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Higher-order identities for balancing numbers

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Abstract: Let B_n be the n-th balancing number. In this paper, we give some explicit expressions of $\sum_{l=0}^{2r-3}(-1)^l\binom{2r-3}{l}\sum_{\substack{j_1+\dots+j_r=n-2l\\ j_1,\dots,j_r\ge 1}} B_{j_1}\cdots B_{j_r}$ and $\sum_{\substack{j_1+\dots+j_r=n\\ j_1,\dots,j_r\ge 1}} B_{j_1}\cdots B_{j_r}$. We also consider the convolution identities with binomial coefficients: $\sum_{\substack{k_1+\dots+k_r=n\\ k_1,\dots,k_r\ge 1}}\binom{n}{k_1,\dots,k_r}B_{k_1}\cdots B_{k_r}$. This type can be generalized, so that B_n is a special case of the number u_n, where $u_n = au_{n-1} + bu_{n-2}$ $(n \ge 2)$ with $u_0 = 0$ and $u_1 = 1$.
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1 Introduction

Higher-order convolutions for various types of numbers (or polynomials) have been studied, with or without binomial (or multinomial) coefficients, including Bernoulli, Euler, Genocchi, Cauchy, Stirling, and Fibonacci numbers [1–3, 6–8, 10]. One typical one is due to Euler, given by

$$\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k\mathcal{B}_{n-k} = -n\mathcal{B}_{n-1} - (n-1)\mathcal{B}_n \quad (n \ge 0)\,,$$

where $\mathcal{B}_n$ are Bernoulli numbers, defined by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty}\mathcal{B}_n\frac{t^n}{n!} \quad (|t| < 2\pi)\,.$$

A positive integer x is called *balancing number* if

$$1 + 2 + \dots + (x-1) = (x+1) + \dots + (y-1) \tag{1}$$

holds for some integer $y \geq x+2$. The problem of determining all balancing numbers leads to a Pell equation, whose solutions in x can be described by the recurrence $B_n = 6B_{n-1} - B_{n-2}$ ($n \geq 2$) with $B_0 = 0$ and $B_1 = 1$ (see [4, 5]). One of the most general extensions of balancing numbers is when (1) is being replaced by

$$1^k + 2^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l\,, \tag{2}$$

where the exponents k and l are given positive integers. In the work of Liptai et al. [12] effective and non-effective finiteness theorems on (2) are proved. In [9] a balancing problem of ordinary binomial coefficients is studied. Using the method of linear forms in logarithms, 1 is the only Fibonacci number [11]. Some more results can be seen in [13, 14].

The generating function $f(x)$ of balancing numbers B_n is given by

$$f(x) := \frac{x}{1-6x+x^2} = \sum_{n=0}^\infty B_n x^n\,.$$

Then $f(x)$ satisfies the relation:

$$f(x)^2 = \frac{x^2}{1-x^2} f'(x) \tag{3}$$

or

$$(1-x^2) f(x)^2 = x^2 f'(x)\,. \tag{4}$$

The left-hand side of (4) is

$$\begin{aligned}&(1-x^2)\left(\sum_{u=0}^\infty B_u x^u\right)\left(\sum_{v=0}^\infty B_v x^v\right)\\ &= \sum_{n=0}^\infty \sum_{j=0}^n B_j B_{n-j} x^n - \sum_{n=2}^\infty \sum_{j=0}^{n-2} B_j B_{n-j-2} x^n\,.\end{aligned}$$

The right-hand side of (4) is

$$x^2 \sum_{n=1}^\infty n B_n x^{n-1} = \sum_{n=0}^\infty (n-1) B_{n-1} x^n\,.$$

Comparing the coefficients of both sides, we get

$$\begin{aligned}(n-1)B_{n-1} &= \sum_{j=0}^n B_j B_{n-j} - \sum_{j=0}^{n-2} B_j B_{n-j-2}\\ &= \sum_{j=1}^{n-1} (B_j B_{n-j} - B_{j-1} B_{n-j-1})\,.\end{aligned}$$

Here, notice that $B_0 = 0$. By changing n by $n+1$, we get the following.

