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On the Generalized Fibonacci-circulant-Hurwitz numbers

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Abstract: The theory of Fibonacci-circulant numbers was introduced by Deveci et al. (see [5]). In this paper, we define the Fibonacci-circulant-Hurwitz sequence of the second kind by Hurwitz matrix of the generating function of the Fibonacci-circulant sequence of the second kind and give a fair generalization of the sequence defined, which we call the generalized Fibonacci-circulant-Hurwitz sequence. First, we derive relationships between the generalized Fibonacci-circulant-Hurwitz numbers and the generating matrices for these numbers. Also, we give miscellaneous properties of the generalized Fibonacci-circulant-Hurwitz numbers such as the Binet formula, the combinatorial, permanental, determinantal representations, the generating function, the exponential representation and the sums.

Keywords: Fibonacci-circulant-Hurwitz Sequence, Circulant matrix, Hurwitz matrix, Representation.
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1 Introduction

The k-step Fibonacci sequence $\left\{F_n^k\right\}$ is defined by initial values $F_0^k = F_1^k = F_{k-2}^k = 0$, $F_{k-1}^k = 1$ and recurrence relation

$$F_{n+k}^k = F_{n+k-1}^k + F_{n+k-2}^k + \cdots + F_n^k \text{ for } n \geq 0.$$

For detailed information about the k-step Fibonacci sequence, see [9, 21].

In [5], Deveci et al. defined the Fibonacci-circulant sequence of the second kind as shown:

$$x_1^2 = \cdots = x_4^2 = 0,\ x_5^2 = 1 \text{ and } x_n^2 = -x_{n-3}^2 + x_{n-4}^2 - x_{n-5}^2 \text{ for } n \geq 6.$$

Note that the characteristic polynomial of the Fibonacci-circulant sequence of the second kind is as follows:

$$f(x) = -x^5 + x^2 + x - 1.$$

Let an n-th degree real polynomial f be given by

$$f^2(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1}x + c_n.$$

In [8], the Hurwitz matrix $H_n = [h_{i,j}]_{n\times n}$ associated to the polynomial f was defined as shown:

$$H_n = \begin{bmatrix} c_1 & c_3 & c_5 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ c_0 & c_2 & c_4 & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ 0 & c_1 & c_3 & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ \vdots & c_0 & c_2 & \ddots & \ddots & \ddots & 0 & \vdots & \vdots \\ \vdots & 0 & c_1 & \ddots & \ddots & \ddots & c_n & \vdots & \vdots \\ \vdots & \vdots & c_0 & \ddots & \ddots & \ddots & c_{n-1} & 0 & \vdots \\ \vdots & \vdots & 0 & \cdots & \cdots & \cdots & c_{n-2} & c_n & \vdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & c_{n-3} & c_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & c_{n-4} & c_{n-2} & c_n \end{bmatrix}.$$

Consider the k-step homogeneous linear recurrence sequence $\{a_n\}$,

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [9], Kalman derived a number of closed-form formulas for the sequence $\{a_n\}$ by matrix method as follows:

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

, where

$$A = [a_{i,j}]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Number theoretic properties such as these obtained from Fibonacci numbers relevant to this paper have been studied by many authors [1, 4, 7, 11, 12, 20, 23, 27, 28]. Now we define the generalized Fibonacci-circulant-Hurwitz numbers and then, we obtain their miscellaneous properties using the generating matrix and the generating function of these numbers.

2 Significance

As it is well-known that recurrence sequences, circulant matrix and Hurwitz matrix appear in modern research in many fields from mathematics, physics, computer science, architecture to nature and art (see, for example, [6, 10, 13, 14, 17, 18, 19, 22, 24, 25, 26]). This paper is expanded the concept to the generalized Fibonacci-circulant-Hurwitz sequence which is defined by using circulant and Hurwitz matrices.

