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Some binomial-sum identities for the generalized bi-periodic Fibonacci sequences

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Abstract: A bi-periodic sequence is a sequence which satisfies different recurrence relations depending on whether the n-th term considered is odd or even. In this paper, we investigate the properties of the generalized bi-periodic Fibonacci sequences. It is a generalization of the bi-periodic Fibonacci sequences defined by Edson and Yayenie. We derive binomial-sum identities for the generalized bi-periodic Fibonacci sequences by matrix method. Our identities generalize binomial-sum identities derived by Edson and Yayenie for the case of bi-periodic Fibonacci sequences.
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1 Introduction

The *generalized bi-periodic Horadam sequence* $(w_n)_{n\geq 0} := (w_n(w_0, w_1; a, b, c))_{n\geq 0}$ is defined by the following recurrence relations:

$$w_n = \begin{cases} aw_{n-1} + cw_{n-2}, & \text{if } n \text{ is even} \\ bw_{n-1} + cw_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2$$

with arbitrary initial conditions w_0 and w_1, and nonzero real numbers a, b and c. The numbers u_n defined by $(u_n)_{n\geq 0} := (w_n(0, 1; a, b, c))_{n\geq 0}$ are called the *generalized bi-periodic Fibonacci* *numbers*. The numbers v_n defined by $(v_n)_{n\geq 0} := (v_n(2, b; a, b, c))_{n\geq 0}$ are called the *generalized bi-periodic Lucas numbers*.

The generalized bi-periodic Horadam sequence $(w_n)_{n\geq 0}$ is a natural generalization of the Horadam sequence $(h_n)_{n\geq 0} := (w_n(w_0, w_1; a, a, c))_{n\geq 0}$ [6], the bi-periodic Fibonacci sequence $(q_n)_{n\geq 0} := (w_n(0, 1; a, b, 1))_{n\geq 0}$ [3], the classical Fibonacci numbers F_n, the Lucas numbers L_n, etc.

For various generalizations of the classical Fibonacci numbers, it is well-known that many combinatorial identities and summation identities can be derived by the matrix method. Its origin comes from the *Fibonacci Q-matrix* Q_n which is defined as follows:

$$Q_n := \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

It has the following interesting property:

$$Q_n = Q_1^n \tag{1}$$

and hence one readily gets the Cassini's identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ by taking determinants on both sides of (1). By the Cayley–Hamilton theorem on Q_1, we have $Q_1^2 - Q_1 - I = 0$. We easily derive various interesting binomial-sum identities for the Fibonacci numbers by the method of binomial expansion. For example, by considering $(Q_1^2)^n = (Q_1 + I)^n$, we easily get the following well-known identity by comparing the top right entries on both sides of the matrix equation:

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k.$$

Gould's paper [4] serves as an excellent reference for the history of the method of Fibonacci Q-matrix. The matrix method is further explored by Bicknell and Hoggatt [5], Deveci [2], Khmovsky [7], Ekin and Tan [9], Tan [10], Waddill [12] to derive identities for the Fibonacci numbers and its generalizations. Bacon, Cook, and Graves [1] gave a recent account of a generalization of this method.

In this paper, we derive some binomial-sum identities for the generalized bi-periodic Fibonacci numbers u_n and the generalized bi-periodic Lucas numbers v_n by the matrix method.

2 Main results

The Binet's formula for the generalized bi-periodic Fibonacci numbers u_n is as follows [13, Theorem 8]:

$$u_n = \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right). \tag{2}$$

The function $\zeta(n)$ is the *parity function* of n, i.e., $\zeta(n) = 0$ if n is even; and $\zeta(n) = 1$ if n is odd. The variables α and β are the roots of the polynomial $x^2 - abx - abc$. That is,

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 4abc}}{2}, \quad \beta = \frac{ab - \sqrt{a^2b^2 + 4abc}}{2}.$$

We note the following algebraic properties of α and β:

$$\alpha+\beta=ab, \quad \alpha-\beta=\sqrt{a^2b^2+4abc}, \quad \alpha\beta=-abc. \tag{3}$$

Also, the Binet's formula for the generalized bi-periodic Lucas numbers v_n is as follows (see (36) in Section 3. Appendix):

$$v_n=\frac{b^{\zeta(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}\left(\alpha^n+\beta^n\right). \tag{4}$$

We state some identities for the numbers u_n and v_n.

