Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 26, 2020, No. 1, 45–58
DOI: 10.7546/nntdm.2020.26.1.45-58

t-cobalancing numbers and t-cobalancers

Ahmet Tekcan[1] and Alper Erdem[2]

[1] Bursa Uludag University, Faculty of Science
Department of Mathematics, Bursa, Turkey
e-mail: tekcan@uludag.edu.tr

[2] Bursa Uludag University, Faculty of Science
Department of Mathematics, Bursa, Turkey
e-mail: alper.erdem@outlook.com

Received: 11 April 2019 **Revised:** 29 January 2020 **Accepted:** 30 January 2020

Abstract: In this work, we determine the general terms of t-cobalancers, t-cobalancing numbers and Lucas t-cobalancing numbers by solving the Pell equation $2x^2 - y^2 = 2t^2 - 1$ for some fixed integer $t \geq 1$.
Keywords: Cobalancing numbers, Cobalancers, t-cobalancers, t-cobalancing numbers, Lucas t-cobalancing numbers, Pell equation.
2010 Mathematics Subject Classification: 11B37, 11B39, 11D09, 11D79.

1 Introduction

A positive integer n is called a balancing number [2] if the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{1}$$

holds for some positive integer r which is called balancer corresponding to n. If n is a balancing number with balancer r, then from (1)

$$n^2 = \frac{(n+r)(n+r+1)}{2} \text{ and } r = \frac{-2n-1+\sqrt{8n^2+1}}{2}. \tag{2}$$

Hence from (2) we get that n is a balancing number if and only if n^2 is a triangular number (triangular numbers denoted by T_n are the numbers of the form $T_n = \frac{n(n+1)}{2}$ for $n \geq 1$) and

$8n^2+1$ is a perfect square. Though the definition of balancing numbers suggests that no balancing number should be less than 2. But from (2), $8(0)^2+1=1$ and $8(1)^2+1=3^2$ are perfect squares. So we accept 0 and 1 to be balancing numbers. A balancing number is denoted by B_n and hence $B_0=0, B_1=1, B_2=6$ and $B_{n+1}=6B_n-B_{n-1}$ for $n\geq 2$.

Later Panda and Ray [16] defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1+2+\cdots+n=(n+1)+(n+2)+\cdots+(n+r) \tag{3}$$

holds for some positive integer r which is called cobalancer corresponding to n. If n is a cobalancing number with cobalancer r, then from (3)

$$n(n+1)=\frac{(n+r)(n+r+1)}{2} \text{ and } r=\frac{-2n-1+\sqrt{8n^2+8n+1}}{2}. \tag{4}$$

Hence from (4) we get that n is a cobalancing number if and only if $n(n+1)$ is a triangular number and $8n^2+8n+1$ is a perfect square. Since $8(0)^2+8(0)+1=1$ is a perfect square, we accept 0 to be a cobalancing number, just like Behera and Panda accepted $0,1$ balancing numbers. A cobalancing number is denoted by b_n and $b_0=b_1=0, b_2=2$ and $b_{n+1}=6b_n-b_{n-1}+2$ for $n\geq 2$.

It is clear from (1) and (3) that every balancing number is a cobalancer and every cobalancing number is a balancer, that is, $B_n=r_{n+1}$ and $R_n=b_n$ for $n\geq 1$, where R_n is the n-th balancer and r_n is the n-th cobalancer. Since $R_n=b_n$, we get from (1) that

$$b_n=\frac{-2B_n-1+\sqrt{8B_n^2+1}}{2} \text{ and } B_n=\frac{2b_n+1+\sqrt{8b_n^2+8b_n+1}}{2}. \tag{5}$$

Thus from (5), we see that B_n is a balancing number if and only if $8B_n^2+1$ is a perfect square and b_n is a cobalancing number if and only if $8b_n^2+8b_n+1$ is a perfect square. Thus

$$C_n=\sqrt{8B_n^2+1} \text{ and } c_n=\sqrt{8b_n^2+8b_n+1} \tag{6}$$

are integers which are called the n-th Lucas-balancing number and n-th Lucas-cobalancing number, respectively.