Theorem 1. *For $n \geq 1$, we have*

$$nB_n = \sum_{j=1}^n (B_j B_{n-j+1} - B_{j-1} B_{n-j})\,.$$

Differentiating both sides of (3) by x and dividing them by 2, we obtain

$$f(x)f'(x)=\frac{x}{(1-x^2)^2}f'(x)+\frac{x^2}{2(1-x^2)}f''(x)\,. \tag{5}$$

By (3) and (5), we get

$$\begin{aligned} f(x)^3&=\frac{x^2}{1-x^2}f(x)f'(x)\\ &=\frac{x^3}{(1-x^2)^3}f'(x)+\frac{x^4}{2(1-x^2)^2}f''(x) \end{aligned} \tag{6}$$

or

$$(1-x^2)^3 f(x)^3=x^3f'(x)+\frac{1}{2}x^4(1-x^2)f''(x)\,. \tag{7}$$

The left-hand side of (7) is equal to equal to

$$\begin{aligned} &(1-3x^2+3x^4-x^6)\sum_{n=0}^\infty\sum_{\substack{j_1+j_2+J_3=n\\ j_1,j_2,j_3\ge 0}}B_{j_1}B_{j_2}B_{j_3}x^n\\ &=\sum_{l=0}^3\sum_{n=2l}^\infty(-1)^l\binom{3}{l}\sum_{\substack{j_1+j_2+J_3=n-2l\\ j_1,j_2,j_3\ge 1}}B_{j_1}B_{j_2}B_{j_3}x^n\,. \end{aligned}$$

The right-hand side of (7) is

$$\begin{aligned} &x^3\sum_{n=1}^\infty nB_n x^{n-1}+\frac{x^4}{2}\sum_{n=2}^\infty n(n-1)B_n x^{n-2}-\frac{x^6}{2}\sum_{n=2}^\infty n(n-1)B_n x^{n-2}\\ &=\sum_{n=2}^\infty\frac{(n-1)(n-2)}{2}B_{n-2}x^n-\sum_{n=4}^\infty\frac{(n-4)(n-5)}{2}B_{n-4}x^n\,. \end{aligned}$$

Comparing the coefficients of both sides, we get the following result.

Theorem 2. *For $n\ge 4$, we have*

$$\sum_{l=0}^3(-1)^l\binom{3}{l}\sum_{\substack{j_1+j_2+J_3=n-2l\\ j_1,j_2,j_3\ge 1}}B_{j_1}B_{j_2}B_{j_3}=\binom{n-1}{2}B_{n-2}-\binom{n-4}{2}B_{n-4}\,.$$

In this paper, we give some explicit expression of a more general case

$$\sum_{l=0}^{2r-3}(-1)^l\binom{2r-3}{l}\sum_{\substack{j_1+\cdots+j_r=n-2l\\ j_1,\dots,j_r\ge 1}}B_{j_1}\cdots B_{j_r}$$

and

$$\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}}B_{j_1}\cdots B_{j_r}.$$

We also consider a different type for more general numbers:

$$\sum_{\substack{k_1+\cdots+k_r=n\\ k_1,\dots,k_r\ge 1}}\binom{n}{k_1,\dots,k_r}u_{k_1}\cdots u_{k_r}\,,$$

where $u_n = au_{n-1} + bu_{n-2}$ ($n \ge 2$) with $u_0 = 0$ and $u_1 = 1$. When $a = 6$ and $b = -1$, this is reduced to the convolution identity for balancing numbers. When $a = b = 1$, this is reduced to the convolution identity for Fibonacci numbers. The corresponding identities for Lucas-balancing numbers are also given.

2 Main results

First, as a general case of (3) and (6), we can have the following.

Lemma 1. *For $r \ge 2$, we have*

$$f(x)^r = \frac{x^{2r-2}f^{(r-1)}(x)}{(r-1)!(1-x^2)^{r-1}} + \sum_{k=1}^{r-2}\frac{\sum_{j=0}^{k-1}\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k+2j-2}}{k(r-k-2)!(1-x^2)^{r+k-1}}f^{(r-k-1)}(x)\,. \tag{8}$$

Proof. The proof is done by induction. It is trivial to see that the identity holds for $r = 2$. Suppose that the identity holds for some r. Differentiating both sides by x, we obtain

