Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 26, 2020, No. 1, 59–69
DOI: 10.7546/nntdm.2020.26.1.59-69

Classical pairs in Z_n

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Received: 14 July 2019 **Revised:** 26 December 2019 **Accepted:** 8 January 2020

Abstract: The interplay between algebraic structures and their elements have been the most famous and productive area of the algebraic theory of numbers. Generally, the greatest common divisor and least common multiple of any two positive integers are dependably non-zero elements. In this paper, we introduce a new pair of elements, called classical pair in the ring Z_n whose least common multiple is zero and concentrate the properties of these pairs. We establish a formula for determining the number of classical pairs in Z_n for various values of n. Further, we present an algorithm for determining all these pairs in Z_n.
Keywords: Greatest common divisor, Least common multiple, Euler-totient function, Classical pairs.
2010 Mathematics Subject Classification: 97K20, 97F60, 11A07.

1 Introduction

Common divisors and multiples of numbers are two focal classes of positive integers which have appreciated incredible regard in the hypothesis of numbers. For any two positive integers a and b, the greatest common divisor and least common multiple of a and b are denoted by (a,b) and $[a,b]$, respectively. But a and b are relatively prime if and only if $(a,b)=1$, and these relatively prime integers assume a huge job in the investigation of the theory of numbers and their outcomes. When dealing with a positive integer, it is clearly helpful to know its prime factorization. Spreading something into its smaller parts allows further insight into how each part and contributes to the behaviour of the whole numbers. We accomplish this decomposition with the help of the following result, founded in [1].

Theorem 1.1. Let a and b be any two positive integers. Then $(a,b)[a,b]=ab$.

Given a positive integer $n>1$, the set $\Phi(n)=\{k\in N: 0<k<n \text{ and } (k,n)=1\}$ represents the numbers which are relatively prime to n and the number of elements in $\Phi(n)$ is $|\Phi(n)|$ which is defined as $\varphi(n)$, the Euler-totient function [2]. The function $\varphi(n)$ satisfies the following properties.

1. $\varphi(n)=n-1$ if and only if n is prime.
2. If $n=p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r}$, then $\varphi(n)=n\prod_{p_i|n}\left(1-\frac{1}{p_i}\right)$.
3. $\sum_{d|n}\varphi(d)=n$.
4. $\varphi(mn)=\varphi(m)\varphi(n)$ if and only if $(m,n)=1$.

In this paper, we are working the elements in the finite ring of integers modulo n which will be represented by Z_n. Now we are going to represent more on definitions and terminologies associated with Z_n and in particular the finite sets of units and zero divisors are also defined. First, we can generally define a ring R. Let R be a non-empty set. Then the algebraic structure $(R, +, \cdot)$ is said to form a ring R as an abelian group with respect to addition $(+)$ together with multiplication $(\cdot)$ such that $(R, \cdot)$ is semigroup and satisfies distributive laws $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$ for all a and b in R, see [3] for more details of a ring R. If $ab=ba$ for all $a,b\in R$, then R is said to be commutative, and similarly, if $ab=0$ for all $a,b\in R$, then R is called a zero ring [5], it is denoted by R^0. For any finite commutative ring R with unity, we have R is exactly a union of three disjoint non-empty subsets. So one method is to take as simply as a subset $U(R)$ of R that consists only of multiplicative inverse elements called units, that is, $a\in U(R)$ implies that there exists $b\in U(R)$ such that $ab=1=ba$. Other than $U(R)$, there is another non-empty subset $Z(R)$ in R which does not contain zero elements '0', that is, $a\in Z(R)$ means that there exists $b\in Z(R)$ such that $ab=ba=0$. These two concepts show that $R=U(R)\bigcup\{0\}\bigcup Z(R)$ if and only if R a finite commutative ring with unity is.

Now we turn our attention to the elements in the finite commutative ring Z_n with unity 1, where $Z_n=\{0,1, 2,...,n-1\}$. It is important that $Z_n=U(Z_n)\bigcup\{0\}\bigcup Z(Z_n)$. In [4], Shan and Wang defined mutual multiplies in Z_n and establish a formula for enumerating the number of unordered mutual multiple pairs in Z_n for all positive integers $n>1$. By this motivation, we define and count the set of all *classical pairs* of elements in the ring Z_n of integers modulo n.

We conclude this section by stating two identities of $U(Z_n)$ and $Z(Z_n)$ which significantly helped in finding and enumerating the set of classical pairs of elements in Z_n. For any positive integer $n>1$, $|U(Z_n)|=\varphi(n)$ and $|Z(Z_n)|=n-\varphi(n)-1$.

