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Number of tuples with a given least common multiple

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Abstract: In this paper, for a given natural number n, we count the number of k-tuples $(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k$ with certain conditions such that $\operatorname{lcm}(x_1, x_2, \ldots, x_k) = n$. In the process, we derived different arithmetic functions.

Keywords: Arithmetic function, Multiplicative function, Least common multiple, Stirling numbers of the second kind.
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1 Introduction and preliminaries

Let $n = \prod_{i=1}^{m} p_i^{\alpha_i}$ be the prime factorization of positive integer n and

$$\Delta_n(k) = \{(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k \mid \forall j, x_j | n\}.$$

For a given $(x_1, x_2, \ldots, x_k) \in \Delta_n(k)$ we associate a k-tuple $(\beta_1, \beta_2, \ldots, \beta_k)$ such that $p_i^{\beta_j} || x_j$ for each p_i $1 \leq i \leq m$. Since $0 \leq \beta_j \leq \alpha_i$, the total number of possible k-tuples $(x_1, x_2, \ldots, x_k)$ corresponding to p_i is $(\alpha_i + 1)^k$. Thus $|\Delta_n(k)| = \prod_{i=1}^{m} (\alpha_i + 1)^k$. If $k = 1$, then $\Delta_n(1) \subseteq \mathbb{N}$ is exactly the set of positive divisors of n. We denote $|\Delta_n(1)|$ as $\tau(n)$, number of positive divisors of n. Hence $|\Delta_n(k)| = \tau(n)^k$.

The following result, also proved by O. Bagdasar [2] we are giving the proof for the sake of completeness. For elementary properties of divisor function, lcm, $\gcd$ refer any one of the following [1, 3, 6, 7].

Theorem 1.1 ([2]). *Let* $n = \prod_{i=1}^{m} p_i^{\alpha_i}$ *and*

$$A_n(k) = \{(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k | \, \mathrm{lcm}(x_1, x_2, \ldots, x_k) = n\}.$$

Then $|A_n(k)| = \prod_{i=1}^{m} \left((\alpha_i + 1)^k - \alpha_i^k\right)$.

Proof. First note that $A_n(k) \subseteq \Delta_n(k)$. In order to have $\mathrm{lcm}(x_1, x_2, \ldots, x_k) = n$, at least one of x_j should be equal to $p_i^{\alpha_i}$. Corresponding to each p_i, the number of elements in $\Delta_n(k) \setminus A_n(k)$ are α_i^k. Thus the total number of valid cases for p_i is $(\alpha_i + 1)^k - (\alpha_i)^k$. Hence the result follows from the product rule. □

Example 1.2. *If* $n = 12$ *and* $k = 2$*, then we have* $A_{12}(2) = \{(1, 12), (2, 12), (3, 4), (3, 12), (4, 3), (4, 6), (4, 12), (6, 4), (6, 12), (12, 1), (12, 2), (12, 3), (12, 4), (12, 6), (12, 12)\}$ *and*

$$|A_{12}(2)| = ((2 + 1)^2 - 2^2)((1 + 1)^2 - 1^2).$$

Let P_k denote the product of first k primes. For example $P_1 = 2, P_2 = 6, P_3 = 30$. The sequence whose n-th term is P_n is called *primorial* and P_n is called n-th primorial number.

Corollary 1.3. *Let* P_n *be the* n*-th primorial number. Then* $|A_{P_n}(k)| = (2^k - 1)^n$.

It is easy to see that $\Delta_n(k)$ and $A_n(k)$ are multiplicative functions in n. Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is multiplicative if $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$.

Let $S_n(k) = \{(x_1, x_2, \ldots, x_k) \in A_n(k) | \, 1 < x_1 < \cdots < x_k < n\}$. Then it is clear that $S_{p^t}(k)$ is an empty set, where p is a prime number and $t \in \mathbb{N}$. Further, $2 \le k \le \tau(n) - 2$. For example

$$S_{30}(2) = \{(2, 15), (3, 10), (5, 6), (6, 10), (6, 15), (10, 15)\}.$$

Our goal is to find out $|S_n(k)|$ for a given n and k. Before stating main result, we state and prove a couple of results.

Lemma 1.4. *Let* $n = \prod_{i=1}^{m} p_i^{\alpha_i}$ *and* $B_n(k) = \{(x_1, x_2, \ldots, x_k) \in A_n(k) | \; \forall i \; x_i < n\}$*. Then* $|B_n(k)| = |A_n(k)| - \left(\tau(n)^k - (\tau(n) - 1)^k\right)$.