Binet formulas for all balancing numbers are $B_n=\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}, b_n=\frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}}-\frac{1}{2}, C_n=\frac{\alpha^{2n}+\beta^{2n}}{2}$ and $c_n=\frac{\alpha^{2n-1}+\beta^{2n-1}}{2}$ for $n\geq 1$, where $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$ which are the the roots of the characteristic equation for Pell numbers P_n (see also [6–8, 14, 15, 19, 21, 24]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [11], Liptai proved that there is no Fibonacci balancing number except 1 and in [12], he proved that there is no Lucas balancing number. In [23], Szalay considered the same problem and obtained some nice results by a different method. In [9], Kovács, Liptai and Olajos extended the concept of balancing numbers to the (a,b)-balancing numbers defined as follows: Let $a>0$ and $b\geq 0$ be coprime integers. If

$$(a+b)+\cdots+(a(n-1)+b)=(a(n+1)+b)+\cdots+(a(n+r)+b)$$

for some positive integers n and r, then $an+b$ is an (a,b)-balancing number. The sequence of (a,b)-balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$. In [10], Liptai, Luca, Pintér and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let $y,k,l \in \mathbb{Z}^+$ such that $y \geq 4$. Then a positive integer x with $x \leq y-2$ is called a (k,l)-power numerical center for y if

$$1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l.$$

They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for (k,l)-power numerical centers. For positive integers k,x, let

$$\Pi_k(x) = x(x+1)\ldots(x+k-1).$$

Then it was proved in [9] that the equation $B_m = \Pi_k(x)$ for a fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2,3,4\}$ all solutions were determined. In [26], Tengely considered the case $k=5$, that is, $B_m = x(x+1)(x+2)(x+3)(x+4)$ and proved that this Diophantine equation has no solution for $m \geq 0$ and $x \in \mathbb{Z}$. In [4], Frontczak considered the sums of balancing and Lucas-balancing numbers with binomial coefficients and in [5] he considered balancing polynomials. In [18], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers. In [20], Patel, Irmak and Ray considered incomplete balancing and Lucas-balancing numbers and in [22], Ray considered the sums of balancing and Lucas-balancing numbers by matrix methods. In [17], Panda and Panda defined almost balancing numbers. A natural number n is called an almost balancing number if the Diophantine equation

$$|[(n+1)+(n+2)+\cdots+(n+r)] - [1+2+\cdots+(n-1)]| = 1$$

holds for some positive integer r which is called the almost balancer. In [25], the first author derived some new results on almost balancing numbers, triangular numbers and square triangular numbers.

Now let $t \geq 1$ be an integer. By considering (3), a positive integer n is called a t-cobalancing number if the Diophantine equation

$$1+2+\cdots+n = (n+1+t)+(n+2+t)+\cdots+(n+r+t) \tag{7}$$

holds for some positive integer r which is called t-cobalancer corresponding to n.

Let b_n^t denote the n-th t-cobalancing number and let r_n^t denote the n-th t-cobalancer. Then from (7), we get

$$r_n^t = \frac{-2b_n^t - 2t - 1 + \sqrt{8(b_n^t)^2 + 8(t+1)b_n^t + (2t+1)^2}}{2} \tag{8}$$

and

$$b_n^t = \frac{2r_n^t - 1 + \sqrt{8(r_n^t)^2 + 8tr_n^t + 1}}{2}. \tag{9}$$

Thus from (8), we notice that b_n^t is the n-th t-cobalancing number if and only if

$$8(b_n^t)^2 + 8(t+1)b_n^t + (2t+1)^2$$

is a perfect square. So

$$c_n^t = \sqrt{8(b_n^t)^2 + 8(t+1)b_n^t + (2t+1)^2} \tag{10}$$

is an integer which is called the n-th Lucas t-cobalancing number.