$$\begin{aligned}
&rf(x)^{r-1}f'(x)\\
&= \frac{x^{2r-2}f^{(r)}(x)}{(r-1)!(1-x^2)^{r-1}} + \frac{(2r-2)x^{2r-3}f^{(r-1)}(x)}{(r-1)!(1-x^2)^{r}}\\
&\quad + \sum_{k=1}^{r-2}\frac{\sum_{j=0}^{k-1}\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{k(r-k-2)!(1-x^2)^{r+k-1}}f^{(r-k)}(x)\\
&\quad + \sum_{k=1}^{r-2}\frac{\sum_{j=0}^{k-1}(2r-k-2+2j)\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-3+2j}}{k(r-k-2)!(1-x^2)^{r+k}}f^{(r-k-1)}(x)\\
&\quad + \sum_{k=1}^{r-2}\frac{\sum_{j=0}^{k-1}(3k-2j)\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-1+2j}}{k(r-k-2)!(1-x^2)^{r+k}}f^{(r-k-1)}(x)\\
&= \frac{x^{2r-2}f^{(r)}(x)}{(r-1)!(1-x^2)^{r-1}} + \frac{2x^{2r-3}f^{(r-1)}(x)}{(r-2)!(1-x^2)^{r}}\\
&\quad + \sum_{k=1}^{r-2}\frac{\sum_{j=0}^{k-1}\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{k(r-k-2)!(1-x^2)^{r+k-1}}f^{(r-k)}(x)\\
&\quad + \sum_{k=2}^{r-1}\frac{\sum_{j=0}^{k-2}(2r-k-1+2j)\binom{k-1}{j}\binom{r-2}{k-j-2}x^{2r-k-2+2j}}{(k-1)(r-k-1)!(1-x^2)^{r+k-1}}f^{(r-k)}(x)\\
&\quad + \sum_{k=2}^{r-1}\frac{\sum_{j=1}^{k-1}(3k-2j-1)\binom{k-1}{j-1}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{(k-1)(r-k-1)!(1-x^2)^{r+k-1}}f^{(r-k)}(x)
\end{aligned}$$

$$= \frac{x^{2r-2} f^{(r)}(x)}{(r-1)!(1-x^2)^{r-1}} + r \sum_{k=1}^{r-1} \frac{\sum_{j=0}^{k-1} \binom{k}{j} \binom{r-1}{k-j-1} x^{2r-k-2+2j}}{k(r-k-1)!(1-x^2)^{r+k-1}} f^{(r-k)}(x) .$$

Together with (3), we get

$$\begin{aligned} f(x)^{r+1} &= \frac{x^2}{1-x^2} f(x)^{r-1} f'(x) \\ &= \frac{x^2}{1-x^2} \left(\frac{x^{2r-2} f^{(r)}(x)}{r!(1-x^2)^{r-1}} + \sum_{k=1}^{r-1} \frac{\sum_{j=0}^{k-1} \binom{k}{j} \binom{r-1}{k-j-1} x^{2r-k-2+2j}}{k(r-k-1)!(1-x^2)^{r+k-1}} f^{(r-k)}(x) \right) \\ &= \frac{x^{2r} f^{(r)}(x)}{r!(1-x^2)^r} + \sum_{k=1}^{r-1} \frac{\sum_{j=0}^{k-1} \binom{k}{j} \binom{r-1}{k-j-1} x^{2r-k+2j}}{k(r-k-1)!(1-x^2)^{r+k}} f^{(r-k)}(x) . \end{aligned}$$

□

In general, we can state the following.

Theorem 3. *Let $r \geq 2$. Then for $n \geq 3r-5$, we have*

$$\begin{aligned} &\sum_{l=0}^{2r-3} (-1)^l \binom{2r-3}{l} \sum_{\substack{j_1+\cdots+j_r=n-2l \\ j_1,\dots,j_r \geq 1}} B_{j_1} \cdots B_{j_r} \\ &\quad = \sum_{k=1}^{r-1} (-1)^{k-1} \frac{n-2k-r+3}{r-1} \binom{n-2k+1}{r-k-1} \binom{n-k-2r+3}{k-1} B_{n-2k-r+3} . \end{aligned}$$

Proof. By Lemma 1 we get

$$\begin{aligned} (1-x^2)^{2r-3} f(x)^r &= (1-x^2)^{r-2} \frac{x^{2r-2} f^{(r-1)}(x)}{(r-1)!} \\ &\quad + \sum_{k=1}^{r-2} (1-x^2)^{r-k-2} \frac{\sum_{j=0}^{k-1} \binom{k}{j} \binom{r-2}{k-j-1} x^{2r-k-2+2j}}{k(r-k-2)!} f^{(r-k-1)}(x) . \quad (9) \end{aligned}$$

Since $B_0 = 0$, the left-hand side of (9) is equal to

$$(1-x^2)^{2r-3} \sum_{n=0}^{\infty} \sum_{\substack{j_1+\cdots+j_r=n \\ j_1,\dots,j_r \geq 0}} B_{j_1} \cdots B_{j_r} x^n = \sum_{l=0}^{2r-3} \sum_{n=2l}^{\infty} (-1)^l \binom{2r-3}{l} \sum_{\substack{j_1+\cdots+j_r=n-2l \\ j_1,\dots,j_r \geq 1}} B_{j_1} \cdots B_{j_r} x^n .$$