Proof. Let $\Delta_n^n(k) = \{(x_1, x_2, \ldots, x_k) \in \Delta_n(k) | x_j = n \text{ for some } j\}$. Then $\Delta_n^n(k) \subseteq A_n(k)$ and $B_n(k) = A_n(k) \setminus \Delta_n^n(k)$. It is easy to see that $|\Delta_n(k) \setminus \Delta_n^n(k)| = (\tau(n) - 1)^k$. Hence the result follows. □

Example 1.5. *Let* $n = 12$ *and* $k = 2$*. Then* $B_{12}(2) = \{(3, 4), (4, 3), (4, 6), (6, 4)\}$.
$|B_{12}(2)| = 15 - (6^2 - (6 - 1)^2)$.

The following result follows from Corollary 1.3 and $|B_{P_n}(2)| = |A_{P_n}(2)| - (2^{2n} - (2^n - 1)^2)$.

Corollary 1.6. *Let P_n be the n-th primorial number. Then*

$$|B_{P_n}(2)| = 3^n - 2^{n+1} + 1 = 2\left\{ {n+1 \atop 3} \right\},$$

where $\left\{ {n \atop k} \right\}$ is the Stirling numbers of the second kind.

Lemma 1.7. *Let $n = \prod_{i=1}^{m} p_i^{\alpha_i}$ and $C_n(k) = \{(x_1, x_2, \ldots, x_k) \in B_n(k) | \forall i, x_i > 1\}$. Then*

$$|C_n(k)| = \left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} |A_n(k-i)| \right) - \left((\tau(n)-1)^k - (\tau(n)-2)^k \right).$$

Proof. We have that $|C_n(k)| = |B_n(k)|$ is the number of tuples in $B_n(k)$ that contain 1. Using the principle of inclusion and exclusion, we get that number of tuples in $B_n(k)$ that contain 1 is

$$\sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} |B_n(k-i)|$$

$$\begin{aligned}
&= \left(\sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} |A_n(k-i)| \right) - \left(\sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} t^{k-i} \right) + \left(\sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} (t-1)^{k-i} \right) \\
&= \left(\sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} |A_n(k-i)| \right) + \left(\sum_{i=1}^{k} (-1)^{i} \binom{k}{i} t^{k-i} \right) - \left(\sum_{i=1}^{k} (-1)^{i} \binom{k}{i} (t-1)^{k-i} \right) \\
&= \left(\sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} |A_n(k-i)| \right) + \left((t-1)^k - t^k \right) - \left((t-2)^k - (t-1)^k \right) \\
&= \left(\sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} |A_n(k-i)| \right) - \left(t^k - 2(t-1)^k + (t-2)^k \right).
\end{aligned}$$

Therefore, $|C_n(k)|$

$$\begin{aligned}
&= |B_n(k)| - \left(\sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} |A_n(k-i)| \right) + \left(t^k - 2(t-1)^k + (t-2)^k \right) \\
&= |A_n(k)| - \left(t^k - (t-1)^k \right) + \left(\sum_{i=1}^{k} (-1)^{i} \binom{k}{i} |A_n(k-i)| \right) + \left(t^k - 2(t-1)^k + (t-2)^k \right) \\
&= \left(\sum_{i=0}^{k} (-1)^{i} \binom{k}{i} |A_n(k-i)| \right) - \left((t-1)^k - (t-2)^k \right).
\end{aligned}$$

This completes the proof. □

Example 1.8. *Since $C_{12}(2) = \{(3,4), (4,3), (4,6), (6,4)\}$, we have $|C_{12}(2)| = 4$. One can verify that $|C_{12}(2)| = (A_{12}(2) - 2A_{12}(1) + A_{12}(0)) - (5^2 - 4^2)$.*

When $k = 2$, we have that $B_n(2) = C_n(2)$. Thus from Corollary 1.6 we have $|C_{P_n}(2)| = 2\left\{ {n+1 \atop 3} \right\}$.

2 Main results

Theorem 2.1. *Let* $n = \prod_{i=1}^{m} p_i^{\alpha_i}$. *Then*

$$|S_n(k)| = \binom{\tau(n)-2}{k} + \sum_{d|n,\ d\neq 1} \left(\mu(d) \binom{\tau\left(\frac{n}{d}\right)-1}{k} \right),$$

where $\mu(.)$ *denotes well-known Möbius function.*

We use the following lemma to prove above theorem. We omit the proof, as it is easy to derive.