2 Properties of Classical pairs in Z_n

In this section, we define classical pairs of elements which are in Z_n. We also show a connection between the classical pairs of elements in the sets of units and non-zero zero-divisors of Z_n, when $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$ with $\alpha_i \geq 1, \forall 1 \leq i \leq r$.

Definition 2.1. Let $n > 1$ be a positive integer and let Z_n be the commutative ring of integers modulo *n*. Then two distinct non-zero elements a and b of Z_n are said to form a **classical pair** if $[a,b] \equiv 0 (\bmod n)$, where $'0'$ is additive identity in Z_n. The classical pair in Z_n is denoted by $\{a,b\}$ which is a **2-element** subset in Z_n and the set of all classical pairs in Z_n is denoted by $\zeta_n = \{\{a, b\} : [a, b] \equiv 0 \bmod n\}$ with cardinality $|\zeta_n|$.

Note that the additive identity $'0'$ cannot form a classical pair with any (non-zero) element in Z_n. Also, if $n = p^\alpha$, $\alpha \geq 1$ is a power of a prime, then clearly any two non-zero elements of Z_n does not form a classical pair, and thus $|\zeta_{p^\alpha}| = 0$.

Following is a more substantial example for existing established classical pairs in Z_n.

Example 2.2. In the ring Z_6, the numbers 2 and 3 form a classical pair, since $[2,3] = 6 \equiv 0 (\bmod 6)$. Similarly, 3 and 4 forms another classical pair, since $[3,4] \equiv 0 (\bmod 6)$. On the other hand, 2 and 4 do not form a classical pair, since $[2,4] = 8 \not\equiv 0 (\bmod 6)$. Hence $|\zeta_6| = 2$.

Here, clearly observe that when two elements *a* and *b* which are in Z_n does not form a classical pair $\{a, b\}$ if either *a* divides *b*, or, *b* divides *a*. Another way, if $[a, b] \equiv 0 (\bmod n)$, then *a* does not divides *b* and *b* does not divides *a*. But the converse of this observation may not be true. For instance, 2 does not divide 3 in Z_{12} and $[2, 3] = 6 \equiv 0 (\bmod 12)$.

Lemma 2.3. If u and v are two distinct units of the ring Z_n, then $\{u,v\}$ is not a classical pair of Z_n.

Proof. Suppose $\{u,v\}$ is a classical pair in Z_n. Then, by Definition 2.1, $[u,v] \equiv 0 (\bmod n)$. Consequently, *n* divides $[u,v]$. There exists $q \in Z_n$ such that $[u,v] = nq$. In view of Theorem 1.1,

$$nq(u,v) = uv \Rightarrow (u,v) = \frac{uv}{nq} \Rightarrow \left(\frac{u}{uv/nq}, \frac{v}{uv/nq} \right) = 1 \Rightarrow \left(\frac{nq}{v}, \frac{nq}{u} \right) = 1.$$

This means that, $\frac{nq}{v}$ and $\frac{nq}{u}$ are relative prime. Therefore, *u* and *v* are divisors of *nq*. It is clear that *u* and *v* are not units of Z_n. So, our assumption is not true, and hence $[u,v] \not\equiv 0 (\bmod n)$. □

By Lemma 2.3, we conclude that the elements of $U(Z_n)$ does not form a classical pair. So, our required classical pair exists in $Z(Z_n)$ only. For general positive integers a and b in Z_n, we have $(a, b) \neq 0$ and $[a, b] \neq 0$. But for some pairs of a and b in Z_n the condition $[a, b] \equiv 0 (\bmod n)$ may be satisfied, while the condition $(a, b) \not\equiv 0 (\bmod n)$ is not satisfied.

Lemma 2.4. If a and b are any two non-zero elements of Z_n, then $(a, b) \not\equiv 0 (\bmod n)$.

Proof. Let $a < n$ and $b < n$, since $a, b \in Z_n$. Then $(a, b) < n$ and hence $(a, b) \not\equiv 0 (\bmod n)$. □

Recall that a ring R^0 is called a zero ring if $ab = 0$ for all $a, b \in R^0$ and 0 is additive identity in R^0. In [5], the author Buck introduced zero rings and studied its basic properties. However, Z_n^0 is a zero ring if and only if $n = p^2$. Now we prove that each non-zero pair of elements in Z_n^0 form a classical pair.

Lemma 2.5. Every pair of non-zero elements in Z_n^0 form a classical pair.