On the other hand,

$$\begin{aligned} &(1-x^2)^{r-2} \frac{x^{2r-2} f^{(r-1)}(x)}{(r-1)!} \\ &= \sum_{i=0}^{r-2} \binom{r-2}{i} x^{2i} \frac{x^{2r-2}}{(r-1)!} \sum_{n=r-1}^{\infty} \frac{n!}{(n-r+1)!} B_n x^{n-r+1} \\ &= \frac{1}{(r-1)!} \sum_{i=0}^{r-2} \binom{r-2}{i} \sum_{n=2r+2i-2}^{\infty} \frac{(n-r-2i+1)!}{(n-2r-2i+2)!} B_{n-r-2i+1} x^n . \end{aligned}$$

For $i = r-2$, we have

$$\begin{aligned}
&\frac{(-1)^{r-2}}{(r-1)!}\sum_{n=4r-6}^{\infty}\frac{(n-3r+5)!}{(n-4r+6)!}B_{n-3r+5}x^n\\
&=(-1)^{r-2}\sum_{n=3r-5}^{\infty}\frac{n-3r+5}{r-1}\binom{n-3r+4}{r-2}B_{n-3r+5}x^n\,,
\end{aligned}$$

which yields the term for $k = r-1$ on the right-hand side of the identity in Theorem 3. Notice that

$$\binom{\gamma'}{\gamma}=0\quad(\gamma'<\gamma)\,.$$

The second term of the right-hand side of (9) is

$$\begin{aligned}
&\sum_{k=1}^{r-2}\sum_{i=0}^{r-k-2}(-1)^i\binom{r-k-2}{i}x^{2i}\frac{1}{k(r-k-2)!}\sum_{j=0}^{k-1}\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}\\
&\quad\times\sum_{n=r-k-1}^{\infty}\frac{n!}{(n-r+k+1)!}B_n x^{n-r+k+1}\\
&=\sum_{i=0}^{r-3}\sum_{j=0}^{r-i-3}\sum_{k=j}^{r-i-3}\frac{(-1)^i}{(k+1)(r-k-3)!(n-2r+k-2i-2j+3)!}\binom{r-k-3}{i}\\
&\quad\times\binom{k+1}{j}\binom{r-2}{k-j}\sum_{n=2r+2i+2j-k-3}(n-r-2i-2j+1)!B_{n-r-2i-2j+1}x^n\\
&=\sum_{i=0}^{r-3}\sum_{\kappa=i+1}^{r-2}\sum_{k=\kappa-i-1}^{r-i-3}\frac{(-1)^i}{(k+1)(r-k-3)!(n-2r+k-2\kappa+5)!}\binom{r-k-3}{i}\\
&\quad\times\binom{k+1}{\kappa-i-1}\binom{r-2}{k-\kappa+i+1}\sum_{n=2r+2\kappa-k-5}(n-r-2\kappa+3)!B_{n-r-2\kappa+3}x^n\,.
\end{aligned}$$

Together with the first term of the right-hand side of (9) we can prove that

$$\begin{aligned}
&\frac{(-1)^{k-1}}{(r-1)!}\binom{r-2}{k-1}\frac{(n-r-2k+3)!}{(n-2r-2k+4)!}\\
&\quad+\sum_{i=0}^{k-1}\sum_{l=k-i-1}^{r-i-3}\frac{(-1)^{k-1}}{(l+1)(r-l-3)!(n-2r+l-2k+5)!}\\
&\qquad\times\binom{r-l-3}{i}\binom{l+1}{k-i-1}\binom{r-2}{l-k+i+1}(n-r-2k+3)!\\
&=(-1)^{k-1}\frac{n-2k-r+3}{r-1}\binom{n-2k+1}{r-k-1}\binom{n-k-2r+3}{k-1}\,.
\end{aligned}\tag{10}$$

Then the proof is done. □

3 Another result

In this section, we shall give an expression of $\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}} B_{j_1}\cdots B_{j_r}$.

The left-hand side of (3) is equal to

$$\left(\sum_{u=0}^\infty B_u x^u\right)\left(\sum_{v=0}^\infty B_v x^v\right)=\sum_{n=0}^\infty\sum_{j=0}^n B_j B_{n-j}x^n\,.$$

The right-hand side of (3) is equal to

$$\begin{aligned}&x^2\left(\sum_{j=0}^\infty x^{2j}\right)\left(\sum_{m=1}^\infty mB_m x^{m-1}\right)\\&=x\sum_{n=0}^\infty\left(\sum_{m=0}^n\frac{1+(-1)^m}{2}(n-m)B_{n-m}\right)x^n\\&=\sum_{n=1}^\infty\left(\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}(n-2m-1)B_{n-2m-1}\right)x^n\,.\end{aligned}$$

Comparing the coefficients of both sides, we have the following.

