}}{k(k-1)\binom{2k-1}{k-1}} P_\ell(O'^{(1)}_k, \dots, O'^{(\ell)}_k),$$

where $O'^{(s)}_n = O_n^{(s)} - 1$. Now we have a more efficient form for Example 5.3:

Corollary 5.4. *Let s, k be two positive integers. Then*

$$\sum_{n=1}^{\infty} \frac{O_n^{(s)}}{n\binom{n+k}{k}} = \frac{(-1)^{s+1} 2^{2k-1}}{(2k-1)\binom{2k-2}{k-1}} \log(2) P_{s-1}(O_k^{(1)}, \dots, O_k^{(s-1)}) \tag{5.4}$$
$$+ \sum_{\ell=0}^{s-2} \frac{(-1)^\ell 2^{2k-1}}{(2k-1)\binom{2k-2}{k-1}} t(s-\ell) P_\ell(O_k^{(1)}, \dots, O_k^{(\ell)}) + \sum_{r=0}^{k-1} \binom{k-1}{r} \frac{(-1)^{r+s+1}}{(2r+1)^s} H_r.$$

Let s, k be two positive integers with $k \geq 2$. Then

$$\sum_{n=1}^{\infty} \frac{O_n^{(s)}}{\binom{n+k}{k}} = \frac{(-1)^{s+1} 2^{2k-2}}{(k-1)\binom{2k-1}{k-1}} \log(2) P_{s-1}(O'^{(1)}_k, \dots, O'^{(s-1)}_k) \tag{5.5}$$
$$+ \sum_{\ell=0}^{s-2} \frac{(-1)^\ell 2^{2k-2}}{(k-1)\binom{2k-1}{k-1}} t(s-\ell) P_\ell(O'^{(1)}_k, \dots, O'^{(\ell)}_k) + k \sum_{r=0}^{k-2} \binom{k-2}{r} \frac{(-1)^{r+s+1}}{(2r+3)^s} H_{r+1},$$

where $O'^{(s)}_n = O_n^{(s)} - 1$.

At last we list some examples for special values of $s = 1, 2$:

$$\sum_{n=1}^{\infty} \frac{O_n}{n\binom{n+k}{k}} = \frac{2^{2k-1}\log(2)}{(2k-1)\binom{2k-2}{k-1}} + \sum_{r=0}^{k-1} \binom{k-1}{r} \frac{(-1)^r H_r}{2r+1}, \tag{5.6}$$

$$\sum_{n=1}^{\infty} \frac{O_n^{(2)}}{n\binom{n+k}{k}} = \frac{2^{2k-4}\pi^2}{(2k-1)\binom{2k-2}{k-1}} - \frac{2^{2k-1}\log(2)}{(2k-1)\binom{2k-2}{k-1}} O_k + \sum_{r=0}^{k-1} \binom{k-1}{r} \frac{(-1)^{r+1} H_r}{(2r+1)^2}, \tag{5.7}$$

$$\sum_{n=1}^{\infty} \frac{O_n}{\binom{n+k}{k}} = \frac{2^{2k-2}\log(2)}{(k-1)\binom{2k-1}{k-1}} + k\sum_{r=0}^{k-2} \binom{k-2}{r} \frac{(-1)^r H_{r+1}}{2r+3}, \tag{5.8}$$

$$\sum_{n=1}^{\infty} \frac{O_n^{(2)}}{\binom{n+k}{k}} = \frac{2^{2k-5}\pi^2}{(k-1)\binom{2k-1}{k-1}} - \frac{2^{2k-2}\log(2)}{(k-1)\binom{2k-1}{k-1}} O'_k + k\sum_{r=0}^{k-2} \binom{k-2}{r} \frac{(-1)^{r+1} H_{r+1}}{(2r+3)^2}. \tag{5.9}$$

Using a similar method, we can derive closed forms for $\sum_{n=1}^{\infty} \frac{H_n^{(s)}}{n\binom{n+k}{k}}$ and $\sum_{n=1}^{\infty} \frac{H_n^{(s)}}{\binom{n+k}{k}}$. Since our explicit formulas are different from the known formulas in [20], we state them as our final conclusions. For $s, k \in \mathbb{N}$, we have two explicit formulas

$$\sum_{n=1}^{\infty} \frac{H_n^{(s)}}{n\binom{n+k}{k}} = \zeta(s+1) + \sum_{r=1}^{k-1} \binom{k-1}{r} \frac{(-1)^{r+s+1} H_r}{r^s} + \sum_{\ell=2}^{s} \zeta(\ell) \sum_{r=1}^{k-1} \binom{k-1}{r} \frac{(-1)^{r+s+\ell}}{r^{s+1-\ell}} \tag{5.10}$$

$$= \sum_{r=1}^{k-1} \binom{k-1}{r} \frac{(-1)^{r+s+1} H_r}{r^s} + \sum_{\ell=0}^{s-1} (-1)^\ell \zeta(s+1-\ell) P_\ell(H_{k-1}^{(1)}, \ldots, H_{k-1}^{(\ell)}). \tag{5.11}$$

Eq. (5.10) also appeared in [5, Example 5.2]. On the other hand, for positive integers s, k with $k \geq 2$, we have

$$\sum_{n=1}^{\infty} \frac{H_n^{(s)}}{\binom{n+k}{k}} = k\sum_{r=0}^{k-2} \binom{k-2}{r} \frac{(-1)^{r+s+1} H_{r+1}}{(r+1)^s} + k\sum_{\ell=0}^{s-2} \zeta(s-\ell) \sum_{r=0}^{k-2} \binom{k-2}{r} \frac{(-1)^{r+\ell}}{(r+1)^{\ell+1}} \tag{5.12}$$

$$= k\sum_{r=0}^{k-2} \binom{k-2}{r} \frac{(-1)^{r+s+1} H_{r+1}}{(r+1)^s} + \frac{k}{k-1} \sum_{\ell=0}^{s-2} (-1)^\ell \zeta(s-\ell) P_\ell(H_{k-1}^{(1)}, \ldots, H_{k-1}^{(\ell)}). \tag{5.13}$$

Acknowledgements

The first author was funded by the Ministry of Science and Technology, Taiwan, R.O.C., under Grant MOST 107-2115-M-845-003. The authors would like to thank the referees for some useful comments and suggestions.

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