Proof. For each pair a and b of non-zero elements in Z_n^0, by Lemma 2.3 and Lemma 2.4, $ab = 0 \Leftrightarrow ab \equiv 0 (\bmod n) \Leftrightarrow [a,b](a,b) \equiv 0 (\bmod n) \Leftrightarrow [a,b] \equiv 0 (\bmod n)$, since $(a, b) \not\equiv 0 (\bmod n)$ $\Leftrightarrow \{a, b\}$ is a classical pair of Z_n^0. □

Example 2.6. The set of all classical pairs in the zero-ring $Z_{25}^0 = \{0, 5, 10, 15, 20\}$ is $\{\{5,10\}, \{5,15\}, \{5,20\}, \{10,15\}, \{10,20\}, \{15,20\}\}$.

3 Enumeration of classical pairs in Z_n

In this section, we determine and enumerate all classical pairs which are 2-element subsets of the ring Z_n and the zero-ring Z_n^0. First, we think the set ζ_n of all classical pairs of Z_n is $\zeta_n = \{\{a, b\} : [a,b] \equiv 0 (\bmod n)\}$ and its cardinality $|\zeta_n|$. By the previous section, $\zeta_{p^\alpha} = 0$ for every prime power p^α, $\alpha \geq 1$. But $\zeta_{pq} \neq 0$ for two distinct primes p and q. We generalize the enumeration process of classical pairs in Z_n and obtain a formula for enumerating the number of classical pairs in Z_n when $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, $r > 1$.

Recently, the authors Sajana and Bharathi explored many results in [6]. The set $Z(Z_n)$ of all non-zero zero divisors of Z_n can be written as the disjoint union of the sets S_d's for all d in D, where $S_d = \{x \in Z_n : (x) = (d)\}$ and the set D denotes the set of all non-trivial proper divisors of the positive integer n. They obtained the result $|S_d| = \varphi\left(\frac{n}{d}\right), \forall\, d \in D$.

The set D can be written as the disjoint union of the sets D_1 and D_2, where $D_1 = \{d_1 \in D : [d_1, d] \not\equiv 0 (\bmod n), \forall\, d \neq d_1 \in D\}$ and $D_2 = \{d_2 \in D : [d_2, d] \equiv 0 (\bmod n),$ for some $d \neq d_2 \in D\}$, the set $Z(Z_n)$ can be written as the disjoint union of the sets $I_1(D_1)$ and $I_2(D_2)$, where $I_1(D_1) = \{x \in Z_n : (x) = (d), d \in D_1\}$ and $I_2(D_2) = \{x \in Z_n : (x) = (d), d \in D_2\}$. Similarly in D_1, every element in $I_1(D_1)$ having the least common multiple incongruent to zero modulo

n with every other element in Z_n. This implies that the elements in the classical pairs are the elements in the set $I_2(D_2)$.

First, we determine a formula for counting the number of classical pairs in the zero-ring Z_n^0. Define $Z_n^0 = \{a \in Z_n : ab \equiv 0(\bmod n) \text{ for all } b \in Z_n\}$. Therefore, $Z_n^0 = \{0\}$ if and only if $n \neq p^2$ and $Z_{p^2}^0 = \{0, p, 2p, 3p, ..., p(p-1)\}$.

Theorem 3.1. The number of classical pairs in the zero-ring Z_n^0 is $\binom{p-1}{2}$, where $n = p^2$.

Proof. Without loss of generality, we have the non-trivial zero-ring Z_n^0 is isomorphic $Z_{p^2}^0$ and $\left|Z_{p^2}^0\right| = p$. In view of the Lemma 2.5, every pair of non-zero elements in $Z_{p^2}^0$ form a classical pair, and the total number of non-zero elements in $Z_{p^2}^0$ is $p-1$. Since, $ab \equiv 0(\bmod n)$ if and only if $[a,b] \equiv 0(\bmod n)$. It follows that each pair $\{a,b\}$ in $Z_{p^2}^0$ satisfies the condition $[a,b] \equiv 0(\bmod n)$. Hence the number of classical pairs in $Z_{p^2}^0$ is $\binom{p-1}{2} = \frac{(p-1)(p-2)}{2}$. □

The following Lemma gives the cardinality of the set of all classical pairs of the ring Z_n for p^α, $\alpha \geq 1$.

Lemma 3.2. The cardinality of ζ_{p^α}, the set of all classical pairs of the ring Z_{p^α}, $\alpha \geq 1$ is $\left|\zeta_{p^\alpha}\right| = 0$.