3 The main resutls

By the polynomial $f^2(x)$, we can write the following Hurwitz matrix:

$$M^2 = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

Using the matrix M^2, we define the Fibonacci-circulant-Hurwitz sequence of the second kind as shown:

$$a_1^2 = \cdots = a_4^2 = 0,\ a_5^2 = 1 \text{ and } a_{n+1}^2 = -a_n^2 + a_{n-1}^2 + a_{n-2}^2 + a_{n-4}^2 \text{ for } n \geq 5.$$

Now we consider a new sequence which is a generalized form of the the Fibonacci-circulant-Hurwitz sequence of the second kind and is called the generalized Fibonacci-circulant-Hurwitz sequence. The sequence is defined by integer constants $a_1^k = \cdots = a_{k-1}^k = 0$, $a_k^k = 1$ and the recurrence relation

$$a_{n+1}^k = -a_n^k + a_{n-1}^k + \cdots + a_{n-k+3}^k + a_{n-k+1}^k \tag{1}$$

for $n \geq k$, where k is a positive integer such that $k \geq 4$.

From (1), we may write the following matrix:

$$M_k = [m_{i,j}]_{k\times k} = \begin{bmatrix} -1 & 1 & 1 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \tag{2}$$

The matrix M_k is called the generalized Fibonacci-circulant-Hurwitz matrix.

Note that $\det(M_k) = (-1)^{k+1}$ for $k \geq 4$.

By induction on n, we get

$$(M_4)^n = \begin{bmatrix} a^4_{n+4} & a^4_{n+3} + a^4_{n+1} & a^4_{n+2} & a^4_{n+3} \\ a^4_{n+3} & a^4_{n+2} + a^4_{n} & a^4_{n+1} & a^4_{n+2} \\ a^4_{n+2} & a^4_{n+1} + a^4_{n-1} & a^4_{n} & a^4_{n+1} \\ a^4_{n+1} & a^4_{n} + a^4_{n-1} & a^4_{n-1} & a^4_{n} \end{bmatrix},$$

$$(M_5)^n = \begin{bmatrix} a^5_{n+5} & a^5_{n+6} + a^5_{n+5} & a^5_{n+4} + a^5_{n+2} & a^5_{n+3} & a^5_{n+4} \\ a^5_{n+4} & a^5_{n+5} + a^5_{n+4} & a^5_{n+3} + a^5_{n+1} & a^5_{n+2} & a^5_{n+3} \\ a^5_{n+3} & a^5_{n+4} + a^5_{n+3} & a^5_{n+2} + a^5_{n} & a^5_{n+1} & a^5_{n+2} \\ a^5_{n+2} & a^5_{n+3} + a^5_{n+2} & a^5_{n+1} + a^5_{n-1} & a^5_{n} & a^5_{n+1} \\ a^5_{n+1} & a^5_{n+2} + a^5_{n+1} & a^5_{n} + a^5_{n-2} & a^5_{n-1} & a^5_{n} \end{bmatrix}$$

and

$$(M_k)^n = \begin{bmatrix} a^k_{n+k} & a^k_{n+k+1} + a^k_{n+k} & & a^k_{n+k-1} + a^k_{n+k-3} & a^k_{n+k-2} & a^k_{n+k-1} \\ a^k_{n+k-1} & a^k_{n+k} + a^k_{n+k-1} & & a^k_{n+k-2} + a^k_{n+k-4} & a^k_{n+k-3} & a^k_{n+k-2} \\ a^k_{n+k-2} & a^k_{n+k-1} + a^k_{n+k-2} & (M_k)^* & a^k_{n+k-3} + a^k_{n+k-5} & a^k_{n+k-4} & a^k_{n+k-3} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ a^k_{n+1} & a^k_{n+2} + a^k_{n+1} & & a^k_{n} + a^k_{n-2} & a^k_{n-1} & a^k_{n} \end{bmatrix}_{k\times k} \tag{3}$$

for $k \geq 6$, where $(M_k)^*$ is a matrix with k row and $k-5$ column given below:

$$\begin{bmatrix} a^k_{n+k-1} + \cdots + a^k_{n+4} + a^k_{n+2} & a^k_{n+k-1} + \cdots + a^k_{n+5} + a^k_{n+3} & \cdots & a^k_{n+k-1} + a^k_{n+k-2} + a^k_{n+k-4} \\ a^k_{n+k-2} + \cdots + a^k_{n+3} + a^k_{n+1} & a^k_{n+k-2} + \cdots + a^k_{n+4} + a^k_{n+2} & \cdots & a^k_{n+k-2} + a^k_{n+k-3} + a^k_{n+k-5} \\ \vdots & \vdots & \ddots & \vdots \\ a^k_{n} + \cdots + a^k_{n-k+4} + a^k_{n-k+2} & a^k_{n} + \cdots + a^k_{n-k+5} + a^k_{n-k+3} & \cdots & a^k_{n} + a^k_{n-1} + a^k_{n-3} \end{bmatrix}.$$

Lemma 3.1. *The characteristic equation of all the generalized Fibonacci-circulant-Hurwitz numbers* $x^k + x^{k-1} - x^{k-2} - \cdots - x^2 - 1 = 0$ *does not have multiple roots for* $k \geq 4$.

Proof. Let $f(x) = x^k + x^{k-1} - x^{k-2} - \cdots - x^2 - 1$. We easily see that $f(1) \neq 1$. Consider $h(x) = (x-1) f(x)$. Since $f(1) \neq 1$, 1 is root but not a multiple root of $h(x)$. Assume that u a multiple root of $h(x)$. Then $h(u) = 0$ and $h'(u) = 0$. So we get

$$(1-k) u^4 + ku^3 + (k-7) u^2 + (4-2k) u + 2(k-1) = 0.$$

Using appropriate softwares such as Wolfram Mathematica 10.0 [29], one can see that this last equation does not have a solution which is a contradiction. This contradiction proves that the equation $f(x)$ does not have multiple roots. □

If $x_1, x_2, \ldots, x_k$ are the eigenvalues of the generalized Fibonacci-circulant-Hurwitz matrix M_k, then by Lemma 3.1, it is known that $x_1, x_2, \ldots, x_k$ are distinct. Let a $k \times k$ Vandermonde matrix V^k be given by

$$V^k = \begin{bmatrix} (x_1)^{k-1} & (x_2)^{k-1} & \cdots & (x_k)^{k-1} \\ (x_1)^{k-2} & (x_2)^{k-2} & \cdots & (x_k)^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_k \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Now assume that $W^k(i)$ is a $(p+2) \times 1$ matrix as shown:

$$W^k(i) = \begin{bmatrix} (x_1)^{n+k-i} \\ (x_2)^{n+k-i} \\ \vdots \\ (x_{p+2})^{n+k-i} \end{bmatrix}$$

and $V^k(i,j)$ is a $k \times k$ matrix derived from the Vandermonde matrix V^k by replacing the j-th column of V^k by matrix $W^k(i)$.

Now we give the Binet formulas for the generalized Fibonacci-circulant-Hurwitz numbers by the following Theorem.

Theorem 3.1. *Let k be a positive integer such that $k \geq 4$ and let $(M_k)^\alpha = \left[m_{i,j}^{(\alpha)}\right]$ for $\alpha \geq 1$, then*

$$m_{i,j}^{(\alpha)} = \frac{\det V^k(i,j)}{V^k}.$$

Proof. Since the eigenvalues of the generalized Fibonacci-circulant-Hurwitz matrix M_k are distinct, M_k is diagonalizable. Then, we may write $M_k V^k = V^k D_k$, where $D_k = diag(x_1, x_2, \ldots, x_k)$. Since $\det V^k \neq 0$, we get

$$\left(V^k\right)^{-1} M_k V^k = D_k.$$

It will thus be seen that the matrices M_k and D_k are similar. Then we can write the matrix equation $(M_k)^\alpha V^k = V^k (D_k)^\alpha$ for $\alpha \geq 1$. Since $(M_k)^\alpha = \left[m_{i,j}^{(\alpha)}\right]$, we get