Lemma 2.2. *Let* $n \geq 2$, $G_n(k) = \{(x_1, x_2, \ldots, x_k) | 1 < x_1 < \cdots < x_k < n,\ \forall i\ x_i | n\}$ *and* $H_n(k) = \{(x_1, x_2, \ldots, x_k) | 1 < x_1 < \cdots < x_k \leq n,\ \forall i\ x_i | n\}$. *Then*

$$|G_n(k)| = \binom{\tau(n)-2}{k}, \quad |H_n(k)| = \binom{\tau(n)-1}{k}.$$

Proof. Proof of Theorem 2.1. First note that $S_n(k) \subseteq G_n(k)$. Let $(x_1, \ldots, x_k) \in G_n(k)$. If $\operatorname{lcm}(x_1, x_2, \ldots, x_k) = n$, then $(x_1, \ldots, x_k) \in S_n(k)$. Let us assume that $\operatorname{lcm}(x_1, \ldots, x_k) = l < n$. Then there exists a prime p such that $p | \frac{n}{l}$. Hence $(x_1, \ldots, x_k) \in H_{\frac{n}{p}}(k)$ and for every $p | n$, $H_{\frac{n}{p}}(k) \subseteq G_n(k)$. Hence

$$S_n(k) = G_n(k) \backslash \left(\cup_{p|n} H_{\frac{n}{p}}(k) \right).$$

Therefore,

$$|S_n(k)| = |G_n(k)| - \left| \left(\cup_{p|n} H_{\frac{n}{p}}(k) \right) \right|.$$

Since the prime factors of n are $p_1, p_2, \ldots, p_m$, after applying principle of inclusion and exclusion, we get

$$\begin{aligned}
-&\left| \left(\cup_{p|n} H_{\frac{n}{p}}(k) \right) \right| \\
&= -\sum_{p_i} \left| H_{\frac{n}{p_i}}(k) \right| + \cdots + (-1)^x \sum_{p_{i_1} < \ldots < p_{i_x}} \left| H_{\frac{n}{p_{i_1} \cdots p_{i_x}}}(k) \right| + \cdots + (-1)^m \left| H_{\frac{n}{p_{i_1} \cdots p_{i_m}}}(k) \right| \\
&= \sum_{d|n, d\neq 1} \left(\mu(d) |H_{\frac{n}{d}}(k)| \right) = \sum_{d|n, d\neq 1} \left(\mu(d) \binom{\tau\left(\frac{n}{d}\right)-1}{k} \right).
\end{aligned}$$

Therefore

$$|S_n(k)| = \binom{\tau(n)-2}{k} + \sum_{d|n,\ d\neq 1} \left(\mu(d) \binom{\tau\left(\frac{n}{d}\right)-1}{k} \right). \qquad \square$$

Example 2.3. *For* $n = 30$ *and* $k = 3$, *we have* $S_{30}(3) = \{(2,3,5), (2,3,10), (2,3,15), (2,5,6), (2,5,15), (2,6,10), (2,6,15), (2,10,15), (3,5,6), (3,5,10), (3,6,10), (3,6,15), (3,10,15), (5,6,10), (5,6,15), (5,10,15), (6,10,15)\}$. *Hence* $|S_{30}(3)| = 17$.

Lemma 2.4. *Let* $n \geq 2$ *and* $Q_n(k) = \{(x_1, x_2, \ldots, x_k) | 1 \leq x_1 < \cdots < x_k < n,\ \operatorname{lcm}(x_1, \ldots, x_k) = n\}$. *Then*

$$|Q_n(k)| = \binom{\tau(n)-1}{k} + \sum_{d|n,\ d \neq 1} \left(\mu(d) \binom{\tau\left(\frac{n}{d}\right)}{k} \right).$$

Proof. If $1 < x_1$, then $(x_1, \ldots, x_k) \in S_n(k)$. If $1 = x_1$, then $(x_2, \ldots, x_k) \in S_n(k-1)$. Therefore,

$$\begin{aligned}
|Q_n(k)| &= |S_n(k)| + |S_n(k-1)| \\
=&\binom{\tau(n)-2}{k} + \sum_{d|n,\ d\neq 1} \left(\mu(d) \binom{\tau\left(\frac{n}{d}\right)-1}{k} \right) + \binom{\tau(n)-2}{k-1} + \sum_{d|n,\ d\neq 1} \left(\mu(d) \binom{\tau\left(\frac{n}{d}\right)-1}{k-1} \right) \\
=&\binom{\tau(n)-2}{k} + \binom{\tau(n)-2}{k-1} + \sum_{d|n,\ d\neq 1} \mu(d) \left(\binom{\tau\left(\frac{n}{d}\right)-1}{k} + \binom{\tau\left(\frac{n}{d}\right)-1}{k-1} \right) \\
=&\binom{\tau(n)-1}{k} + \sum_{d|n,\ d\neq 1} \left(\mu(d) \binom{\tau\left(\frac{n}{d}\right)}{k} \right).
\end{aligned}$$