In order to determine the general terms of t-cobalancers, t-cobalancing numbers and Lucas t-cobalancing numbers, we have to determine the set of all (positive) integer solutions of the Pell equation ([1, 3, 13])

$$2x^2 - y^2 = 2t^2 - 1. \tag{11}$$

From (9), we see that r_n^t is a t-cobalancer if and only if $8(r_n^t)^2 + 8tr_n^t + 1$ is a perfect square. So we set

$$8(r_n^t)^2 + 8tr_n^t + 1 = y^2 \tag{12}$$

for some integer $y \geq 1$. Then $2(2r_n^t + t)^2 - y^2 = 2t^2 - 1$ and putting

$$x = 2r_n^t + t, \tag{13}$$

we get the Pell equation in (11). To get the set of all integer solutions of (11), we need some notations.

Let $F(x,y) = ax^2 + bxy + cy^2$ be an indefinite integral quadratic form [3] of discriminant $\Delta = b^2 - 4ac$ and let m be any integer. Then the Δ-order O_Δ is defined for nonsquare discriminant Δ to be the ring $O_\Delta = \{x + y\rho_\Delta : x, y \in \mathbb{Z}\}$, where $\rho_\Delta = \sqrt{\frac{\Delta}{4}}$ if $\Delta \equiv 0 \pmod 4$ or $\frac{1+\sqrt{\Delta}}{2}$ if $\Delta \equiv 1 \pmod 4$. So O_Δ is a subring of $\mathbb{Q}(\sqrt{\Delta}) = \{x + y\sqrt{\Delta} : x, y \in \mathbb{Q}\}$. The unit group O_Δ^u is defined to be the group of units of the ring O_Δ. We can rewrite F to be

$$F(x,y) = \frac{(xa + y\frac{b+\sqrt{\Delta}}{2})(xa + y\frac{b-\sqrt{\Delta}}{2})}{a}.$$

So the module M_F of F is

$$M_F = \{xa + y\frac{b+\sqrt{\Delta}}{2} : x, y \in \mathbb{Z}\} \subset \mathbb{Q}(\sqrt{\Delta}).$$

Therefore, we get

$$(u + v\rho_\Delta)(xa + y\frac{b+\sqrt{\Delta}}{2}) = x'a + y'\frac{b+\sqrt{\Delta}}{2},$$

where

$$[x' \quad y'] = \begin{cases} [x \quad y]\begin{bmatrix} u - \frac{b}{2}v & av \\ -cv & u + \frac{b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 0 \pmod 4, \\ \\ [x \quad y]\begin{bmatrix} u + \frac{1-b}{2}v & av \\ -cv & u + \frac{1+b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 1 \pmod 4. \end{cases} \tag{14}$$

So there is a bijection $\Psi : \{(x,y) : F(x,y) = m\} \to \{\gamma \in M_F : N(\gamma) = am\}$ for solving $F(x,y) = m$. The action of $O_{\Delta,1}^u = \{\alpha \in O_\Delta^u : N(\alpha) = 1\}$ on $\{(x,y) : F(x,y) = m\}$ of integral solutions of the equation $F(x,y) = m$ is most interesting when Δ is a positive nonsquare since $O_{\Delta,1}^u$ is infinite. Therefore, the orbit of each solution will be infinite and so the set $\{(x,y) :$

$F(x,y)=m\}$ is either empty or infinite. Since $O^u_{\Delta,1}$ can be explicitly determined, the set $\{(x,y):F(x,y)=m\}$ is satisfactorily described by the representation of such a list, called a set of representatives of the orbits. Let $\varepsilon_\Delta>1$ be the smallest unit of O_Δ and let $\tau_\Delta=\varepsilon_\Delta$ if $N(\varepsilon_\Delta)=1$ or ε^2_Δ if $N(\varepsilon_\Delta)=-1$. Then every $O^u_{\Delta,1}$ orbit of integral solutions of $F(x,y)=m$ contains a solution $(x,y)\in\mathbb{Z}\times\mathbb{Z}$ such that $0\le y\le U$, where

$$U=\begin{cases}\left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}}(1-\frac{1}{\tau_\Delta}) & \text{if } am>0\\ \left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}}(1+\frac{1}{\tau_\Delta}) & \text{if } am<0.\end{cases}$$