$$\begin{cases} m_{i,1}^{(\alpha)} (x_1)^{k-1} + m_{i,2}^{(\alpha)} (x_1)^{k-2} + \cdots + m_{i,k}^{(\alpha)} = (x_1)^{\alpha+k-i} \\ m_{i,1}^{(\alpha)} (x_2)^{k-1} + m_{i,2}^{(\alpha)} (x_2)^{k-2} + \cdots + m_{i,k}^{(\alpha)} = (x_2)^{\alpha+k-i} \\ \vdots \\ m_{i,1}^{(\alpha)} (x_k)^{k-1} + m_{i,2}^{(\alpha)} (x_k)^{k-2} + \cdots + m_{i,k}^{(\alpha)} = (x_k)^{\alpha+k-i} \end{cases}$$

So we conclude that

$$m_{i,j}^{(\alpha)} = \frac{\det V^k(i,j)}{V^k}$$

for each $i, j = 1, 2, \ldots, k$. □

Thus by Theorem 3.1 and the matrix $(M_k)^n$, we have the following useful results.

Corollary 3.1. *Let a_n^k be the n-th element of the generalized Fibonacci-circulant-Hurwitz sequence, then*

$$a_n^k = \frac{\det V^k(k,k)}{V^k} = \frac{\det V^k(k-1,k-1)}{V^k}$$

for $k \geq 4$.

Now we consider the combinatorial representations for all the generalized Fibonacci-circulant-Hurwitz numbers.

Let a $k \times k$ companion matrix $C(c_1, c_2, \ldots, c_k)$ be given by

$$C(c_1, c_2, \ldots, c_k) = \begin{bmatrix} c_1 & c_2 & \cdots & c_k \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

For more details on the companion type matrices, see [15, 16].

Theorem 3.2 (Chen and Louck [3])**.** *The (i,j) entry $c_{i,j}^{(\alpha)}(c_1, c_2, \ldots, c_k)$ in the matrix $C^\alpha(c_1, c_2, \ldots, c_k)$ is given by the following formula:*

$$c_{i,j}^{(\alpha)}(c_1, c_2, \ldots, c_k) = \sum_{(t_1, t_2, \ldots, t_k)} \frac{t_j + t_{j+1} + \cdots + t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k} c_1^{t_1} \cdots c_k^{t_k} \tag{4}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + kt_k = \alpha - i + j$, $\binom{t_1+\cdots+t_k}{t_1,\ldots,t_k} = \frac{(t_1+\cdots+t_k)!}{t_1!\cdots t_k!}$ is a multinomial coefficient, and the coefficients in (4) are defined to be 1 if $\alpha = i - j$.

Corollary 3.2. *Let k be a positive integer such that $k \geq 4$ and let a_n^k be the n-th element of the generalized Fibonacci-circulant-Hurwitz sequence, then*

$$\begin{aligned} a_n^k &= \sum_{(t_1, t_2 \ldots, t_k)} \frac{t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k} \\ &= \sum_{(t_1, t_2 \ldots, t_{p+2})} \frac{t_{k-1} + t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k} \end{aligned}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + kt_k = n$.

Proof. In Theorem 3.2, if we choose $i = j = k$ and $i = j = k - 1$, then the proof is immediately seen from (3). □

Definition 3.1. *An* $u \times v$ *real matrix* $A = [a_{i,j}]$ *is called a contractible matrix in the* n*-th column (resp. row) if the* n*-th column (resp. row) contains exactly two non-zero entries.*

Let $x_1, x_2, \ldots, x_u$ be row vectors of the matrix A. If A is contractible in the n-th column such that $a_{\tau,n} \neq 0, a_{\sigma,n} \neq 0$ and $\tau \neq \sigma$, then the $(u-1) \times (v-1)$ matrix $A_{\tau,\sigma:n}$ obtained from A by replacing the τ-th row with $a_{\tau,n}x_\sigma + a_{\sigma,n}x_\tau$ and deleting the σ-th row. We call the n-th column the contraction in the n-th column relative to the τ-th row and the σ-th row.