□

Example 2.5. *For* $n = 12$ *and* $k = 3$, *we have* $Q_{12}(3) = \{(1, 3, 4), (1, 4, 6), (2, 3, 4), (2, 4, 6), (3, 4, 6)\}$.

$$|Q_{12}(3)| = 10 + (-4 - 1 + 0 + 0 + 0).$$

Corollary 2.6. *Let* P_n *denote the* n*-th primorial number. Then* $|Q_{P_n}(2)| = \left\{ {n+1 \atop 3} \right\}$.

Proof. We have $|Q_{P_n}(2)| = \binom{\tau(P_n)-1}{2} + \sum_{d|P_n,\ d\neq 1} \left(\mu(d) \binom{\tau\left(\frac{P_n}{d}\right)-1}{2} \right)$. Thus

$$\begin{aligned}
|Q_{P_n}(2)| &= \binom{2^n - 1}{2} + \sum_{i=1}^{n} (-1)^i \binom{n}{i} \binom{2^{n-i}}{2} \\
&= \binom{2^n - 1}{2} - \binom{2^n}{2} + \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{2^{n-i}}{2} \\
&= \binom{2^n - 1}{2} - \binom{2^n}{2} + \frac{1}{2} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (2^{n-i})(2^{n-i} - 1) \\
&= \binom{2^n - 1}{2} - \binom{2^n}{2} + \frac{1}{2} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (4^{n-i} - 2^{n-i}) \\
&= \binom{2^n - 1}{2} - \binom{2^n}{2} + \frac{1}{2} 3^n - \frac{1}{2} = \frac{3^n - 2^{n+1} + 1}{2} = \left\{ {n+1 \atop 3} \right\}.
\end{aligned}$$

□

Theorem 2.7. *Let* $R_n(k) = \{(x_1, x_2, \ldots, x_k) \in S_n(k) \mid \gcd(x_1, \ldots, x_k) = 1\}$. *Then*

$$|R_n(k)| = |S_n(k)| + \sum_{d|n\ d\neq 1} \mu(d) |Q_{\frac{n}{d}}(k)|.$$

Proof. Let $(x_1, \ldots, x_k) \in S_n(k)$ and $\gcd(x_1, \ldots, x_k) = d > 1$. Then there exists a prime p such that $p|d$. Hence we can write $(x_1, \ldots, x_k) = p(\frac{x_1}{p}, \ldots, \frac{x_k}{p})$ and $(x_1, \ldots, x_k) \in pQ_{\frac{n}{p}}(k)$. Since for every prime p, $pQ_{\frac{n}{p}}(k) \subseteq S_n(k)$, $R_n(k) = S_n(k) \setminus \left(\cup_{p|n} pQ_{\frac{n}{p}}(k) \right)$. Hence we have

$$|R_n(k)| = |S_n(k)| - |\left(\cup_{p|n} pQ_{\frac{n}{p}}(k) \right)|.$$

Let the prime factors of n be $\{p_1, p_2, \ldots, p_m\}$. By applying principle of inclusion and exclusion we get

$$-\left|\left(\cup_{p|n} p Q_{\frac{n}{p}}(k)\right)\right|$$
$$= -\sum_{p_i}\left|Q_{\frac{n}{p_i}}(k)\right| + \cdots + (-1)^x \sum_{p_{i_1}<\ldots<p_{i_x}}\left|Q_{\frac{n}{p_{i_1}\cdots p_{i_x}}}(k)\right| + \cdots + (-1)^m\left|Q_{\frac{n}{p_{i_1}\cdots p_{i_m}}}(k)\right|$$
$$= \sum_{d|n, d\neq 1}\left(\mu(d)|Q_{\frac{n}{d}}(k)|\right).$$

Therefore

$$|R_n(k)| = |S_n(k)| + \sum_{d|n\ d\neq 1}\mu(d)|Q_{\frac{n}{d}}(k)|.$$ □

We noticed that the sequence $|S_{P_n}(2)|$ coincide with the sequence in OEIS: A000392 (`https://oeis.org/A000392`). The following result establishes the same correspondence.