So for finding the a set of representatives of the $O^u_{\Delta,1}$ orbits of integral solutions of $F(x,y)=m$, we must find for each integer y_0 such that $0\le y_0\le U$, all integers x_0 that satisfy $F(x_0,y_0)=m$. If $F(x_0,y_0)=m$, then

$$ax_0^2+bx_0y_0+cy_0^2=m\Leftrightarrow\Delta y_0^2+4am=(2ax_0+by_0)^2$$

and hence

$$x_0=\frac{-by_0\pm\sqrt{\Delta y_0^2+4am}}{2a}.$$

Consequently, we get the set of representatives Rep $=\{[x_0\ \ y_0]\}$. Thus for the matrix M defined in (14), the set of all integer solutions of $F(x,y)=m$ is

$$\{\pm(x,y):[x\ \ y]=[x_0\ \ y_0]M^n,n\in\mathbb{Z}\}.$$

2 Main results

For the set of all integer solutions of (11), the indefinite form $F(x,y)=2x^2-y^2$ of discriminant $\Delta=8$. So $\tau_8=3+2\sqrt{2}$ and

$$M=\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix} \tag{15}$$

by (14). In order to determine the set of all integer solutions of (11), we have to consider our problem in three cases: $t=1$, $2t^2-1$ is a perfect square or not a perfect square for $t\ge 2$.

2.1 Case 1: $t=1$

Theorem 2.1. *If $t=1$, then*

1. *The set of all integer solutions of $2x^2-y^2=1$ is $\{(-2B_n+C_n,4B_n-C_n):n\ge 1\}$.*
2. *The general terms of 1-cobalancers, 1-cobalancing numbers and Lucas 1-cobalancing numbers are*

   $$r^1_n=b_{n+1},\ b^1_n=B_{n+1}-1 \ \text{and}\ c^1_n=C_{n+1}$$

   for $n\ge 1$.

Proof. **(1)** Let $t = 1$. Then for the Pell equation $2x^2 - y^2 = 1$, in the range

$$0 \le y_0 \le U = \left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}} (1 - \frac{1}{\tau_\Delta}) = \left|\frac{2(1)(3+2\sqrt{2})}{8}\right|^{\frac{1}{2}} \left(\frac{2+2\sqrt{2}}{3+2\sqrt{2}}\right) = 1,$$

$\Delta y_0^2 + 4am = 8y_0^2 + 8$ is a perfect square only for $y_0 = 1$ and hence

$$x_0 = \frac{-by_0 \pm \sqrt{\Delta y_0^2 + 4am}}{2a} = \frac{\pm\sqrt{8(1)^2 + 8}}{4} = \pm 1.$$

So the set of representatives is Rep $= \{[\pm 1 \quad 1]\}$ and in this case $[1 \quad -1]M^n$ (where M is defined in (15)) generates all integer solutions (x_n, y_n) for $n \ge 1$. Since the n-th power of M is

$$M^n = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix}$$

for $n \ge 1$, we conclude that the set of all integer solutions is $\{(-2B_n + C_n, 4B_n - C_n) : n \ge 1\}$.

(2) Recall that $x = 2r_n^t + t$ by (13). So from (1), we easily deduce that

$$\begin{aligned} r_n^1 &= \frac{-2B_{n+1} + C_{n+1} - 1}{2} \\ &= \frac{-2(\frac{\alpha^{2n+2}-\beta^{2n+2}}{4\sqrt{2}}) + \frac{\alpha^{2n+2}+\beta^{2n+2}}{2} - 1}{2} \\ &= \alpha^{2n+2}(-\frac{1}{4\sqrt{2}} + \frac{1}{4}) + \beta^{2n+2}(\frac{1}{4\sqrt{2}} + \frac{1}{4}) - \frac{1}{2} \\ &= \frac{\alpha^{2n+2}(\sqrt{2}-1) + \beta^{2n+2}(\sqrt{2}+1)}{4\sqrt{2}} - \frac{1}{2} \\ &= \frac{\alpha^{2n+1} - \beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2} \\ &= b_{n+1}. \end{aligned}$$

