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Restrictive factor and extension factor

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Abstract: Restrictive factor and extension factor are two arithmetic functions, introduced by the authors. In the paper, some of their extensions are introduced and some of the basic properties of the newly defined functions are studied.
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1 Introduction

In a series of papers, published over the last 35 years, the authors introduced some new arithmetic functions. Two of them were called "Restrictive Factor" [1] and "Extension Factor" [2]. For each natural number $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where k, α_1, α_2, …, $\alpha_k \geq 1$ are natural numbers and $p_1, p_2, \dots, p_k$ are different prime numbers, these factors are defined, respectively, by:

$$RF(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1},$$
$$RF(1) = 1$$

and

$$EF(n) = \prod_{i=1}^{k} p_i^{\alpha_i+1},$$
$$EF(1) = 1.$$

In the present paper, for each natural number n, of the above form we will introduce new arithmetic functions, related to the above mentioned ones.

2 First round of generalizations

Let

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

with k, α_1, α_2, $\ldots$, $\alpha_k \geq 1$, is the prime factorization of $n > 1$. Define

$$EF_s(n) = \prod_{i=1}^{r} p_i^{s\alpha_i+1}$$

and

$$RF_s(n) = \prod_{i=1}^{r} p_i^{s\alpha_i-1},$$

where $s \in \mathcal{R}$, and $\mathcal{R}$ is the set of real numbers.

Then, clearly

$$EF_1(n) = EF(n),$$
$$RF_1(n) = RF(n)$$

and

$$EF_s(n).RF_s(n) = n^{2s}. \tag{1}$$

So, using the inequality $x + y \geq 2\sqrt{xy}$, from (1) we get

$$EF_s(n) + RF_s(n) \geq 2n^s. \tag{2}$$

We have

$$\begin{aligned} EF_s(n) &= n^s.\underline{\text{mult}}(n), \\ RF_s(n) &= \frac{n^s}{\underline{\text{mult}}(n)}. \end{aligned} \tag{3}$$

We get

$$\begin{aligned} EF_s(n) &\geq EF(n), \\ RF_s(n) &\geq RF(n), \end{aligned} \tag{4}$$

where $s \geq 1$ and

$$\begin{aligned} EF_s(n) &\leq EF(n), \\ RF_s(n) &\leq RF(n), \end{aligned} \tag{5}$$

for $0 < s \leq 1$

Similarly, as

$$\frac{n^s}{\underline{\text{mult}}(n)} \geq \left[\frac{n}{\underline{\text{mult}}(n)}\right]^s$$

for $s \geq 1$, we get by (3) that

$$\begin{aligned} EF_s(n) &\geq (EF(n))^s, \\ RF_s(n) &\geq (RF(n))^s, \end{aligned} \tag{6}$$

for $s \geq 1$. For $0 < s \leq 1$, the reverse inequalities hold true. Now, we prove

Theorem 1. *Let J_s denote the Jordan totient function. Then we have for $n > 1$:*

$$RF_s(n) \leq (\underline{\text{mult}}(n))^{s-1}(n^s - J_s(n)), \tag{7}$$

for $s > 0$.

Proof. We have

$$J_s(n) = n^s \prod_{i=1}^{s} \left(1 - \frac{1}{p_i^s}\right).$$

Now, first we prove that

$$\prod_{i=1}^{s} \left(1 - \frac{1}{p_i^s}\right) \leq 1 - \frac{1}{\prod\limits_{i=1}^{s} p_i^s} \tag{8}$$

or equivalently,

$$\prod_{i=1}^{s} (p_i^s - 1) \leq \prod_{i=1}^{s} p_i^s - 1.$$

Put $p_i^s - 1 = x_i$ for $i = 1, 2, \ldots, r$. Then we have to prove that

$$\prod_{i=1}^{s} x_i \leq \prod_{i=1}^{s} (x_i + 1) - 1,$$

or

$$\prod_{i=1}^{s} (x_i + 1) \geq \prod_{i=1}^{s} x_i + 1.$$

This holds true, as $x_i > 0$ by $p_i \geq 2^s > 1$ for $s > 0$. For $r = 1$, we have equality. Now, by (8) we can write

$$J_s(n) \leq n^k - \frac{n^s}{\underline{\text{mult}}(n)} . \frac{1}{(\underline{\text{mult}}(n))^{s-1}} = n^s - RF_s(n) . \frac{1}{(\underline{\text{mult}}(n))^{s-1}}$$

and equality (7) follows.