Proof. We have $Z_n = U(Z_n) \bigcup \{0\} \bigcup Z(Z_n)$ and from Lemma 2.3 no pair of elements in $U(Z_n)$ form a classical pair. Let $n = p^\alpha$, then we have $Z(Z_n) = \phi$, if $\alpha = 1$ and $Z(Z_n) = I_1(D_1)$, if $\alpha > 1$, see [6]. By the definition of $I_1(D_1)$, no pair of elements form a classical pair. Therefore, $|\zeta_n| = 0$. □

Next, we generalize the formula for enumerating the number of classical pairs in Z_n, when $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$, $r > 1$. Note that,

$$\left|\bigcup_{0 \leq \beta_2 < \alpha_2} S_{p_1^{\alpha_1} p_2^{\beta_2}}\right| = \left|S_{p_1^{\alpha_1}}\right| + \left|S_{p_1^{\alpha_1} p_2}\right| + ... + \left|S_{p_1^{\alpha_1} p_2^{\alpha_2 - 1}}\right|$$

$$= \varphi\left(\frac{n}{p_1^{\alpha_1}}\right) + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2}\right) + ... + \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2 - 1}}\right)$$

$$= \sum_{0 \leq \beta_2 < \alpha_2} \varphi\left(\frac{n}{p_1^{\alpha_1} p_2^{\beta_2}}\right), \text{ since } |S_d| = \varphi\left(\frac{n}{d}\right), \forall d \in D.$$ □

Theorem 3.3. If $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$, $r > 1$ and $\alpha_i \geq 1$ for all $1 \leq i \leq r$, then the cardinality of the Z_n set of all classical pairs in the ring is

$$|\zeta_n| = (2^{r-1} - 1)(p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1)...(p_r^{\alpha_r} - 1) + (2^{r-2} - 1)\sum (p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1)...(p_{m-1}^{\alpha_{m-1}} - 1) +$$
$$... + (2^{r-(r-1)} - 1)\sum (p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1).$$

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$, $r > 1$, then the set $I_2(D_2)$ can be written as the disjoint union of the following sets:

$$\bigcup_{\substack{0\le\beta_i<\alpha_i,\\ i=2,3,\dots,r}} S_{p_1^{\alpha_1} p_2^{\beta_2} p_3^{\beta_3} ... p_r^{\beta_r}} = \{x \in Z_n : (x) = (p_1^{\alpha_1} p_2^{\beta_2} p_3^{\beta_3} ... p_r^{\beta_r}), 0 \le \beta_i < \alpha_i, i = 2,3,...,r\}$$

$$\bigcup_{\substack{0\le\beta_i<\alpha_i,\\ i=1,3,\dots,r}} S_{p_1^{\beta_1} p_2^{\alpha_2} p_3^{\beta_3} ... p_r^{\beta_r}} = \{x \in Z_n : (x) = (p_1^{\beta_1} p_2^{\alpha_2} p_3^{\beta_3} ... p_r^{\beta_r}), 0 \le \beta_i < \alpha_i, i = 1,3,...,r\} ,...,$$

$$\bigcup_{\substack{0\le\beta_i<\alpha_i,\\ i=1,2,\dots,r-1}} S_{p_1^{\beta_1} p_2^{\beta_2} ... p_{r-1}^{\beta_{r-1}} p_r^{\alpha_r}} = \{x \in Z_n : (x) = (p_1^{\beta_1} p_2^{\beta_2} ... p_{r-1}^{\beta_{r-1}} p_r^{\alpha_r}), 0 \le \beta_i < \alpha_i, i = 1,2,...,r-1\},$$

$$\bigcup_{\substack{0\le\beta_i<\alpha_i,\\ i=3,4,\dots,r}} S_{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\beta_3} p_4^{\beta_4} ... p_r^{\beta_r}} = \{x \in Z_n : (x) = (p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\beta_3} p_4^{\beta_4} ... p_r^{\beta_r}), 0 \le \beta_i < \alpha_i, i = 3,4,...,r\} ,$$

$$\bigcup_{\substack{0\le\beta_i<\alpha_i,\\ i=2,4,\dots,r}} S_{p_1^{\alpha_1} p_2^{\beta_2} p_3^{\alpha_3} p_4^{\beta_4} ... p_r^{\beta_r}} = \{x \in Z_n : (x) = (p_1^{\alpha_1} p_2^{\beta_2} p_3^{\alpha_3} p_4^{\beta_4} ... p_r^{\beta_r}), 0 \le \beta_i < \alpha_i, i = 2,4,...,r\},...,$$