Theorem 4. *For $n\ge 2$, we have*

$$\sum_{j=1}^{n-1}B_j B_{n-j}=\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}(n-2m-1)B_{n-2m-1}\,.$$

In general, we have the following.

Theorem 5. *For $n\ge r\ge 2$, we have*

$$\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}} B_{j_1}\cdots B_{j_r}=\sum_{m=0}^{\lfloor\frac{n-r+1}{2}\rfloor}\binom{n-m-1}{r-2}\binom{m+r-2}{r-2}\frac{n-2m-r+1}{r-1}B_{n-2m-r+1}\,.$$

Proof. The left-hand side of (8) in Lemma 1 is equal to

$$\sum_{n=0}^\infty\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}} B_{j_1}\cdots B_{j_r}x^n\,.$$

The first term on the right-hand side of (8) in Lemma 1 is equal to

$$\begin{aligned}&\frac{x^{2r-2}f^{(r-1)}(x)}{(r-1)!(1-x^2)^{r-1}}\\&=\frac{x^{2r-2}}{(r-1)!}\sum_{i=0}^\infty\binom{i+r-2}{r-2}x^{2i}\sum_{m=0}^\infty\frac{(m+r-1)!}{m!}B_{m+r-1}x^m\end{aligned}$$

$$
\begin{aligned}
&= \frac{x^{2r-2}}{(r-1)!}\sum_{k=0}^{\infty}\frac{1}{(r-2)!2^{r-2}}\frac{(k+2r-4)!!}{k!!}\frac{1+(-1)^k}{2}x^k\\
&\quad\times\sum_{m=0}^{\infty}\frac{(m+r-1)!}{m!}B_{m+r-1}x^m\\
&= \frac{1}{(r-1)!(r-2)!2^{r-2}}\sum_{n=2r-2}^{\infty}\sum_{m=r-1}^{n-r+1}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}\\
&\quad\times\frac{1+(-1)^{n-m-r+1}}{2}\frac{m!}{(m-r+1)!}B_m x^n\,.
\end{aligned}
$$

Concerning the second term, we have

$$
\begin{aligned}
&\frac{\sum_{j=0}^{k-1}\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{(1-x^2)^{r+k-1}}f^{(r-k-1)}(x)\\
&= \sum_{j=0}^{k-1}\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}\sum_{i=0}^{\infty}\binom{i+r+k-2}{r+k-2}x^{2i}\\
&\quad\times\sum_{m=0}^{\infty}\frac{(m+r-k-1)!}{m!}B_{m+r-k-1}x^m\\
&= \frac{1}{(r+k-2)!2^{r+k-2}}\sum_{j=0}^{k-1}\binom{k}{j}\binom{r-2}{k-j-1}\sum_{n=2r-k-2+2j}^{\infty}\sum_{m=0}^{n-2r+k+2-2j}\\
&\quad\frac{(n-m+3k-2-2j)!!}{(n-m-2r+k+2-2j)!!}\frac{1+(-1)^{n-m+k}}{2}\frac{(m+r-k-1)!}{m!}B_{m+r-k-1}x^n\,.
\end{aligned}
$$

Since

$$
\frac{(n-m+r+2k-3-2j)!!}{(n-m-r+k+3-2j)!!}=0
$$

if $m=n-2r+k+2-2j$ ($j=1,2,\dots,k-2$), this is equal to

$$
\begin{aligned}
&\frac{1}{(r+k-2)!2^{r+k-2}}\sum_{j=0}^{k-1}\binom{k}{j}\binom{r-2}{k-j-1}\sum_{n=2r-k-2}^{\infty}\sum_{m=0}^{n-2r+k+2}\\
&\quad\times\frac{(n-m+3k-2-2j)!!}{(n-m+k)!!}\frac{1+(-1)^{n-m+k}}{2}\frac{(m+r-k-1)!}{m!}B_{m+r-k-1}x^n\\
&= \frac{1}{(r+k-2)!2^{r+k-2}}\sum_{n=2r-k-2}^{\infty}\sum_{m=r-k-1}^{n-r+1}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}\binom{r+k-2}{k-1}\\
&\quad\times\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\frac{1+(-1)^{n-m-r+1}}{2}\frac{m!}{(m-r+k+1)!}B_m x^n\,.
\end{aligned}
$$