Lemma 2.1. *Let m and n be any non-negative integers. The following identities are true:*

$$u_m v_n+u_n v_m=2\left(\frac{b}{a}\right)^{\zeta(n)\zeta(m)}u_{n+m}, \tag{5}$$

$$u_{n+m}+(-c)^m u_{n-m}=\left(\frac{a}{b}\right)^{\zeta(n)\zeta(m)}u_n v_m, \tag{6}$$

$$u_{n+m}-(-c)^m u_{n-m}=\left(\frac{a}{b}\right)^{\zeta(n)\zeta(m)}u_m v_n. \tag{7}$$

Proof. It can be proved by the Binet's formulas for the numbers u_n (see (2)) and v_n (see (4)). We will do it only for the case of even n and odd m in (7). The other cases would be left as exercises to the readers. Let $m=2k+1$ and n be even. Then,

$$\begin{aligned}
&u_{n+(2k+1)}+c^{2k+1}u_{n-(2k+1)}\\
&=\frac{1}{(ab)^{\frac{n}{2}+k}}\cdot\frac{\alpha^{n+(2k+1)}-\beta^{n+(2k+1)}}{\alpha-\beta}+\frac{c^{2k+1}}{(ab)^{\frac{n}{2}-k-1}}\cdot\frac{\alpha^{n-(2k+1)}-\beta^{n-(2k+1)}}{\alpha-\beta}\\
&=\frac{1}{(ab)^{\frac{n}{2}+k}\cdot(\alpha-\beta)}\left(\alpha^{n+(2k+1)}-\beta^{n+(2k+1)}+(abc)^{2k+1}\left(\alpha^{n-(2k+1)}-\beta^{n-(2k+1)}\right)\right)\\
&=\frac{1}{(ab)^{\frac{n}{2}+k}\cdot(\alpha-\beta)}\left(\alpha^{n+(2k+1)}-\beta^{n+(2k+1)}-(\alpha\beta)^{2k+1}\left(\alpha^{n-(2k+1)}-\beta^{n-(2k+1)}\right)\right)\\
&=\frac{1}{(ab)^{\frac{n}{2}+k}\cdot(\alpha-\beta)}\left(\alpha^{n+(2k+1)}-\beta^{n+(2k+1)}-\left(\alpha^n\beta^{2k+1}-\alpha^{2k+1}\beta^n\right)\right)\\
&=\frac{1}{(ab)^k}\cdot\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}\cdot\frac{1}{(ab)^{\frac{n}{2}}}\cdot\left(\alpha^n+\beta^n\right)=u_{2k+1}v_n.
\end{aligned}$$

□

Remark 1. *The identity (5) is a generalization of the classical identity $F_nL_n=F_{2n}$ which involves both the Fibonacci numbers F_n and the Lucas numbers L_n; while the identities (6), (7) are generalizations of the identities for the numbers F_n and L_n first given by Ruggles [8].*

Let k be any positive integer. Let R_{2k} be the following matrix:

$$R_{2k}:=\begin{pmatrix} v_{2k} & 1\\ -c^{2k} & 0\end{pmatrix}=\frac{1}{u_{2k}}\begin{pmatrix} u_{4k} & u_{2k}\\ -c^{2k}u_{2k} & 0\end{pmatrix}. \tag{8}$$

The last equality is due to the identity (5). We state the following property for the matrix R_{2k}.

Lemma 2.2. *Let $n \geq 0$. Then, we have*

$$R_{2k}^n = \frac{1}{u_{2k}} \begin{pmatrix} u_{(2n+2)k} & u_{2nk} \\ -c^{2k}u_{2nk} & -c^{2k}u_{(2n-2)k} \end{pmatrix}.$$