Theorem 2.8. *Let P_n be the n-th primorial number. Then*

$$|S_{P_n}(2)| = \left\{ {n+1 \atop 3} \right\}.$$

Proof. We have

$$|S_{P_n}(2)|$$
$$= \binom{2^n-2}{2} + \left(\sum_{i=1}^{n}(-1)^i\binom{n}{i}\binom{2^{n-i}-1}{2}\right)$$
$$= (2^{n-1}-1)(2^n-3) + \left(\sum_{i=1}^{n}(-1)^i\binom{n}{i}(2^{n-i}-1)(2^{n-i-1}-1)\right)$$
$$= -2(2^{n-1}-1) + (-1)^0\binom{n}{0}(2^n-1)(2^{n-1}-1) + \left(\sum_{i=1}^{n}(-1)^i\binom{n}{i}(2^{n-i}-1)(2^{n-i-1}-1)\right)$$
$$= -2(2^{n-1}-1) + \left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}(2^{n-i}-1)(2^{n-i-1}-1)\right)$$
$$= -2(2^{n-1}-1) + \left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}(\frac{1}{2}4^{n-i} - \frac{3}{2}2^{n-i} + 1)\right)$$
$$= -2(2^{n-1}-1) + \frac{1}{2}\left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}(4^{n-i})\right) - \frac{3}{2}\left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}(2^{n-i})\right) + \left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}\right)$$
$$= -2(2^{n-1}-1) + \frac{3^n-3}{2}$$
$$= \frac{3^n - 2^{n+1} + 1}{2}$$
$$= \left\{ {n+1 \atop 3} \right\}.$$ □

Theorem 2.9. *Let* $n = \prod_{i=1}^{m} p_i^{\alpha_i}$, $k \leq m$ *and* $F_n(k) = \{(x_1, x_2, \ldots, x_k) \in A_n(k) | x_i \neq 1$ *and* $\gcd(x_i, x_j) = 1\}$. *Then* $|F_n(k)| = \left\{ {m \atop k} \right\} k!$.

Proof. Let $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$ be an onto function. Let $f(i)$ denote the position of prime power $p_i^{\alpha_i}$ in the k-tuple. Since f is onto every entry in the k-tuple is a non-unit. Therefore, the number of onto functions is equal to the number of required k-tuples. The number of onto functions from a set of size m to a set of size k is given by $\left\{ {m \atop k} \right\} k!$. Hence $|F_n(k)| = \left\{ {m \atop k} \right\} k!$. □

Example 2.10. $F_{210}(3) = \{(2, 3, 35), (2, 5, 21), (2, 7, 15), (2, 15, 7), (2, 21, 5), (2, 35, 3), (3, 2, 35), (3, 5, 14), (3, 7, 10), (3, 10, 7), (3, 14, 5), (3, 35, 2), (5, 2, 21), (5, 3, 14), (5, 6, 7), (5, 7, 6), (5, 14, 3), (5, 21, 2), (6, 5, 7), (6, 7, 5), (7, 2, 15), (7, 3, 10), (7, 5, 6), (7, 6, 5), (7, 10, 3), (7, 15, 2), (10, 3, 7), (10, 7, 3), (14, 3, 5), (14, 5, 3), (15, 2, 7), (15, 7, 2), (21, 2, 5), (21, 5, 2), (35, 2, 3), (35, 3, 2)\}$. *Hence* $|F_{210}(3)| = 36$.

Corollary 2.11. *Let* $F'_n(k) = \{(x_1, x_2, \ldots, x_k) \in F_n(k) | x_1 < \cdots < x_k < n\}$.

Proof. Each tuple in $F'_n(k)$ corresponds to $k!$ tuples in $F_n(k)$. Hence

$$|F'_n(k)| = \frac{|F_n(k)|}{k!} = \left\{ {m \atop k} \right\}.$$ □

Example 2.12. $F'_{210}(3) = \{(2, 3, 35), (2, 5, 21), (2, 7, 15), (3, 5, 14), (3, 7, 10), (5, 6, 7)\}$. *Hence* $|F_{210}(3)| = 6$. *It is easy to verify that* $R_n(2) = F'_n(2)$.

3 Conclusion

In this article, for a given natural numbers n and k, we derived different arithmetic functions of the form $f_n(k)$ which count the numbers elements in $\mathbb{N}^k$ satisfying few conditions such that whose lcm is n. We associate these functions with Stirling numbers of the second kind for certain values of n and k. In future we will work on applications of these functions on the multiplicative representation of integers studied in [5, 8] in particular, Theorem 2.9. One can also explore sequences obtained by iterating these functions as studied in the recent paper [4].

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