In [2], it was shown that $per\,(A) = per\,(B)$ if A is a real matrix of order $u > 1$ and the matrix B is a contraction of A.

Let $u \geq k$ and let a $u \times u$ super-diagonal matrix $N_u^k = \left[n_{i,j}^k\right]$ be given by

$$n_{i,j}^p = \begin{cases} & \text{if } i = s \text{ and } j = s+1 \text{ for } 1 \leq s \leq u-1, \\ & i = s \text{ and } j = s+2 \text{ for } 1 \leq s \leq u-2, \\ & \quad\vdots \qquad\qquad\qquad \vdots \\ 1 & i = s \text{ and } j = s+k-3 \text{ for } 1 \leq s \leq u-k+3, \\ & i = s \text{ and } j = s+k-1 \text{ for } 1 \leq s \leq u-k+1 \\ & \text{and} \\ & i = s+1 \text{ and } j = s \text{ for } 1 \leq s \leq u-1, \\ -1 & \text{if } i = s \text{ and } j = s \text{ for } 1 \leq s \leq u, \\ 0 & \text{otherwise,} \end{cases}$$

where $k \geq 4$.

Now we give the permanental representations for the generalized Fibonacci-circulant-Hurwitz numbers by the following Theorems.

Theorem 3.3. *Let* a_n *be the* n*-th element of the generalized Fibonacci-circulant-Hurwitz sequence, then*

$$per\left(N_u^k\right) = a_{u+k}^k$$

for $u \geq k$.

Proof. The assertion may be proved by induction on u. Assume that the result hold for any integer grater than or equal to k. Then we show the equation holds for $u+1$. Expanding the $per\left(N_u^k\right)$ by the Laplace expansion of permanent according to the first row gives us

$$per\left(N_{u+1}^k\right) = -per\left(N_u^k\right) + per\left(N_{u-1}^k\right) + \cdots + per\left(N_{u-k+3}^k\right) + per\left(N_{u-k+1}^k\right).$$

Since

$$per\left(N_u^k\right) = a_{u+k}^k,\ per\left(N_{u-1}^k\right) = a_{u+k-1}^k, \ldots, per\left(N_{u-k+3}^k\right) = a_{u+3}^k,\ per\left(N_{u-k+1}^k\right) = a_{u+1}^k,$$

by using the recurrence relation of the generalized Fibonacci circulant-Hurwitz numbers, we obtain $per\left(N_{u+1}^k\right) = a_{u+k+1}^k$. □

Suppose that $u > k$ and the $u \times u$ matrices $H_u^k = \left[h_{i,j}^k\right]$ and $T_u^k = \left[t_{i,j}^k\right]$ are defined by

$$h_{i,j}^k = \begin{cases} & \text{if } i = s \text{ and } j = s+\rho \text{ for } 1 \le s \le u-k+2, \\ & \text{and } 1 \le \rho \le k-3, \\ 1 & i = s \text{ and } j = s+k-1 \text{ for } 1 \le s \le u-k+1 \\ & \text{and} \\ & i = s+1 \text{ and } j = s \text{ for } 1 \le s \le u-1, \\ -1 & \text{if } i = s \text{ and } j = s \text{ for } 1 \le s \le u-k+1, \\ 0 & \text{otherwise} \end{cases} .$$

and

$$T_u^k = \begin{bmatrix} 1 & \cdots & \overset{\overset{(u-k)\text{-th}}{\downarrow}}{1} & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & H_{u-1}^k & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix},$$

$k \ge 4$.

Using the matrices $H_u^k = \left[h_{i,j}^k\right]$ and $T_u^k = \left[t_{i,j}^k\right]$ and the above results we can give more general permanental representations.