Proof. We prove it by induction on n. The base case for $n = 1$ is clear by (8). For the inductive step, we do the following computation:

$$\begin{aligned} R_{2k}^n \cdot R_{2k} &= \frac{1}{u_{2k}} \begin{pmatrix} u_{(2n+2)k} & u_{2nk} \\ -c^{2k}u_{2nk} & -c^{2k}u_{(2n-2)k} \end{pmatrix} \begin{pmatrix} v_{2k} & 1 \\ -c^{2k} & 0 \end{pmatrix} \\ &= \frac{1}{u_{2k}} \begin{pmatrix} u_{(2n+2)k}v_{2k} - c^{2k}u_{2nk} & u_{(2n+2)k} \\ -c^{2k}u_{2nk}v_{2k} + c^{4k}u_{(2n-2)k} & -c^{2k}u_{2nk} \end{pmatrix} \\ &= \frac{1}{u_{2k}} \begin{pmatrix} u_{(2n+4)k} & u_{(2n+2)k} \\ -c^{2k}u_{(2n+2)k} & -c^{2k}u_{2nk} \end{pmatrix}. \end{aligned}$$

The last equality is due to the identity (6) and hence the inductive step is complete. □

The following identities for the numbers u_n are true due to an application of a theorem in the paper [11] (see Theorem 3.2 in Section 3. Appendix):

$$u_{2k+s}u_{2k+2nk} - u_{2k}u_{2nk+2k+s} = c^{2k}u_s u_{2nk}, \tag{9}$$

$$u_s u_{2k+2nk} - u_{2k}u_{2nk+s} = c^{2k}u_{2nk}u_{s-2k}, \tag{10}$$

$$u_{2k+s}u_{2nk} - u_{2k}u_{2nk+s} = c^{2k}u_s u_{(2n-2)k}. \tag{11}$$

Let $s \geq 2k$. Let Q_s be the following matrix:

$$Q_s := \begin{pmatrix} u_{s+2k} & u_s \\ -c^{2k}u_s & -c^{2k}u_{s-2k} \end{pmatrix}.$$

By Lemma 2.2, we do the following matrix computation:

$$\begin{aligned} R_{2k}^n Q_s &= \frac{1}{u_{2k}} \begin{pmatrix} u_{(2n+2)k} & u_{2nk} \\ -c^{2k}u_{2nk} & -c^{2k}u_{(2n-2)k} \end{pmatrix} \begin{pmatrix} u_{s+2k} & u_s \\ -c^{2k}u_s & -c^{2k}u_{s-2k} \end{pmatrix} \\ &= \frac{1}{u_{2k}} \begin{pmatrix} u_{(2n+2)k}u_{s+2k} - c^{2k}u_{2nk}u_s & u_{(2n+2)k}u_s - c^{2k}u_{2nk}u_{s-2k} \\ -c^{2k}\left(u_{2nk}u_{s+2k} - c^{2k}u_s u_{(2n-2)k}\right) & -c^{2k}\left(u_{2nk}u_s - c^{2k}u_{(2n-2)k}u_{s-2k}\right) \end{pmatrix} \\ &= \begin{pmatrix} u_{2nk+2k+s} & u_{2nk+s} \\ -c^{2k}u_{2nk+s} & -c^{2k}u_{2nk-2k+s}. \end{pmatrix} \end{aligned} \tag{12}$$

The last equality is due to the identities (9), (10) and (11).

By the Cayley–Hamilton theorem, the characteristic equation for the matrix R_{2k} (see (8)) is

$$R_{2k}^2 - v_{2k}R_{2k} + c^{2k}I = 0 \quad \Longrightarrow \quad \left(R_{2k} \pm c^k I\right)^2 = \left(v_{2k} \pm 2c^k\right) R_{2k}. \tag{13}$$

For the expression $\left(v_{2k} \pm 2c^k\right)$, we have the following identities:

Lemma 2.3. *Let $k \geq 0$. Then, we have*

$$\begin{aligned} v_{4k} + 2c^{2k} &= v_{2k}^2, \\ v_{4k} - 2c^{2k} &= \frac{(ab)^2 + 4abc}{a^2} u_{2k}^2, \\ v_{4k+2} + 2c^{2k+1} &= (ab + 4c)\, u_{2k+1}^2, \\ v_{4k+2} - 2c^{2k+1} &= \frac{a}{b} v_{2k+1}^2. \end{aligned}$$