Therefore, the right-hand side of the relation in Theorem 4 is

$$
\begin{aligned}
&\frac{1}{(r-1)!(r-2)!2^{r-2}}\sum_{n=2r-2}^{\infty}\sum_{m=r-1}^{n-r+1}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}\frac{1+(-1)^{n-m-r+1}}{2}\\
&\quad\times\frac{m!}{(m-r+1)!}B_m x^n
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{r-1}\frac{1}{k(r-k-2)!}\frac{1}{(r+k-2)!2^{r+k-2}}\sum_{n=2r-k-2}^{\infty}\\
&\quad\sum_{m=r-k-1}^{n-r+1}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}\binom{r+k-2}{k-1}\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\\
&\qquad\times\frac{1+(-1)^{n-m-r+1}}{2}\frac{m!}{(m-r+k+1)!}B_m x^n\\
&=\frac{1}{(r-1)!(r-2)!2^{r-2}}\sum_{n=2r-2}^{\infty}\sum_{m=r-1}^{n-r+1}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}\frac{1+(-1)^{n-m-r+1}}{2}\\
&\quad\times\frac{m!}{(m-r+1)!}B_m x^n\\
&\quad+\sum_{n=r-1}^{\infty}\frac{1}{(r-1)!2^{r-2}}\sum_{m=1}^{r-2}\sum_{k=r-m-1}^{r-2}\frac{1}{k!(r-k-2)!2^k}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}\\
&\quad\times\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\frac{1+(-1)^{n-m-r+1}}{2}\frac{m!}{(m-r+k+1)!}B_m x^n\\
&\quad+\sum_{n=r-1}^{\infty}\frac{1}{(r-1)!2^{r-2}}\sum_{m=r-1}^{n-r+1}\sum_{k=1}^{r-2}\frac{1}{k!(r-k-2)!2^k}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}\\
&\quad\times\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\frac{1+(-1)^{n-m-r+1}}{2}\frac{m!}{(m-r+k+1)!}B_m x^n\,.
\end{aligned}
$$

Since for $1\le m\le r-2$ we have

$$
\begin{aligned}
&\frac{1}{(r-1)!2^{r-2}}\sum_{k=r-m-1}^{r-2}\frac{1}{k!(r-k-2)!2^k}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}\\
&\quad\times\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\frac{m!}{(m-r+k+1)!}\\
&=\frac{1}{(r-1)!(r-2)!2^{2r-4}}\frac{(n+m+r-3)!!}{(n+m-r+1)!!}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}m
\end{aligned}
$$

and for $r-1\le m\le n-r+1$ we have

$$
\begin{aligned}
&\frac{1}{(r-1)!(r-2)!2^{r-2}}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}\frac{m!}{(m-r+1)!}\\
&\quad+\frac{1}{(r-1)!2^{r-2}}\sum_{k=r-m-1}^{r-2}\frac{1}{k!(r-k-2)!2^k}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}\\
&\quad\times\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\frac{m!}{(m-r+k+1)!}\\
&=\frac{1}{(r-1)!(r-2)!2^{2r-4}}\frac{(n+m+r-3)!!}{(n+m-r+1)!!}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}m\,,
\end{aligned}
$$

By comparing the coefficients, we have

$$
\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}} B_{j_1}\cdots B_{j_r}
$$

$$
\begin{aligned}
&= \frac{1}{(r-1)!(r-2)!2^{2r-4}} \\
&\quad \times \sum_{m=0}^{n-r+1} \frac{(n+m+r-3)!!(n-m+r-3)!!}{(n+m-r+1)!!(n-m-r+1)!!} \frac{1+(-1)^{n-m-r+1}}{2} m F_m \\
&= \frac{1}{(r-1)!(r-2)!2^{2r-4}} \\
&\quad \times \sum_{m=0}^{\lfloor \frac{n-r+1}{2} \rfloor} \frac{(2n-2m-2)!!(2m+2r-4)!!}{(2n-2m-2r+2)!!(2m)!!} (n-2m-r+1) B_{n-2m-r+1} \\
&= \sum_{m=0}^{\lfloor \frac{n-r+1}{2} \rfloor} \binom{n-m-1}{r-2} \binom{m+r-2}{r-2} \frac{n-2m-r+1}{r-1} B_{n-2m-r+1} \,. \qquad \square
\end{aligned}
$$

4 Some other generating functions

Another kinds of the generating functions of balancing numbers and Lucas-balancing numbers are given by

$$
b(t) := \frac{e^{\alpha t} - e^{\beta t}}{4\sqrt{2}} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad c(t) := \frac{e^{\alpha t} + e^{\beta t}}{2} = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!}
$$

because they satisfy the differential equation $y'' - 6y' + y = 0$.

