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Gaussian binomial coefficients

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Abstract: This paper extends Gaussian binomial coefficients (and so-called) Fibonomial coefficients) with identities related to Horadam's generalized binomial coefficients.

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1 Introduction

Gaussian binomial coefficients and associated q-series have been treated in many papers in a variety of applications by Carlitz [5, 6, 7, 8, 10] which display some typical settings for the series, so is there anything more that can be said? These Gaussian numbers are different from, but have some analogous properties to, 'Gaussian Integers' properly so-called [11].
q-series are defined basically by

$$(q)_n = (1-q)(1-q^2)\dots(1-q^n),\ n > 0,\ (q)_0 - 1. \tag{1.1}$$

Arising out of these are Gaussian binomial coefficients

$$\binom{n}{k}_q = \begin{cases} \dfrac{(1-q^n)(1-q^{n-1})\dots(1-q^{n-k+1})}{(1-q)(1-q^2)\dots(1-q^k)}, & k \le n, \\ 0, & k > n, \end{cases}$$
$$= \frac{[n]_q!}{[k]_q!\,[n-k]_q!}, (k \le n)$$

where

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$$
$$= \begin{cases} \dfrac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1. \end{cases}$$

Mercier [18] has already developed some properties in his theorem and its corollaries. It is proposed here to develop some new results with generalized Fibonacci numbers $\{U_n\} \equiv \{U_n(a,b;p,q\}$ [16] where p, q are arbitrary integers, and a, b are initial values.

2 Fibonomial coefficients

Carlitz [9] and Horadam [16] have used them in the form that follows with generating functions for powers of generalized Fibonacci numbers. If we formally let $q = \beta/\alpha$ in the above definition, where α, β, assumed distinct, are the roots of $x^2 - px + q = 0$, then

$$\binom{n}{k}_q = \frac{\left(1-\left(\beta/\alpha\right)^n\right)\ldots\left(1-\left(\beta/\alpha\right)^{n-k+1}\right)}{\left(1-\left(\beta/\alpha\right)^1\right)\left(1-\left(\beta/\alpha\right)^2\right)\ldots\left(1-\left(\beta/\alpha\right)^k\right)}$$

$$= \alpha^{k(n-k)} \frac{U_n U_{n-1} \ldots U_{n-k+1}}{U_1 U_2 \ldots U_k}$$

$$= U_n C_{n,k} \alpha^{k(n-k)},$$

in which the sequence $\{U_n\}$ defined above, and

$$C_{n,k} = \frac{U_{n-1} \ldots U_{n-k+1}}{U_1 U_2 \ldots U_k}. \tag{2.1}$$

The significance of the $C_{n,k}$ can be seen in Hoggatt [13, 14] in which he developed properties for ordinary Fibonacci numbers and the Gaussian binomial coefficients, there called Fibonomial coefficients [12, 17]. Some of these properties were prefigured by Alexanderson [1] and Andrews [4].

In this spirit we obtain

Theorem 1.

$$\binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \frac{2-q^k-q^{n-k}}{1-q^n}\binom{n}{k}_q. \tag{2.2}$$

Proof:

$$\binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \frac{(1-q^{n-1})(1-q^{n-2})\ldots(1-q^{n-k})}{(1-q)(1-q^2)\ldots(1-q^k)} + \frac{(1-q^{n-1})(1-q^{n-2})\ldots(1-q^{n-k})}{(1-q)(1-q^2)\ldots(1-q^{k-1})}$$

$$= \frac{(1-q^{n-1})(1-q^{n-2})\ldots(1-q^{n-k+1})}{(1-q)(1-q^2)\ldots(1-q^{k-1})}\left\{\frac{1-q^{n-k}}{1-q^k} + 1\right\}$$

which yields the desired result. (2.2) is a variation of the relatively well-known identities

$$q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \binom{n}{k}_q.$$

and

$$\binom{n-1}{k}_q + q^{n-k} \binom{n-1}{k-1}_q = \binom{n}{k}_q.$$

See, for instance Andrews [3]. □

3 Connections with generalized Fibonacci numbers

We finish with two results which extend the known formulas for these numbers, irrespective of their initial conditions, namely (3.1) and (3.2) below.

Theorem 2.

$$\left(\frac{2-q^k-q^{n-k}}{1-q^n}\right)\alpha^n U_n = \alpha^{n-k} U_{n-k} + \alpha^k U_k. \tag{3.1}$$

Proof:

$$\binom{n-1}{k}_q = \alpha^{k(n-k-1)} U_{n-k} \frac{U_{n-1}U_{n-2} \dots U_{n-k+1}}{U_1 U_2 \dots U_k}$$

$$= U_{n-k} C_{n,k} \alpha^{k(n-k-1)}$$

$$\binom{n-1}{k-1}_q = \alpha^{(k-1\_(n-k)} U_k \frac{U_{n-1}U_{n-2} \dots U_{n-k+1}}{U_1 U_2 \dots U_k}$$

$$= U_k C_{n,k} \alpha^{(k-1)(n-k)}$$

and the result follows after induction. □

Theorem 3.

$$\binom{n}{m}_q = U_{m+1}\alpha^m \binom{n-1}{m}_q - qU_{n-m+1}\alpha^{n-m}\binom{n-1}{m-1}_q. \tag{3.2}$$

Proof: It can readily be shown [15] that

$$U_n = U_{m+1}U_{n-m} - qU_m U_{n-m-1}$$

and so from Section 2

$$U_n C_{n,k} \alpha^{m(n-m)} = U_{m+1}\left(U_{n-m} C_{n,k} \alpha^{m(n-m-1)}\right)\alpha^m - qU_{n-m-1}\left(U_m C_{n,k} \alpha^{(m-1)(n-m)}\right)\alpha^{n-m}$$

and the required result comes from the use of the definition of $C_{n,k}$. This result is a generalization of equation (F) in [14]. □

4 Concluding comments

There is a three-fold value in searching for elegant generalizations in number theory, namely, to investigate which identities are essential, to discover links with otherwise apparently unrelated results, and to formulate ideas for further research. Thus generalizing q-biomial coefficients to their multinomial analogues has provided the mathematical identities for scientific applications in seemingly unexpected contexts; for example, [2].

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