Proof. By the Binet's formula for the numbers v_n shown in (4) and the equations shown in (3), the identities stated in the lemma can be proved by straightforward algebraic manipulations. As an example, for the first identity,

$$\begin{aligned} v_{2k}^2 &= \frac{1}{(ab)^{2k}} \left(\alpha^{2k} + \beta^{2k}\right)^2 = \frac{1}{(ab)^{2k}} \left(\alpha^{4k} + \beta^{4k} + 2(\alpha\beta)^{2k}\right) \\ &= \frac{1}{(ab)^{2k}} \left(\alpha^{4k} + \beta^{4k}\right) + \frac{1}{(ab)^{2k}} \left(2(abc)^{2k}\right) = v_{4k} + 2c^{2k}. \end{aligned}$$

Other identities can be proved in a similar manner. □

We state the following matrix identities for the matrix R_{2k}.

Theorem 2.4. *Let k, m and n be non-negative integers. We have*

$$R_{4k}^m \left(R_{4k} + c^{2k} I\right)^{2n} = v_{2k}^{2n} R_{4k}^{n+m}, \tag{14}$$

$$R_{4k}^m \left(R_{4k} + c^{2k} I\right)^{2n+1} = v_{2k}^{2n} R_{4k}^{n+m} \left(R_{4k} + c^{2k} I\right), \tag{15}$$

$$R_{4k}^m \left(R_{4k} - c^{2k} I\right)^{2n} = \left(b^2 + \frac{4bc}{a}\right)^n u_{2k}^{2n} R_{4k}^{n+m}, \tag{16}$$

$$R_{4k}^m \left(R_{4k} - c^{2k} I\right)^{2n+1} = \left(b^2 + \frac{4bc}{a}\right)^n u_{2k}^{2n} R_{4k}^{n+m} \left(R_{4k} - c^{2k} I\right), \tag{17}$$

$$R_{4k+2}^m \left(R_{4k+2} + c^{2k+1} I\right)^{2n} = (ab + 4c)^n u_{2k+1}^{2n} R_{4k+2}^{n+m}, \tag{18}$$

$$R_{4k+2}^m \left(R_{4k+2} + c^{2k+1} I\right)^{2n+1} = (ab + 4c)^n u_{2k+1}^{2n} R_{4k+2}^{n+m} \left(R_{4k+2} + c^{2k+1} I\right), \tag{19}$$

$$R_{4k+2}^m \left(R_{4k+2} - c^{2k+1} I\right)^{2n} = \left(\frac{a}{b}\right)^n v_{2k+1}^{2n} R_{4k+2}^{n+m}, \tag{20}$$

$$R_{4k+2}^m \left(R_{4k+2} - c^{2k+1} I\right)^{2n+1} = \left(\frac{a}{b}\right)^n v_{2k+1}^{2n} R_{4k+2}^{n+m} \left(R_{4k+2} - c^{2k+1} I\right). \tag{21}$$

Proof. All of these matrix identities can be shown to be true by essentially the same method. To illustrate it, we do it only for the identity (15). By (13), we get

$$\begin{aligned} & \left(R_{4k} + c^{2k} I\right)^2 = \left(v_{4k} + 2c^{2k}\right) R_{4k} \\ \Longrightarrow & \left(R_{4k} + c^{2k} I\right)^{2n+1} = \left(v_{4k} + 2c^{2k}\right)^n R_{4k}^n \left(R_{4k} + c^{2k} I\right) \\ \Longrightarrow & R_{4k}^m \left(R_{4k} + c^{2k} I\right)^{2n+1} = \left(v_{4k} + 2c^{2k}\right)^n R_{4k}^{n+m} \left(R_{4k} + c^{2k} I\right) \\ \Longrightarrow & \left(R_{4k} + c^{2k} I\right)^{2n+1} = v_{2k}^{2n} R_{4k}^{n+m} \left(R_{4k} + c^{2k} I\right). \end{aligned}$$

The last equality is due to Lemma 2.3. □

We state some binomial-sum identities for the numbers u_n and v_n as follows:

Theorem 2.5. *For the generalized bi-periodic Fibonacci numbers u_n and the generalized bi-periodic Lucas numbers v_n, we have the following binomial-sum identities:*

$$\sum_{j=0}^{2n}\binom{2n}{j}c^{2k(2n-j)}u_{4k(j+m)+s}=v_{2k}^{2n}u_{4k(n+m)+s}, \tag{22}$$

$$\sum_{j=0}^{2n+1}\binom{2n+1}{j}c^{2k(2n+1-j)}u_{4k(j+m)+s}=v_{2k}^{2n+1}u_{4k(n+m)+2k+s}, \tag{23}$$

$$\sum_{j=0}^{2n}\binom{2n}{j}(-c)^{2k(2n-j)}u_{4k(j+m)+s}=\left(b^2+\frac{4bc}{a}\right)^n u_{2k}^{2n}u_{4k(n+m)+s}, \tag{24}$$

$$\sum_{j=0}^{2n+1}\binom{2n+1}{j}(-c)^{2k(2n+1-j)}u_{4k(j+m)+s}=\left(b^2+\frac{4bc}{a}\right)^n u_{2k}^{2n+1}v_{4k(j+m)+2k+s}, \tag{25}$$

$$\sum_{j=0}^{2n}\binom{2n}{j}c^{(2k+1)(2n-j)}u_{(4k+2)(j+m)+s}=(ab+4c)^n u_{2k+1}^{2n}u_{(4k+2)(n+m)+s}, \tag{26}$$

$$\sum_{j=0}^{2n+1}\binom{2n+1}{j}c^{(2k+1)(2n+1-j)}u_{(4k+2)(j+m)+s}=\left(\frac{a}{b}\right)^{\zeta(s+1)}(ab+4c)^n u_{2k+1}^{2n+1}v_{(4k+2)(n+m)+2k+1+s}, \tag{27}$$

$$\sum_{j=0}^{2n}\binom{2n}{j}(-c)^{(2k+1)(2n-j)}u_{(4k+2)(j+m)+s}=\left(\frac{a}{b}\right)^n v_{2k+1}^{2n}u_{(4k+2)(n+m)+s}, \tag{28}$$

$$\sum_{j=0}^{2n+1}\binom{2n+1}{j}(-c)^{(2k+1)(2n+1-j)}u_{(4k+2)(j+m)+s}=\left(\frac{a}{b}\right)^{n+\zeta(s+1)} v_{2k+1}^{2n+1}u_{(4k+2)(n+m)+2k+1+s}. \tag{29}$$

Proof. The derivation is straightforward for (22), (24), (26), (28) by first applying binomial expansions on the identities (14), (16), (18) and (20) respectively; then, by applying the matrices of the equations to the matrix Q_s and by (12), we get the desired identities by comparing the $(1,2)$-entries on both sides of the matrix equations.

For the identities (23), (25), (27), (29), we first apply binomial expansions to the identities (15), (17), (19) and (21), respectively. Then, we need to apply Lemma 2.1 to get the identities. We will show it for the identity (23). The proofs of the remaining identities will be left as exercises for the readers.

For identity (23), by a binomial expansion on (15), we get

$$\begin{aligned}R_{4k}^m\left(R_{4k}+c^{2k}I\right)^{2n+1}&=R_{4k}^m\sum_{j=0}^{2n+1}\binom{2n+1}{j}c^{2k(2n+1-j)}R_{4k}^j=\sum_{j=0}^{2n+1}\binom{2n+1}{j}c^{2k(2n+1-j)}R_{4k}^{j+m}\\&=v_{2k}^{2n}(R_{4k}^{n+m+1}+c^{2k}R_{4k}^{n+m}).\end{aligned}$$

We multiply the matrices on both sides by the matrix Q_s and compare the $(1,2)$-entries on both sides of the matrix equation, we get

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} c^{2k(2n+1-j)} u_{4k(j+m)+s} = v_{2k} \left(u_{4k(n+m+1)+s} + c^{2k} u_{4k(n+m)+s} \right)$$
$$= v_{2k}^{2n+1} u_{4k(n+m)+2k+s}.$$

The last equality is due to the identity (6). □

Remark 2. *If we set $c = 1$ in Theorem 2.5, then we get the corresponding binomial-sum identities for the bi-periodic Fibonacci sequence $(q_n)_{n\geq 0} := (w_n(0,1;a,b,1))_{n\geq 0}$ and bi-periodic Lucas sequence $(p_n)_{n\geq 0} := (w_n(2,b;a,b,1))_{n\geq 0}$ defined by Edson and Yayenie [3]. If we set $a = b = c = 1$, then we get the corresponding binomial-sum identities for the Fibonacci numbers F_n and Lucas numbers L_n originally derived by Bicknell and Hoggatt [5].*

3 Appendix

The results stated in Appendix are presented in the paper [11]. Since it is unpublished, we state and prove some of the results in [11] which are used in this paper.