Obviously, for $s = 1$ we get from (7):

$$RF(n) \leq n - \varphi(n) \tag{9}$$

for $n > 1$. □

Theorem 2. *For $s \geq 1$ we have*

$$EF_s(n) > \sigma_s(n) \tag{10}$$

for $n > 1$. When $n \geq 3$ is odd, then

$$EF_s(n) > \sigma_s(n) + n^s, \tag{11}$$

where $\sigma_s(n)$ denotes the sum of s-th powers of the divisors of n.

Proof. As

$$\sigma_s(n) = \prod_{i=1}^{r} \frac{p_i^{s(\alpha_i+1)} - 1}{p_1^s - 1},$$

for the proof of (10) it will be sufficient to show that

$$p^{sa+1} > \frac{p^{s(a+1)} - 1}{p^s - 1}. \tag{12}$$

Now, (12) is equivalent to $p^{sa+s+1} - p^{sa+1} - p^{sa+s} > -1$ that is valid, because

$$p^s - p^{s-1} = p^{s-1}(p-1) \geq p - 1 \geq 1$$

by $s \geq 1$.

For the proof of (11) we will use the following well-known inequality (see, e.g., [6]) for $s \geq 1$:

$$\sigma_s(n).J_s(n) < n^{2s}.$$

Thus, we get

$$\sigma_s(n) < \frac{n^{2s}}{J_s(n)} < n^s.(\underline{\text{mult}}(n))^s - n^s.$$

The right inequality is equivalent to

$$\frac{n^s}{J_s(n)} < (\underline{\text{mult}}(n))^s - n^s. \tag{13}$$

The inequality (13) can be written also as

$$\frac{x_1 \dots x_r}{(x_1 - 1)\dots(x_r - 1)} < x_1 \dots x_r - 1, \tag{14}$$

where $x_i = p_i^s$ for $i = 1, \dots, r$. When n is odd and $s \geq 1$, then $x_1, \dots, x_r \geq 3$, and the inequality (14) is proved in [2]. Thus, we get the inequality

$$\sigma_s(n) + n^s < (EF(n))^s,$$

which is even stronger than (11), by the second relation of (6). □

Now, we shall use the following lemma (see [4, 5]).

Lemma 1. *If $a_1, \dots, a_r > 0, \alpha_1, \dots, \alpha_r > 0$ and $\alpha_1 + \dots + \alpha_r = 1$, then*

$$\frac{1}{\frac{\alpha_1}{a_1} + \dots + \frac{\alpha_r}{a_r}} \leq a_1^{\alpha_1} \dots a_r^{\alpha_r} \leq \alpha_1 a_1 + \dots + \alpha_r a_r. \tag{15}$$

We will mention that this is the classical Weighted Harmonic Mean – Geometric Mean – Arithmetic Mean inequality.

Let $\omega(n) = r$ (the number of distinct prime factors of $n > 1$), $\Omega(n) = a_1 + \cdots + a_r$ (the number of prime factors of n), $\beta(n) = p_1 + \cdots + p_r$; $B(n) = a_1 p_1 + \cdots + a_r p_r$ (see [6]).

Theorem 3. *For $n > 1$ we have*

$$EF_s(n) \leq \left(\frac{s.B(n) + \beta(n)}{k} \right)^k, \tag{16}$$

where $k = s\Omega(n) + \omega(n)$ and $s > 0$ and

$$RF_s(n) \leq \left(\frac{s.B(n) - \beta(n)}{m} \right)^m, \tag{17}$$

where $m = s\Omega(n) - \omega(n)$ and $s \geq 1$.

Proof. Apply the right-hand side of inequality (15) to $a_1 = p_1, \ldots, a_r = p_r$ and $\alpha_1 = \frac{sa_1 + 1}{k}, \ldots, \alpha_r = \frac{sa_r + 1}{k}$. Then, clearly,

$$\alpha_1 + \cdots + \alpha_r = \frac{s(a_1 + \cdots + a_r) + r}{k} = \frac{s\Omega(n) + \omega(n)}{k} = 1.$$

After elementary computations, we get (16).