Note that $\sqrt{8(r_n^t)^2 + 8tr_n^t + 1} = y_n$ by (12). So from (9), we get

$$\begin{aligned} b_n^1 &= \frac{2r_n^1 - 1 + \sqrt{8(r_n^1)^2 + 8r_n^1 + 1}}{2} \\ &= \frac{2b_{n+1} - 1 + 4B_{n+1} - C_{n+1}}{2} \\ &= \frac{2(\frac{\alpha^{2n+1}-\beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2}) - 1 + 4(\frac{\alpha^{2n+2}-\beta^{2n+2}}{4\sqrt{2}}) - \frac{\alpha^{2n+2}+\beta^{2n+2}}{2}}{2} \\ &= \frac{\alpha^{2n+1}(\frac{1}{2\sqrt{2}} + \frac{\alpha}{\sqrt{2}} - \frac{\alpha}{2}) + \beta^{2n+1}(\frac{-1}{2\sqrt{2}} - \frac{\beta}{\sqrt{2}} - \frac{\beta}{2})}{2} - 1 \\ &= \frac{\alpha^{2n+2} - \beta^{2n+2}}{4\sqrt{2}} - 1 \\ &= B_{n+1} - 1 \end{aligned}$$

and from (10), we conclude that

$$c_n^1 = \sqrt{8(B_{n+1}-1)^2 + 16(B_{n+1}-1) + 9} = \sqrt{8B_{n+1}^2 + 1} = C_{n+1}$$

as we wanted. □

2.2 Case 2: $2t^2 - 1$ is a perfect square

In this section, we assume that $2t^2 - 1$ is a perfect square for $t \geq 2$. Before considering our problem, we can give the following theorem.

Theorem 2.2. *The quadratic equation $2t^2 - 1 = h^2$ is satisfied for $(t_n, h_n) = (P_{2n-1}, c_n)$ for $n \geq 1$.*

Proof. Let $2t^2 - 1 = h^2$ for some positive integer h. Then we get the Pell equation $2t^2 - h^2 = 1$. In this case the set of representatives is Rep $= \{[\pm 1 \quad 1]\}$ and $[1 \quad -1]M^n$ generates all integer solutions (t_n, h_n) for $n \geq 1$. Thus $2t^2 - 1 = h^2$ is satisfied for $(t_n, h_n) = (-2B_n + C_n, 4B_n - C_n)$ for $n \geq 1$. On the other hand it can be easily seen that

$$-2B_n + C_n = P_{2n-1} \text{ and } 4B_n - C_n = c_n.$$

So the quadratic equation $2t^2 - 1 = h^2$ is satisfied for $(t_n, h_n) = (P_{2n-1}, c_n)$ for $n \geq 1$. Indeed, since $P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$ and $c_n = \frac{\alpha^{2n-1} + \beta^{2n-1}}{2}$, we easily get

$$2t_n^2 - 1 = \frac{\alpha^{4n-2} + \beta^{4n-2} + 2(\alpha\beta)^{2n-1}}{4} = (\frac{\alpha^{2n-1} + \beta^{2n-1}}{2})^2 = c_n^2 = h_n^2$$

as we claimed. □

From Theorem 2.2, we see that $2t^2 - 1$ is a perfect square for $t = P_{2n-1}$. Consequently, we determine the general terms of all $P_{2n-1}-$cobalancers, $P_{2n-1}-$cobalancing numbers and Lucas $P_{2n-1}-$cobalancing numbers for $n \geq 2$ (For $n = 1$, we have $t = P_1 = 1$ and clearly we have already considered it in the previous section). In order to determine the set of all integer solutions of (11), we have two cases: #Rep $= 4$ or #Rep > 4.