$$\bigcup_{\substack{0\le\beta_i<\alpha_i,\\ i=1,2,\dots,r-2}} S_{p_1^{\beta_1} p_2^{\beta_2} ... p_{m-2}^{\beta_{m-2}} p_{m-1}^{\alpha_{m-1}} p_r^{\alpha_r}} = \{x \in Z_n : (x) = (p_1^{\beta_1} p_2^{\beta_2} ... p_{m-2}^{\beta_{m-2}} p_{m-1}^{\alpha_{m-1}} p_r^{\alpha_r}), 0 \le \beta_i < \alpha_i,$$

$$i = 1,2,...,r-2\} ,...,$$

$$\bigcup_{0\le\beta_r<\alpha_r} S_{p_1^{\alpha_1} p_2^{\alpha_2} ... p_{r-1}^{\alpha_{r-1}} p_r^{\beta_r}} = \{x \in Z_n : (x) = (p_1^{\alpha_1} p_2^{\alpha_2} ... p_{r-1}^{\alpha_{r-1}} p_r^{\beta_r}), 0 \le \beta_r < \alpha_r\},$$

$$\bigcup_{0\le\beta_{r-1}<\alpha_{r-1}} S_{p_1^{\alpha_1} p_2^{\alpha_2} ... p_{r-2}^{\alpha_{r-2}} p_{r-1}^{\beta_{r-1}} p_r^{\alpha_r}} = \{x \in Z_n : (x) = (p_1^{\alpha_1} p_2^{\alpha_2} ... p_{r-2}^{\alpha_{r-2}} p_{r-1}^{\beta_{r-1}} p_r^{\alpha_r}), 0 \le \beta_{r-1} < \alpha_{r-1}\},...,$$

$$\bigcup_{0\le\beta_1<\alpha_1} S_{p_1^{\beta_1} p_2^{\alpha_2} p_3^{\alpha_3} ... p_r^{\alpha_r}} = \{x \in Z_n : (x) = (p_1^{\beta_1} p_2^{\alpha_2} p_3^{\alpha_3} ... p_r^{\alpha_r}), 0 \le \beta_1 < \alpha_1\}.$$

The cardinality of the set of all classical pairs in the ring Z_n is

$$|\zeta_n| = \frac{1}{2}\Bigg[\sum\Bigg|\bigcup_{\substack{0\le\beta_i<\alpha_i,\\ i=2,3,\dots,r}} S_{p_1^{\alpha_1} p_2^{\beta_2} p_3^{\beta_3} ... p_r^{\beta_r}}\Bigg|\Bigg|\bigcup_{0\le\beta_1<\alpha_1} S_{p_1^{\beta_1} p_2^{\alpha_2} p_3^{\alpha_3} ... p_r^{\alpha_r}}\Bigg| + \sum\Bigg|\bigcup_{\substack{0\le\beta_i<\alpha_i,\\ i=3,4,\dots,r}} S_{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\beta_3} p_4^{\beta_4} ... p_r^{\beta_r}}\Bigg|$$

$$\Bigg[\Bigg|\bigcup_{\substack{0\le\beta_i<\alpha_i,\\ i=1,2}} S_{p_1^{\beta_1} p_2^{\beta_2} p_3^{\alpha_3} p_4^{\alpha_4} ... p_r^{\alpha_r}}\Bigg| + \Bigg|\bigcup_{0\le\beta_1<\alpha_1} S_{p_1^{\beta_1} p_2^{\alpha_2} p_3^{\alpha_3} ... p_r^{\alpha_r}}\Bigg| + \Bigg|\bigcup_{0\le\beta_2<\alpha_2} S_{p_1^{\alpha_1} p_2^{\beta_2} p_3^{\alpha_3} ... p_r^{\alpha_r}}\Bigg|\Bigg] + ... +$$

$$\sum\Bigg|\bigcup_{0\le\beta_r<\alpha_r} S_{p_1^{\alpha_1} p_2^{\alpha_2} ... p_{r-1}^{\alpha_{r-1}} p_r^{\beta_r}}\Bigg|\Bigg[\Bigg|\bigcup_{\substack{0\le\beta_i<\alpha_i,\\ i=1,2,\dots,r-1}} S_{p_1^{\beta_1} p_2^{\beta_2} ... p_{r-1}^{\beta_{r-1}} p_r^{\alpha_r}}\Bigg| +$$