In the same manner, apply the right-hand side of (15) to $a_1 = p_1, \ldots, a_r = p_r$ and $\alpha_1 = \frac{sa_1 - 1}{m}, \ldots, \alpha_r = \frac{sa_r - 1}{m}$. Then

$$\alpha_1 + \cdots + \alpha_r = \frac{s(a_1 + \cdots + a_r) + r}{m} = \frac{s\Omega(n) - \omega(n)}{m}$$

and from $\alpha_1, \ldots, \alpha_r > 0$ by $s.a_i - 1 \geq a_i - 1 \geq 1 > 0$; inequality (17) follows, as well. □

In that follows, we will introduce the following new arithmetic functions: let

$$\beta^*(n) = \frac{1}{p_1} + \cdots + \frac{1}{p_r}$$

and

$$B^*(n) = \frac{a_1}{p_1} + \cdots + \frac{a_r}{p_r}.$$

Theorem 4. *For $n > 1$ we have*

$$EF_s(n) \geq \left(\frac{k}{sB^*(n) + \beta^*(n)} \right)^k \tag{18}$$

for $s > 0$; and

$$RF_s(n) \geq \left(\frac{k}{sB^*(n) - \beta^*(n)} \right)^k \tag{19}$$

for $s \geq 1$, where $k = s\Omega(n) + \omega(n)$ for $s > 0$ and $m = s\Omega(n) - \omega(n)$ for $s \geq 1$.

Proof. Use the left-hand side of inequality (15) to $a_1 = p_1, \dots, a_r = p_r$ and $\alpha_1 = \frac{sa_1+1}{k}, \dots, \alpha_r = \frac{sa_r+1}{k}$ and use the new arithmetic functions β^* and B^*. So, inequality (18) follows. Inequality (19) follows in the same manner. □

From (15), by letting $\alpha_1 = \dots = \alpha_r = \frac{1}{r}$, we get

$$(a_1 + \dots + a_r).\left(\frac{1}{a_1} + \dots + \frac{1}{a_r}\right) \geq r^2 \tag{20}$$

so we get the relation

$$\beta(n).\beta^*(n) \geq (\omega(n))^2. \tag{21}$$

We shall prove the similar inequality

$$B(n).B^*(n) \geq (\Omega(n))^2. \tag{22}$$

For this purpose, apply the classical Cauchy–Bunyakowski inequality (see [4])

$$\left(\sum_{i=1}^{r} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{r} x_i^2\right) \cdot \left(\sum_{i=1}^{r} y_i^2\right)$$

to $x_i = \sqrt{a_i p_i}, y_i = \sqrt{\frac{a_i}{p_i}}$. As $x_i y_i = a_i$, by the given definitions, inequality (22) follows. By $x + y \geq 2\sqrt{xy}$, clearly from (21) and (22), we get:

$$\begin{aligned} \beta(n) + \beta^*(n) &\geq 2\omega(n), \\ B(n) + B^*(n) &\geq 2\Omega(n). \end{aligned} \tag{23}$$

Functions β^* and B^* will be studied in detail in another paper.

3 Second round of generalizations

A second generalization of EF and RF will be given by

$$EF^{(s)}(n) = \prod_{i=1}^{r} p_i^{a_i+s}$$

and

$$RF^{(s)}(n) = \prod_{i=1}^{r} p_i^{a_i-s},$$

where $s \in \mathcal{R}$. Then clearly $EF^{(1)}(n) = EF(n)$ and $RF^{(1)}(n) = RF(n)$.

Now,

$$EF^{(s)}(n).RF^{(s)}(n) = n^2. \tag{24}$$

Thus, we have the inequality similar to (2):

$$EF^{(s)}(n) + RF^{(s)}(n) = 2n. \tag{25}$$

We have

$$EF^{(s)}(n) = n(\underline{\text{mult}}(n))^s,$$
$$RF^{(s)}(n) = \frac{n}{(\underline{\text{mult}}(n))^s}. \tag{26}$$

From (26) and (3) it is clear that

$$EF_s(n) \geq EF^{(s)}(n) \tag{27}$$

for $s \geq 1$ with an equality only when $s = 1$, and

$$RF_s(n) \geq RF^{(s)}(n) \tag{28}$$

for $s \geq 1$ with an equality only when $s = 1$.

For $0 < s \leq 1$, the inequalities in (27) and (28) are reversed.