Since $b'(t) = 3b(t) + c(t)$ and $c'(t) = 8b(t) + 3c(t)$, we have for $n \ge 0$, $B_{n+1} = 3B_n + C_n$ and $C_{n+1} = 8B_n + 3C_n$. Since

$$
c(t)^2 = \frac{e^{2\alpha t} + e^{2\beta t}}{4} + \frac{e^{6t}}{2} \,,
$$

we have

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} C_k C_{n-k} = \frac{1}{2} \sum_{n=0}^{\infty} C_n \frac{(2t)^n}{n!} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(6t)^n}{n!} \,,
$$

yielding

$$
\sum_{k=0}^{n} \binom{n}{k} C_k C_{n-k} = \frac{2^n C_n + 6^n}{2} \quad (n \ge 0) \,.
$$

Similarly, by

$$
b(t)^2 = \frac{e^{2\alpha t} + e^{2\beta t}}{32} - \frac{e^{6t}}{16} \,,
$$

we have

$$
\sum_{k=0}^{n} \binom{n}{k} B_k B_{n-k} = \frac{2^n C_n - 6^n}{16} \quad (n \ge 0) \,.
$$

Since by $2\alpha + \beta = \alpha + 6$ and $\alpha + 2\beta = \beta + 6$

$$
b(t)^3 = \frac{e^{3\alpha t} - e^{3\beta t}}{(4\sqrt{2})^3} - \frac{3(e^{(2\alpha+\beta)t} - e^{(\alpha+2\beta)t})}{(4\sqrt{2})^3}
$$

$$= \frac{1}{32}\sum_{n=0}^{\infty} 3^n B_n \frac{t^n}{n!} - \frac{3}{32}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} 6^{n-k} B_k \frac{t^n}{n!}\,,$$

we have

$$\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\ge 1}} \binom{n}{k_1,k_2,k_3} B_{k_1}B_{k_2}B_{k_3} = \frac{1}{32}\left(3^n B_n - 3\sum_{k=0}^{n}\binom{n}{k}6^{n-k}B_k\right)\,.$$

Notice that $B_0 = 0$. Similarly, we can obtain that

$$\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} C_{k_1}C_{k_2}C_{k_3} = \frac{1}{4}\left(3^n C_n + 3\sum_{k=0}^{n}\binom{n}{k}6^{n-k}C_k\right)\,.$$

Let $r \ge 1$. If r is odd, then

$$\begin{aligned} b(t)^r &= \left(\frac{e^{\alpha t} - e^{\beta t}}{4\sqrt{2}}\right)^r \\ &= \frac{1}{(4\sqrt{2})^r}\sum_{j=0}^{\frac{r-1}{2}}(-1)^j\binom{r}{j}\left(e^{((r-j)\alpha+j\beta)t} - e^{(j\alpha+(r-j)\beta)t}\right) \\ &= \frac{1}{(4\sqrt{2})^{r-1}}\sum_{j=0}^{\frac{r-1}{2}}(-1)^j\binom{r}{j}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(6j)^{n-k}(r-2j)^k B_k\frac{t^n}{n!}\,. \end{aligned}$$

Therefore, we get

$$\sum_{\substack{k_1+\cdots+k_r=n\\ k_1,\dots,k_r\ge 1}} \binom{n}{k_1,\dots,k_r} B_{k_1}\cdots B_{k_r} = \frac{1}{(4\sqrt{2})^{r-1}}\sum_{j=0}^{\frac{r-1}{2}}(-1)^j\binom{r}{j}\sum_{k=0}^{n}\binom{n}{k}(6j)^{n-k}(r-2j)^k B_k\,.$$

Similarly, we get

$$\sum_{\substack{k_1+\cdots+k_r=n\\ k_1,\dots,k_r\ge 0}} \binom{n}{k_1,\dots,k_r} C_{k_1}\cdots C_{k_r} = \frac{1}{2^{r-1}}\sum_{j=0}^{\frac{r-1}{2}}\binom{r}{j}\sum_{k=0}^{n}\binom{n}{k}(6j)^{n-k}(r-2j)^k C_k\,.$$

If r is even, then

$$\begin{aligned} b(t)^r &= \frac{1}{(4\sqrt{2})^r}\left(\sum_{j=0}^{\frac{r}{2}-1}(-1)^j\binom{r}{j}\left(e^{((r-j)\alpha+j\beta)t} + e^{(j\alpha+(r-j)\beta)t}\right)\right. \\ &\qquad \left. +(-1)^{\frac{r}{2}}\binom{r}{\frac{r}{2}}e^{(\frac{r}{2}\alpha+\frac{r}{2}\beta)t}\right) \\ &= \frac{1}{(4\sqrt{2})^r}\left(2\sum_{j=0}^{\frac{r}{2}-1}(-1)^j\binom{r}{j}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(6j)^{n-k}(r-2j)^k C_k\frac{t^n}{n!}\right. \\ &\qquad \left. +(-1)^{\frac{r}{2}}\binom{r}{\frac{r}{2}}\sum_{n=0}^{\infty}(3r)^n\frac{t^n}{n!}\right)\,. \end{aligned}$$