The matrix $A = \begin{pmatrix} ab & abc \\ 1 & 0 \end{pmatrix}$ has the following property:

$$A^n = \begin{pmatrix} ab & abc \\ 1 & 0 \end{pmatrix}^n = (ab)^{\lfloor \frac{n}{2} \rfloor} \begin{pmatrix} b^{\zeta(n)} u_{n+1} & cba^{\zeta(n)} u_n \\ a^{-\zeta(n+1)} u_n & cb^{\zeta(n)} u_{n-1} \end{pmatrix}. \tag{30}$$

If n is even, then we have

$$A^n \begin{pmatrix} w_1 \\ a^{-1} w_0 \end{pmatrix} = (ab)^{\frac{n}{2}} \begin{pmatrix} w_{n+1} \\ a^{-1} w_n \end{pmatrix}, \quad A^n \begin{pmatrix} cbw_2 \\ cw_1 \end{pmatrix} = (ab)^{\frac{n}{2}} \begin{pmatrix} cbw_{n+2} \\ cw_{n+1} \end{pmatrix}. \tag{31}$$

By combining equations (30) and (31), we get

$$\begin{pmatrix} w_{n+1} \\ a^{-1} w_n \end{pmatrix} = \begin{pmatrix} u_{n+1} & cbu_n \\ a^{-1} u_n & cu_{n-1} \end{pmatrix} \begin{pmatrix} w_0 \\ a^{-1} w_1 \end{pmatrix}, \begin{pmatrix} cbw_{n+2} \\ cw_{n+1} \end{pmatrix} = \begin{pmatrix} u_{n+1} & cbu_n \\ a^{-1} u_n & cu_{n-1} \end{pmatrix} \begin{pmatrix} cbw_2 \\ cw_1 \end{pmatrix}. \tag{32}$$

Theorem 3.1. *Let n and p be positive integers. Then,*

$$w_{n+p} = \left(\frac{b}{a}\right)^{\zeta(n+1)\zeta(p)} u_n w_{p+1} + c \left(\frac{b}{a}\right)^{\zeta(n)\zeta(p+1)} u_{n-1} w_p$$

Proof. Let n and p be even. Then, by (31) and (32), we get

$$(ab)^{\frac{n+p}{2}} \begin{pmatrix} w_{n+p+1} \\ a^{-1} w_{n+p} \end{pmatrix} = A^{n+p} \begin{pmatrix} w_1 \\ a^{-1} w_0 \end{pmatrix} = (ab)^{\frac{p}{2}} A^n \begin{pmatrix} w_{p+1} \\ a^{-1} w_p \end{pmatrix} \tag{33}$$

$$= (ab)^{\frac{n+p}{2}} \begin{pmatrix} u_{n+1} & cbu_n \\ a^{-1} u_n & cu_{n-1} \end{pmatrix} \begin{pmatrix} w_{p+1} \\ {}^{-1} w_p \end{pmatrix} \tag{34}$$

By comparing both entries of the matrices on both sides of (33), we get the result as desired. Similarly, we obtain the following equation by (31) and (32),

$$\begin{pmatrix} cbw_{n+p+2} \\ cw_{n+p+1} \end{pmatrix} = \begin{pmatrix} u_{n+1} & cbu_n \\ a^{-1}u_n & cu_{n-1} \end{pmatrix} \begin{pmatrix} cbw_{p+2} \\ cw_{p+1} \end{pmatrix}. \tag{35}$$

By comparing entries of the matrices in (35) and taking $p+1 = q$, we get the result as desired. $\square$