By (26), we get that

$$RF^{(s)}(n) \leq RF(n) \leq n - \varphi(n). \tag{29}$$

Now, we shall introduce an extension of the well-known arithmetic function function σ.

Put

$$\sigma^{(s)}(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+s} - 1}{p_i - 1} \tag{30}$$

for $n > 1$. Clearly, we have $\sigma^{(1)}(n) = \sigma(n)$.

As $s(a_i + 1) \geq a_i + s$ for $s \geq 1$, we get that

$$\sigma_s(n) \geq \sigma^{(s)}(n), \tag{31}$$

where $\sigma_s(n)$ is the sum of the s-th powers of the divisors of n.

Theorem 5. *For $s \geq 1$ we have for $n > 1$:*

$$n < \sigma^{(s)}(n) < EF(s)(n). \tag{32}$$

Proof. The following double inequality can be directly proved:

$$p^a < \frac{p^{a+s} - 1}{p - 1} < p^{a+s}, \tag{33}$$

where $a \geq 1, s \geq 1$. Then (32) follows from the definitions. □

Theorem 6. *For $s \geq 1$ we have:*

$$\frac{(\underline{\text{mult}}(n))^{s-1}}{\zeta(s+1)} < \frac{\varphi(n).\sigma^{(s)}(n)}{n^2} < (\underline{\text{mult}}(n))^{s-1}, \tag{34}$$

where ζ is the Riemann zeta function.

Proof. By (30) we have

$$\sigma^{(s)}(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+s}.\left(1 - \frac{1}{p_i^{a_i+s}}\right)}{p_i - 1} < \prod_{i=1}^{r} \frac{p_i^{a_i+s}}{p_i - 1} = n.\prod_{i=1}^{r} \frac{p_i^{s}}{p_i - 1} = n.\prod_{i=1}^{r} p_i^{s-1}.\prod_{i=1}^{r} \frac{p_i}{p_i - 1}$$

$$= \frac{n^2}{\varphi(n)}.\prod_{i=1}^{r} p_i^{s-1} = \frac{n^2}{\varphi(n)}.\underline{\text{mult}}(n)^{s-1},$$

so the right-hand side of (34) follows.

For the left-hand side of the inequality, let us remark that

$$\frac{\varphi(n)\sigma^{(s)}(n)}{n^2} = \prod_{i=1}^{r} \frac{p_i^{a_i+s} - 1}{p_i^{a_i+1}} = \prod_{i=1}^{r} p_i^{s-1}.\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^{a_i+s}}\right) = \underline{\text{mult}}(n)^{s-1}.\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^{a_i+s}}\right).$$

Now, by Euler's formula we see that

$$\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^{a_i+s}}\right) \geq \prod_{i=1}^{r}\left(1 - \frac{1}{p_i^{s+1}}\right) > \prod_{p \text{ prime}}\left(1 - \frac{1}{p_i^{s+1}}\right) = \frac{1}{\zeta(s+1)}.$$

Thus, the left-hand side of (34) follows, too. □

For $s = 1$ and $n > 1$, we get the classical inequalities (see [6]):

$$\frac{6}{\pi^2} < \frac{\varphi(n)\sigma(n)}{n^2} < 1. \tag{35}$$

Corollary. For $s \geq 1$ we have

$$\frac{EF^{s-1}(n)}{\zeta(s+1)} < \frac{\varphi(n)\sigma^{(s)}(n)}{n} < EF^{(s-1)}(n). \tag{36}$$

Using now Lemma 1, we can obtain results similar to those stated in Theorems 3 and 4:

Theorem 7. *For $n > 1$ we have*

$$EF^{(s)}(n) \leq \left(\frac{B(n) + s\beta(n)}{\Omega(n) + s\omega(n)}\right)^{\Omega(n)+s\omega(n)}, \tag{37}$$

$$EF^{(s)}(n) \geq \left(\frac{\Omega(n) + s\omega(n)}{B(n) + s\beta(n)}\right)^{\Omega(n)+s\omega(n)}, \tag{38}$$

$$RF^{(s)}(n) \leq \left(\frac{B(n) - s\beta(n)}{\Omega(n) - s\omega(n)}\right)^{\Omega(n)-s\omega(n)}, \tag{39}$$

$$EF^{(s)}(n) \geq \left(\frac{\Omega(n) - s\omega(n)}{B(n) - s\beta(n)}\right)^{\Omega(n)-s\omega(n)}. \tag{40}$$