Therefore, we get

$$\sum_{\substack{k_1+\cdots+k_r=n\\k_1,\dots,k_r\ge 1}} \binom{n}{k_1,\dots,k_r} B_{k_1}\cdots B_{k_r}$$
$$= \frac{1}{(4\sqrt{2})^r}\left(2\sum_{j=0}^{\frac{r}{2}-1}(-1)^j\binom{r}{j}\sum_{k=0}^{n}\binom{n}{k}(6j)^{n-k}(r-2j)^k C_k + (-1)^{\frac{r}{2}}\binom{r}{\frac{r}{2}}(3r)^n\right) .$$

Similarly, we get

$$\sum_{\substack{k_1+\cdots+k_r=n\\k_1,\dots,k_r\ge 0}} \binom{n}{k_1,\dots,k_r} C_{k_1}\cdots C_{k_r}$$
$$= \frac{1}{2^r}\left(2\sum_{j=0}^{\frac{r}{2}-1}\binom{r}{j}\sum_{k=0}^{n}\binom{n}{k}(6j)^{n-k}(r-2j)^k C_k + \binom{r}{\frac{r}{2}}(3r)^n\right) .$$

5 More general cases

Let $\{u_n\}_{n\ge 0}$ and $\{v_n\}_{n\ge 0}$ be integer sequences, satisfying the same recurrence relation: $u_n = au_{n-1} + bu_{n-2}$ ($n \ge 2$) and $v_n = av_{n-1} + bv_{n-2}$ ($n \ge 2$) with initial values u_0, u_1, v_0 and v_1. If the general terms are given by

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad v_n = \alpha^n + \beta^n \quad (n \ge 0)$$

where

$$\alpha = \frac{a + \sqrt{a^2+4b}}{2} \quad \text{and} \quad \beta = \frac{a - \sqrt{a^2+4b}}{2},$$

we can set $u_0 = 0$, $u_1 = 1$, $v_0 = 2$ and $v_1 = a$. Then, the generating functions of u_n and v_n are given by

$$u(t) := \frac{e^{\alpha t} - e^{\beta t}}{\sqrt{a^2+4b}} = \sum_{n=0}^{\infty} u_n \frac{t^n}{n!} \quad \text{and} \quad v(t) := e^{\alpha t} + e^{\beta t} = \sum_{n=0}^{\infty} v_n \frac{t^n}{n!},$$

respectively, because they satisfy the differential equation $y'' - ay' - by = 0$.

Our main results can be stated as follows. The proof is similar to that in the previous section and is omitted.

Theorem 6. *If r is odd, then*

$$\sum_{\substack{k_1+\cdots+k_r=n\\k_1,\dots,k_r\ge 1}} \binom{n}{k_1,\dots,k_r} u_{k_1}\cdots u_{k_r}$$
$$= \frac{1}{(\sqrt{a^2+4b})^{r-1}}\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{\frac{r-1}{2}}(-1)^j\binom{r}{j}(aj)^{n-k}(r-2j)^k u_k \quad (11)$$

and

$$\sum_{\substack{k_1+\cdots+k_r=n\\ k_1,\dots,k_r\ge 0}} \binom{n}{k_1,\dots,k_r} v_{k_1}\cdots v_{k_r} = \sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{\frac{r-1}{2}}\binom{r}{j}(aj)^{n-k}(r-2j)^k v_k\,. \tag{12}$$

If r is even, then

$$\sum_{\substack{k_1+\cdots+k_r=n\\ k_1,\dots,k_r\ge 1}} \binom{n}{k_1,\dots,k_r} u_{k_1}\cdots u_{k_r}$$

$$= \frac{1}{(\sqrt{a^2+4b})^r}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{\frac{r}{2}-1}(-1)^j\binom{r}{j}(aj)^{n-k}(r-2j)^k v_k + (-1)^{\frac{r}{2}}\binom{r}{\frac{r}{2}}\left(\frac{ar}{2}\right)^n\right) \tag{13}$$

and

$$\sum_{\substack{k_1+\cdots+k_r=n\\ k_1,\dots,k_r\ge 0}} \binom{n}{k_1,\dots,k_r} v_{k_1}\cdots v_{k_r} = \sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{\frac{r}{2}-1}\binom{r}{j}(aj)^{n-k}(r-2j)^k v_k + \binom{r}{\frac{r}{2}}\left(\frac{ar}{2}\right)^n\,. \tag{14}$$

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