$$\Bigg|\bigcup_{\substack{0\le\beta_i<\alpha_i,\\ i=1,2,\dots,r-2}} S_{p_1^{\beta_1} p_2^{\beta_2} ... p_{r-2}^{\beta_{r-2}} p_{r-1}^{\alpha_{r-1}} p_r^{\alpha_r}}\Bigg| + ... + \Bigg|\bigcup_{0\le\beta_{r-1}<\alpha_{r-1}} S_{p_1^{\alpha_1} p_2^{\alpha_2} ... p_{r-2}^{\alpha_{r-2}} p_{r-1}^{\beta_{r-1}} p_r^{\alpha_r}}\Bigg|\Bigg]\Bigg]$$

$$=\frac{1}{2}\Bigg[\binom{r}{1}\sum_{\substack{0\le\beta_i<\alpha_i,\\ i=2,3,\dots,r}}\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\beta_2}p_3^{\beta_3}\dots p_r^{\beta_r}}\right)\sum_{0\le\beta_1<\alpha_1}\varphi\left(\frac{n}{p_1^{\beta_1}p_2^{\alpha_2}p_3^{\alpha_3}\dots p_r^{\alpha_r}}\right)+$$

$$\binom{r}{2}\sum_{\substack{0\le\beta_i<\alpha_i,\\ i=3,4,\dots,r}}\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\beta_3}p_4^{\beta_4}\dots p_r^{\beta_r}}\right)\sum_{\substack{0\le\beta_i<\alpha_i,\\ i=1,2}}\varphi\left(\frac{n}{p_1^{\beta_1}p_2^{\beta_2}p_3^{\alpha_3}p_4^{\alpha_4}\dots p_r^{\alpha_r}}\right)+$$

$$\binom{r-1}{1}\sum\sum_{\substack{0\le\beta_i<\alpha_i,\\ i=3,4,\dots,r}}\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\beta_3}p_4^{\beta_4}\dots p_r^{\beta_r}}\right)\sum_{0\le\beta_1<\alpha_1}\varphi\left(\frac{n}{p_1^{\beta_1}p_2^{\alpha_2}p_3^{\alpha_3}\dots p_r^{\alpha_r}}\right)+\dots+$$

$$\binom{r}{r-1}\sum_{0\le\beta_r<\alpha_r}\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\dots p_{r-1}^{\alpha_{r-1}}p_r^{\beta_r}}\right)\sum_{\substack{0\le\beta_i<\alpha_i,\\ i=1,2,\dots,r-1}}\varphi\left(\frac{n}{p_1^{\beta_1}p_2^{\beta_2}\dots p_{r-1}^{\beta_{r-1}}p_r^{\alpha_r}}\right)+$$

$$\binom{r-1}{r-2}\sum\sum_{0\le\beta_r<\alpha_r}\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\dots p_{r-1}^{\alpha_{r-1}}p_r^{\beta_r}}\right)\sum_{\substack{0\le\beta_i<\alpha_i,\\ i=1,2,\dots,r-2}}\varphi\left(\frac{n}{p_1^{\beta_1}p_2^{\beta_2}\dots p_{r-2}^{\beta_{r-2}}p_{r-1}^{\alpha_{r-1}}p_r^{\alpha_r}}\right)$$

$$+\dots+$$

$$\binom{r-(r-2)}{1}\sum\sum_{0\le\beta_r<\alpha_r}\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\dots p_{r-1}^{\alpha_{r-1}}p_r^{\beta_r}}\right)\sum_{0\le\beta_{r-1}<\alpha_{r-1}}\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\dots p_{r-2}^{\alpha_{r-2}}p_{r-1}^{\beta_{r-1}}p_r^{\alpha_r}}\right)\Bigg]$$

$$=\frac{1}{2}\Bigg[\left(\binom{r}{1}+\binom{r}{2}+\dots+\binom{r}{r-1}\right)\sum_{\substack{0\le\beta_i<\alpha_i,\\ i=1,2,\dots,r}}\varphi\left(p_1^{\alpha_1-\beta_1}p_2^{\alpha_2-\beta_2}\dots p_r^{\alpha_r-\beta_r}\right)+$$

$$\left(\binom{r-1}{1}+\binom{r-1}{2}+\dots+\binom{r-1}{r-2}\right)\sum\sum_{\substack{0\le\beta_i<\alpha_i,\\ i=1,3,4,\dots,r}}\varphi\left(p_1^{\alpha_1-\beta_1}p_3^{\alpha_3-\beta_3}p_4^{\alpha_4-\beta_4}\dots p_r^{\alpha_r-\beta_r}\right)+$$

$$\binom{r-(r-2)}{1}\sum\sum_{\substack{0\le\beta_i<\alpha_i,\\ i=r-1,r}}\varphi\left(p_{r-1}^{\alpha_{r-1}-\beta_{r-1}}p_r^{\alpha_r-\beta_r}\right)\Bigg].$$