By setting $p = 0$ in Theorem 3.1, we get the relation

$$w_n = u_n w_1 + c \left(\frac{b}{a}\right)^{\zeta(n)} u_{n-1} w_0.$$

Based on this identity and the Binet's formula for the generalized bi-periodic Fibonacci sequence $\{u_n\}$ (see (2)), by setting $w_0 = 2$ and $w_1 = b$, we get the Binet's formula for the generalized bi-periodic Lucas sequence $\{v_n\}$ as follows:

$$v_n = \frac{b^{\zeta(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} (\alpha^n + \beta^n). \tag{36}$$

By Theorem 3.1, we have the following matrix identities for even n and even p:

$$\begin{pmatrix} u_{n+p+1} \\ a^{-1}u_p \end{pmatrix} = \begin{pmatrix} u_{n+1} & bcu_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{p+1} \\ a^{-1}u_p \end{pmatrix}, \tag{37}$$

$$\begin{pmatrix} cbu_{n+p+2} \\ cu_{p+1} \end{pmatrix} = \begin{pmatrix} u_{n+1} & bcu_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} cbu_{p+2} \\ cu_{p+1} \end{pmatrix}. \tag{38}$$

Theorem 3.2. *Let n and p be even integers and q be any integer. Then, we have the following identity for the generalized bi-periodic Horadam sequence:*

$$u_{n+p}u_{n+q} - u_n u_{n+p+q} = (c)^n u_p u_q.$$

Proof. For the case of even n, even p and odd q, we do a computation for

$$(ab)^{n+\frac{p}{2}} \begin{pmatrix} u_{n+q} & a^{-1}u_n \end{pmatrix} \begin{pmatrix} a^{-1}u_{n+p} \\ -u_{n+p+q} \end{pmatrix}.$$

By (37) and (38), we get the following equations:

$$\begin{aligned} (ab)^{\frac{n}{2}} \begin{pmatrix} u_{n+q} & a^{-1}u_n \end{pmatrix} &= (ab)^{\frac{n}{2}} \begin{pmatrix} u_{n+1} & a^{-1}u_n \end{pmatrix} \begin{pmatrix} u_q & 0 \\ bcu_{q-1} & 1 \end{pmatrix} \\ &= \begin{pmatrix} u_1 & a^{-1}u_0 \end{pmatrix} \left(A^T\right)^n \begin{pmatrix} u_q & 0 \\ bcu_{q-1} & 1 \end{pmatrix}. \end{aligned} \tag{39}$$

$$\begin{aligned} (ab)^{\frac{n+p}{2}} \begin{pmatrix} a^{-1}u_{n+p} \\ -u_{n+p+q} \end{pmatrix} &= (ab)^{\frac{n+p}{2}} \begin{pmatrix} 1 & 0 \\ -bcu_{q-1} & u_q \end{pmatrix} \begin{pmatrix} a^{-1}u_{n+p} \\ -u_{n+p+1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -bcu_{q-1} & u_q \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -abc & ab \end{pmatrix}^{n+p} \begin{pmatrix} a^{-1}u_0 \\ -u_1 \end{pmatrix} \end{aligned} \tag{40}$$

We note that

$$\begin{pmatrix} u_q & 0 \\ bcu_{q-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -bcu_{q-1} & u_q \end{pmatrix} = u_q I, \quad \begin{pmatrix} ab & 1 \\ abc & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -abc & ab \end{pmatrix} = -abcI.$$

By (39) and (40), we get

$$(ab)^{n+\frac{p}{2}} \begin{pmatrix} u_{n+q} & a^{-1}u_n \end{pmatrix} \begin{pmatrix} a^{-1}u_{n+p} \\ -u_{n+p+q} \end{pmatrix} = (u_q)(-abc)^n \begin{pmatrix} u_1 & a^{-1}u_0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -abc & ab \end{pmatrix}^p \begin{pmatrix} a^{-1}u_0 \\ -u_1 \end{pmatrix}$$

$$= u_q(abc)^n(ab)^{\frac{p}{2}} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} cu_{p-1} & -a^{-1}u_p \\ -bcu_p & u_{p+1} \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = (ab)^{n+\frac{p}{2}} c^n (a^{-1}) u_p u_q.$$

By expanding the left side of the equation and cancelling common terms on both sides, we get the desired result. For the case of even n, even p and even q, it can be proved in a similar way. □

Remark 3. *In the paper [11], a generalization of Theorem 3.2 is proved for the generalized bi-periodic Horadam numbers w_n.*

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