Proof. For the proof of (37), apply the right-hand side of inequality (15) to $a_1 = p_1, \dots, a_r = p_r$ and $\alpha_1 = \frac{a_1+s}{t}, \dots, \alpha_r = \frac{a_r+s}{t}$, where $t = \Omega(n) + s\omega(n)$. Then,

$$\alpha_1 + \cdots + \alpha_r = \frac{\Omega(n) + s\omega(n)}{t} = 1.$$

So, after elementary computations, we get (37). For the inequality (38), apply the left-hand side of inequality (15), and use the new arithmetic functions β^* and B^* (see the proof of Theorem 4).

Inequalities (39) and (40) can be proved in the same manner, and we omit the details. □

We now state an auxiliary result, which is essentially due to Minkowski [4]:

Lemma 2. *Let $A, B \geq 0$. Then we have:*

$$\left(\prod_{i=1}^{r}(A_i + B_i)\right)^{\frac{1}{r}} \geq \left(\prod_{i=1}^{r} A_i\right)^{\frac{1}{r}} + \left(\prod_{i=1}^{r} B_i\right)^{\frac{1}{r}}. \tag{41}$$

If $A_i \geq B_i$ $(i = 1, \dots, r)$, then

$$\left(\prod_{i=1}^{r}(A_i - B_i)\right)^{\frac{1}{r}} \leq \left(\prod_{i=1}^{r} A_i\right)^{\frac{1}{r}} - \left(\prod_{i=1}^{r} B_i\right)^{\frac{1}{r}}. \tag{42}$$

Proof. (41) is well-known. For the proof of (42), for each i $(i = 1, \dots, r)$ put: $A_i := A_i - B_i$ and $B_i := B_i$ instead of A_i and B_i in (41). Then we get from (41) the inequality (42). □

Theorem 8. *From any $s \in \mathcal{R}$ we have*

$$\left(EF^{(s)}(n)\right)^{\frac{1}{\omega(n)}} + \left(EF^{(s-1)}(n)\right)^{\frac{1}{\omega(n)}} \leq \left((\underline{\text{mult}}(n))^s.\psi(n)\right)^{\frac{1}{\omega(n)}}, \tag{43}$$

$$\left(EF^{(s)}(n)\right)^{\frac{1}{\omega(n)}} - \left(EF^{(s-1)}(n)\right)^{\frac{1}{\omega(n)}} \geq \left((\underline{\text{mult}}(n))^s.\varphi(n)\right)^{\frac{1}{\omega(n)}}, \tag{44}$$

where ψ denotes Dedekind's arithmetic function and φ denotes Euler's totient function.

Proof. Let $A_i = p_i^{a_i+s}, B_i = p_i^{a_i+s-1}$ in (41). Then

$$A_i + B_i = p_i^{a_i+s-1}(p_i + 1) = p_i^s.p_i^{a_i-1}(p_i + 1).$$

As $\psi(n) = \prod_{i=1}^{r} p_i^{a_i-1}(p_i + 1)$, by definitions, we get the desired inequality (43). Inequality (44) can be deduced in the same manner from (42). □

Theorem 9. *From any $s \in \mathcal{R}$ we have*

$$\left(EF_{(s-1)}(n).\psi(n)\right)^{\frac{1}{\omega(n)}} \geq \left(EF_{(s)}(n)\right)^{\frac{1}{\omega(n)}} + n^{\frac{s}{\omega(n)}}, \tag{45}$$

$$\left(EF_{(s-1)}(n).\varphi(n)\right)^{\frac{1}{\omega(n)}} \geq \left(EF_{(s)}(n)\right)^{\frac{1}{\omega(n)}} + n^{\frac{s}{\omega(n)}}, \tag{46}$$

for $n > 1$.

Proof. Apply (41) to $A_i = p_i^{sa_i+1}, B_i = p_i^{sa_i}$. Now,

$$A_i + B_i = p_i^{sa_i}.(p_i+1) = p_i^{a_i-1}.(p_i+1).p_i^{a_i(s-1)+1}.$$

So, by the given definition, inequality (45) follows from (41). The similar proof applies to (46), and we omit the details. □

We can mention that when $A_i > 0, B_i > 0$ hold true for any $s \in \mathcal{R}$ and inequality $A_i \geq B_i$ is equivalent to $p_i \geq 1$, so, we can assume again that s can take any real natural value.