We have $\binom{r}{1}+\binom{r}{2}+\dots+\binom{r}{r-1}=2^r-2$ and simplifying the above, we obtain

$$|\zeta_n|=(2^{r-1}-1)(p_1^{\alpha_1}-1)(p_2^{\alpha_2}-1)\dots(p_r^{\alpha_r}-1)+(2^{r-2}-1)\sum(p_1^{\alpha_1}-1)(p_2^{\alpha_2}-1)\dots(p_{r-1}^{\alpha_{r-1}}-1)+$$
$$(2^{r-(r-1)}-1)\sum(p_1^{\alpha_1}-1)(p_2^{\alpha_2}-1).$$ □

Corollary 3.4. If $n=p_1^{\alpha_1}p_2^{\alpha_2}$ and $\alpha_i\ge 1$ for all $1\le i\le 2$, then the cardinality of the set of all classical pairs in the ring Z_n is $|\zeta_n|=(p_1^{\alpha_1}-1)(p_2^{\alpha_2}-1)$.

Proof. For $n=p_1^{\alpha_1}p_2^{\alpha_2}$, the set $I_2(D_2)$ can be written as the disjoint union of the sets

$$\bigcup_{0\le\beta_2<\alpha_2} S_{p_1^{\alpha_1}p_2^{\beta_2}} = \{x\in Z_n : (x) = (p_1^{\alpha_1}p_2^{\beta_2}),\ 0\le\beta_2<\alpha_2\}$$

and

$$\bigcup_{0\le\beta_1<\alpha_1} S_{p_1^{\beta_1}p_2^{\alpha_2}} = \{x\in Z_n : (x) = (p_1^{\beta_1}p_2^{\alpha_2}),\ 0\le\beta_1<\alpha_1\}.$$

Now the cardinality of the set of all classical pairs in the ring Z_n is

$$|\zeta_n| = \frac{1}{2}\left[\sum\left|\bigcup_{0\le\beta_2<\alpha_2} S_{p_1^{\alpha_1}p_2^{\beta_2}}\right|\left|\bigcup_{0\le\beta_1<\alpha_1} S_{p_1^{\beta_1}p_2^{\alpha_2}}\right|\right]$$

$$= \frac{1}{2}\left[\binom{2}{1}\sum_{0\le\beta_2<\alpha_2}\varphi\left(\frac{n}{p_1^{\alpha_1}p_2^{\beta_2}}\right)\sum_{0\le\beta_1<\alpha_1}\varphi\left(\frac{n}{p_1^{\beta_1}p_2^{\alpha_2}}\right)\right]$$

$$= \frac{1}{2}\left[\binom{2}{1}\sum_{0\le\beta_2<\alpha_2}\varphi\left(p_2^{\alpha_2-\beta_2}\right)\sum_{0\le\beta_1<\alpha_1}\varphi\left(p_1^{\alpha_1-\beta_1}\right)\right]$$

$$= \frac{1}{2}\left[\binom{2}{1}\sum_{\substack{0\le\beta_i<\alpha_i,\\ 1\le i\le 2}}\varphi\left(p_1^{\alpha_1-\beta_1}p_2^{\alpha_2-\beta_2}\right)\right]$$

$$= \frac{1}{2}\left[\binom{2}{1}\Big(\varphi\left(p_1^{\alpha_1}p_2^{\alpha_2}\right)+\varphi\left(p_1^{\alpha_1}p_2^{\alpha_2-1}\right)+\ldots+\varphi\left(p_1p_2\right)+\Big(\varphi\left(p_1^{\alpha_1}\right)+\varphi\left(p_1^{\alpha_1-1}\right)+\ldots+\right.$$

$$\varphi(p_1)+\varphi\left(p_2^{\alpha_2}\right)+\varphi\left(p_2^{\alpha_2-1}\right)+\ldots+\varphi(p_2)+\varphi(1)\Big)-\Big(\varphi\left(p_1^{\alpha_1}\right)+\varphi\left(p_1^{\alpha_1-1}\right)+\ldots+$$

$$\left.\varphi(p_1)+\varphi\left(p_2^{\alpha_2}\right)+\varphi\left(p_2^{\alpha_2-1}\right)+\ldots+\varphi(p_2)+\varphi(1)\Big)\Big)\right]$$