Theorem 3.4. *For $u > k$,*

$$per\left(H_u^k\right) = a_u^k \ ,$$

and

$$per\left(T_u^k\right) = \sum_{\tau=0}^{u-1} a_\tau^k \ .$$

Proof. Consider the first part of the theorem. We prove this by the induction method. Suppose that the equation holds for $u > k$, then we show that the equation holds for $u+1$. If we expand the $per\left(H_u^k\right)$ by the Laplace expansion of permanent according to the first row, then we get

$$\begin{aligned} per\left(H_{u+1}^k\right) &= -per\left(H_u^k\right) + per\left(H_{u-1}^k\right) + \cdots + per\left(H_{u-k+3}^k\right) + per\left(H_{u-k+1}^k\right) \\ &= -a_u^k + a_{u-1}^k + \cdots + a_{u-k+3}^k + a_{u-k+1}^k \\ &= a_{u+1}^k. \end{aligned}$$

Prove the second part of the theorem: Expanding the $per\left(T_u^k\right)$ with respect to the first row, we can write

$$per\left(T_u^k\right) = per\left(T_{u-1}^k\right) + per\left(H_{u-1}^p\right).$$

Thus, by the results and an inductive argument, the proof is easily seen. □

Using the definition of the generalized Fibonacci-circulant-Hurwitz numbers we find the generating function $g(x)$ as shown

$$g(x)=\frac{x^k}{1+x-x^2-\cdots-x^{k-2}-x^k}$$

where $k\geq 4$.

Now we investigate an exponential representation for the generalized Fibonacci-circulant-Hurwitz numbers.

Theorem 3.5. *For $k \geq 4$, the generalized Fibonacci-circulant-Hurwitz numbers have the following exponential representation:*

$$g(x)=x^k\exp\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\left(-1+x+\cdots+x^{k-3}+x^{k-1}\right)^n\right).$$

Proof. We consider the generating function $g(x)=\frac{x^k}{1+x-x^2-\cdots-x^{k-2}-x^k}$. Since

$$\ln g(x)=\ln\left(\frac{x^k}{1+x-x^2-\cdots-x^{k-2}-x^k}\right),$$

$$\ln g(x)=\ln x^k-\ln\left(1+x-x^2-\cdots-x^{k-2}-x^k\right)$$

and

$$\begin{aligned}\ln\left(1+x-x^2-\cdots-x^{k-2}-x^k\right) &= -[x\left(-1+x+x^2+\cdots+x^{k-3}+x^{k-1}\right)\\ &\quad+\frac{1}{2}x^{2\left(-1+x+x^2+\cdots+x^{k-3}+x^{k-1}\right)2}+\cdots\\ &\quad+\frac{1}{i}x^i\left(-1+x+x^2+\cdots+x^{k-3}+x^{k-1}\right)^i+\cdots],\end{aligned}$$

it is clear that

$$\ln\frac{g(x)}{x^k}=\sum_{n=1}^{\infty}\frac{x^n}{n}\left(-1+x+\cdots+x^{k-3}+x^{k-1}\right)^n. \qquad \square$$

Now we consider the sums of all the generalized Fibonacci-circulant-Hurwitz numbers. Let the $k\times k$ matrix M_k be as in (2) and let the sums of the generalized Fibonacci-circulant-Hurwitz numbers from 1 to n, $(n>1)$ be denoted by S_n, that is,

$$S_n=\sum_{i=1}^{n}a_i^k.$$

If we define the $(k+1)\times(k+1)$ matrix Z_k as in the following form:

$$Z_k=\begin{bmatrix}1 & 0 & \cdots & 0\\ 1 & & & \\ 0 & & M_k & \\ \vdots & & & \\ 0 & & & \end{bmatrix},$$

then by using induction on n, we may write

$$(Z_k)^n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ S_{n+k-1} & & & \\ S_{n+k-1} & & (M_k)^n & \\ \vdots & & & \\ S_n & & & \end{bmatrix}.$$

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