Theorem 10. *For $s > -1$ we have*

$$\left(EF^{(s)}(n)\right)^{\frac{1}{\omega(n)}} \geq \left(\prod_{i=1}^{r}(p_i-1)\right)^{\frac{1}{\omega(n)}} . \left(\sigma^s(n)\right)^{\frac{1}{\omega(n)}} + 1. \tag{47}$$

Proof. Apply inequality (42) of Lemma 2 to $A_i = \dfrac{p_i^{a_i+s}}{p_i-1}$ and $B_i = \dfrac{1}{p_i-1}$. Then

$$\prod_{i=1}^{r}(A_i - B_i) = \sigma^{(s)}(n),$$

$$\prod_{i=1}^{r} A_i = \frac{ES^{(s)}(n)}{\prod\limits_{i=1}^{r}(p_i-1)},$$

$$\prod_{i=1}^{r} B_i = \frac{1}{\prod\limits_{i=1}^{r}(p_1-1)},$$

and after elementary transformations, we get inequality (47). □

We will mention that it is immediate that

$$\prod_{i=1}^{r}(p_i-1) = \frac{\underline{\text{mult}}(n).\varphi(n)}{n}, \tag{48}$$

so (47) can be written also in terms of the arithmetic functions mult and φ.

4 Additive analogues

As $\beta(n)$ is an additive analogue of mult(n) and $B(n)$ – of the identity function n, respectively, one can introduce the additive analogues of the functions EF and RF. More generally, let us denote

$$RF_+^{(s)}(n) = \sum_{i=1}^{r} p_i^{a_i-s},$$
$$EF_+^{(s)}(n) = \sum_{i=1}^{r} p_i^{a_i+s}, \tag{49}$$

and similarly,

$$RF_{+,s}(n) = \sum_{i=1}^{r} p_i^{sa_i-1},$$
$$EF_{+,s}(n) = \sum_{i=1}^{r} p_i^{sa_i+1}. \tag{50}$$

They are generalizations of the additive functions:

$$RF_{+}(n) = \sum_{i=1}^{r} p_i^{a_i-1},$$
$$EF_{+}(n) = \sum_{i=1}^{r} p_i^{a_i+1}. \tag{51}$$

Here, the respective conditions for the s-argument of $RF_s(n)$ and $RF_+^{(s)}$ are valid, as above.

We will study first the arithmetic functions (51), as these have not been studied in the literature up to now. First, we prove the following theorem.

Theorem 11. *For* $n > 1$*,*

$$RF_+(n) \geq \omega(n)(RF(n))^{\frac{1}{\omega(n)}}, \tag{52}$$

$$EF_+(n) \geq \omega(n)(EF(n))^{\frac{1}{\omega(n)}}, \tag{53}$$

$$\left(\frac{B^1(n) - RF_+(n)}{\omega(n)}\right) \geq \varphi(n), \tag{54}$$

$$\left(\frac{B^1(n) + RF_+(n)}{\omega(n)}\right) \geq \psi(n), \tag{55}$$

$$\left(\frac{EF_+(n) - RF_+(n)}{\omega(n)}\right) \geq \frac{\varphi(n)\psi(n)}{RF(n)}. \tag{56}$$

Proof. Inequality (52) follows by applying the arithmetic-geometric mean inequality

$$\sum_{i=1}^{r} x_i \geq r\left(\sqrt[r]{\prod_{i=1}^{r} x_i}\right), \tag{57}$$

for $x_i = p_i^{a_i-1} (i = 1, \ldots, r), r = \omega(n)$. For (53) put $x_i = p_i^{a_i+1}$; for (54) remark that $p^a - p^{a-1} = p^{a-1}(p-1)$ and $\prod_{i=1}^{r}(p_i^{a_i} - p_i^{a_i-1}) = \varphi(n)$. Let $x_i = p_i^{a_i} - p_i^{a_i-1}$ in (57). As $\sum_{i=1}^{r} p_i^{a_i} = B(n)$ and $\sum_{i=1}^{r} p_i^{a_i-1} = RF_+(n)$, (54) follows.