$$= \frac{1}{2}\left[\binom{2}{1}\Big(p_1^{\alpha_1}p_2^{\alpha_2}-\Big(\varphi\left(p_1^{\alpha_1}\right)+\varphi\left(p_1^{\alpha_1-1}\right)+\ldots+\varphi(p_1)+\varphi(1)+\varphi\left(p_2^{\alpha_2}\right)+\varphi\left(p_2^{\alpha_2-1}\right)+\right.$$

$$\left.\ldots+\varphi(p_2)+\varphi(1)-\varphi(1)\Big)\Big)\right] \qquad \text{(since } \sum_{d|n}\varphi(d)=n\text{)}$$

$$= \frac{1}{2}\left[\left(2^2-2\right)\left(p_1^{\alpha_1}p_2^{\alpha_2}-(p_1^{\alpha_1}+p_2^{\alpha_2})+1\right)\right] \qquad \text{(by using } \binom{2}{1}=2^2-2,\ \varphi(1)=1\text{)}$$

$$= \left(p_1^{\alpha_1}-1\right)\left(p_2^{\alpha_2}-1\right). \qquad \square$$

Example 3.5. For the ring $Z_{10}=\{0,1,2,3,4,5,6,7,8,9\}$, $10=2.5$, the set $I_2(D_2)=S_2\cup S_5$, where $S_2=\{x\in Z_{10}:\ (x)=(2)\}=\{2,4,6,8\}$ and $S_5=\{x\in Z_{10}:\ (x)=(5)\}=\{5\}$. Clearly, every element in S_2 having the least common multiple congruent to zero modulo 10 with every element in S_5 and also these are the only classical pairs in Z_{10}. So, the set of all classical pairs in Z_{10} is $\zeta_{10}=\{\{2,5\},\{4,5\},\{5,6\},\{5,8\}\}$ with cardinality 4. Also from the above formula, we have $|\zeta_{10}|=(2-1)(5-1)=4$.

4 Algorithm

In this section, we present an algorithm for determining all the classical pairs in Z_n depends on the value of n and gave the outputs when running the program in C-language for various values of n based on the algorithm.

Algorithm 4.1.
Step 1: Start
Step 2: Initialize variables n, i, j, a, b, minMultiple, lcm, r
Step 3: Read the value of n
Step 4: i ← 2
Step 5: j ← i+1
Step 6: a ← i, b ← j, minMultiple ← (a > b) ? a : b
Step 7: While always be true
Step 8: If (minMultiple % a=0) and (minMultiple % b =0), then goto Step 9 else goto Step 14
Step 9: lcm ← minMultiple
Step 10: r ← (lcm % n)
Step 11: If (r = 0), then goto Step 12 else goto Step 13
Step 12: Print Classical pair
Step 13: Break
Step 14: Increment minMultiple
Step 15: Goto Step 8
Step 16: If (j < n), then goto Step 17 else goto Step 19
Step 17: j ← j+1
Step 18: Goto Step 6
Step 19: If (i < n), then goto Step 20 else goto Step 22
Step 20: i ← i+1
Step 21: Goto Step 5
Step 22: Stop

Outputs 4.2. The obtained outputs for various values of n are:
(i). For given number $n = 10$, then the output is
Classical pairs:{2,3},{3,4}
(ii). For given number $n = 12$, then the output is
Classical pairs:{3,4},{3,8},{4,6},{4,9},{6,8},{8,9}.

5 Conclusion

In this paper, we characterized and examined the classical pairs of a finite commutative ring. Additionally, we acquired a recipe for finding the cardinality of the arrangement of every classical pair of a Z_n for all values of n. At long last, the outcomes were confirmed with appropriate precedents by utilizing the calculation of C-program.

Acknowledgements

All the authors thank to the peer reviewers for their valuable suggestions to improve the presentation of this paper.

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Appendix

Here, we present a program in C-language for finding all the Classical pairs in Z_n for various values of n.

```
#include<stdio.h>
int main()
{
    int n, i, j, a, b, minMultiple, lcm, r;
    printf("enter n value:");
          scanf("%d", &n);
          printf("Classical Pairs:");
    for(i=2; i<n; i++)
    {
    for(j=i+1; j<n; j++)
    {
          a = i;
    b = j;
            // maximum number between a and b is stored in minMultiple
    minMultiple = (a>b) ? a : b;
        // Always true
        while(1)
        {
        if (minMultiple%a==0 && minMultiple%b==0)
        {
                //lCM of the two numbers will be stored in minMultiple
            lcm = minMultiple;
            r = (lcm%n);
      if( r == 0 )
            {
               printf("{%d,%d},",a,b);
             }
            break;
        }
        ++minMultiple;
        }
    }
    }
return 0;
}
```