Apply (57) for $x_i = p_i^{a_i} + p_i^{a_i-1}$ to deduce (55).

Finally, as $p_i^{a_i+1} - p_i^{a_i-1} = p_i^{a_i-1}(p_i-1)(p_i+1)$, we get

$$\prod_{i=1}^{r}(p_i^{a_i+1} - p_i^{a_i-1}) = \frac{\varphi(n)\psi(n)}{RF(n}$$

and (56) follows by applying (57) to $x_i = p_i^{a_i+1} - p_i^{a_i-1}$. □

Theorem 12. *For $n > 1$ we have*

$$(B_1(n))^2 \leq RF_+(n).EF_+(n), \tag{58}$$

$$(RF_+(n))^2 \leq \omega(n).(RF(n))^{\frac{2}{\omega(n)}} + (\omega(n)-1).RF_+\left(\frac{m}{\underline{\text{mult}}(n)}\right), \tag{59}$$

$$(EF_+(n))^2 \leq \omega(n).(EF(n))^{\frac{2}{\omega(n)}} + (\omega(n)-1).EF_+\left(\frac{m}{\underline{\text{mult}}(n)}\right). \tag{60}$$

Proof. For the proof of (58) apply the classical Cauchy–Bunyakovski–Schwarz inequality (see [4]) for $x_i = \sqrt{p_i^{a_i-1}}, y_i = \sqrt{p_i^{a_i+1}}$. Then, the inequality (58) follows.

For the proof of (59) and (60), we will use the following inequality due to T. Popoviciu and V. Cîrtoaje (see [3]).

If $I \subseteq \mathcal{R}$ is an interval and $f : I \to \mathcal{R}$ is a convex function, and $a_1, \ldots, a_r \in I$ for $r > 2$, then

$$\sum_{i=1}^{r} f(a_i) + \frac{r}{r-2}.f\left(\frac{\sum_{i=1}^{r} a_i}{r}\right) \geq \frac{2}{r-2}.\sum_{1\leq i<j\leq r} f\left(\frac{a_i+a_j}{2}\right). \tag{61}$$

Put $f(x) = e^x$ in(61) and then, let $a_i = 2\log x_i$ for $x_i > 0$. As

$$\sum_{1\leq i<j\leq r} x_i x_j = \frac{1}{r-2}\left(\left(\sum_{i=1}^{r} x_i\right)^2 - \sum_{i=1}^{r} x_i^2\right),$$

after some transformations, we get from (61):

$$(r-1)\sum_{i=1}^{r} x_i^2 + r\sqrt[\frac{1}{r}]{\prod_{i=1}^{r} x_i^2} \geq \left(\sum_{i=1}^{r} x_i\right)^2. \tag{62}$$

Now, apply first the inequality (62) for $x_i = p_i^{a_i-1}$. As $\frac{n^2}{\underline{\text{mult}}(n)} = \prod_{i=1}^{r} p_i^{2a_i-1}$, we get that

$$\sum_{i=1}^{r} x_i^2 = \sum_{i=1}^{r} p_i^{a_i-1} = RF_+\left(\frac{n^2}{\underline{\text{mult}}(n)}\right),$$

and (59) follows. In the same manner, apply (62) to $x_i = p_i^{a_i+1}$. As

$$EF_+\left(\frac{n^2}{\underline{\text{mult}}(n)}\right) = \sum_{i=1}^{r} p_i^{2a_i+2},$$

inequality (60) follows. □

5 Conclusion

In Section 3 we introduced an extension of the sum-of-divisor function $\sigma^{(s)}$

$$\sigma^{(s)}(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+s} - 1}{p_i - 1}.$$

We note that a similar extension can be introduced, namely

$$\sigma_{(s)}(n) = \prod_{i=1}^{r} \frac{p_i^{sa_i+1} - 1}{p_i - 1}.$$

Both functions are new – and distinct – from the classical function

$$\sigma_s(n) = \prod_{i=1}^{r} \frac{p_i^{s(a_i+1)} - 1}{p_i - 1}.$$

The properties of the new σ-functions, and their connections with other arithmetic functions can be studied, and these will be the object of future research.

Theorems 11 and 12 may be extended to the general functions $RF_{+}^{(s)}$, $RF_{+,s}(n)$, etc.

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