Theorem 2.3. *If #Rep $= 4$, then*

1. *The set of all integer solutions of (11) is*

$$\{(x_{3n+1}, y_{3n+1}), (x_{3n+2}, y_{3n+2}) : n \geq 0\} \cup \{(x_{3n}, y_{3n}) : n \geq 1\},$$

where

$$\begin{aligned}(x_{3n+1}, y_{3n+1}) &= (2B_n + tC_n, 4tB_n + C_n)\\ (x_{3n+2}, y_{3n+2}) &= (2hB_n + hC_n, 4hB_n + hC_n)\\ (x_{3n}, y_{3n}) &= (-2B_n + tC_n, 4tB_n - C_n).\end{aligned}$$

2. *The general terms of t-cobalancers, t-cobalancing numbers and Lucas t-cobalancing numbers are*

$$\begin{aligned}r_{3n}^t &= \frac{2B_n + tC_n - t}{2}\\ r_{3n-1}^t &= \frac{-2B_n + tC_n - t}{2}\\ b_{3n}^t &= \frac{2t(B_n + b_{n+1}) + 2B_n + C_n - 1}{2}\\ b_{3n-1}^t &= \frac{2t(B_n + b_{n+1}) - 2B_n - C_n - 1}{2}\end{aligned}$$

$$c_{3n}^t = \sqrt{8(b_{3n}^t)^2 + 8(t+1)b_{3n}^t + (2t+1)^2}$$
$$c_{3n-1}^t = \sqrt{8(b_{3n-1}^t)^2 + 8(t+1)b_{3n-1}^t + (2t+1)^2}$$

for $n \geq 1$ *and*

$$r_{3n+1}^t = \frac{2hB_n + hC_n - t}{2}$$
$$b_{3n+1}^t = \frac{2hB_{n+1} - t - 1}{2}$$
$$c_{3n+1}^t = \sqrt{8(b_{3n+1}^t)^2 + 8(t+1)b_{3n+1}^t + (2t+1)^2}$$

for $n \geq 0$.

Proof. **(1)** If $\#\text{Rep} = 4$, then the set of representatives is $\text{Rep} = \{[\pm t \quad 1], [\pm h \quad h]\}$ and in this case

1. $[t \quad 1]M^n$ generates all integer solutions (x_{3n+1}, y_{3n+1}) for $n \geq 0$,
2. $[t \quad -1]M^n$ generates all integer solutions (x_{3n}, y_{3n}) for $n \geq 1$,
3. $[h \quad h]M^n$ generates all integer solutions (x_{3n+2}, y_{3n+2}) for $n \geq 0$.

Thus the the set of all integer solutions is $\{(2B_n + tC_n, 4tB_n + C_n), (2hB_n + hC_n, 4hB_n + hC_n) : n \geq 0\} \cup \{(-2B_n + tC_n, 4tB_n - C_n) : n \geq 1\}$.

(2) From (1), we easily get

$$r_{3n}^t = \frac{2B_n + tC_n - t}{2}$$

for $n \geq 1$. Thus from (9), we get

$$\begin{aligned} b_{3n}^t &= \frac{2B_n + tC_n - t - 1 + 4tB_n + C_n}{2} \\ &= \frac{t(4B_n + C_n - 1) + 2B_n + C_n - 1}{2} \\ &= \frac{2t(B_n + b_{n+1}) + 2B_n + C_n - 1}{2} \end{aligned}$$

for $n \geq 1$ since $4B_n + C_n - 1 = 2(B_n + b_{n+1})$. From (10), we observe that

$$c_{3n}^t = \sqrt{8(b_{3n}^t)^2 + 8(t+1)b_{3n}^t + (2t+1)^2}$$

for $n \geq 1$. The other cases can be proved similarly. ☐

Theorem 2.4. *If* $\#Rep = 2k > 4$, *then*

1. *The set of all integer solutions of (11) is*

$$\{(x_{(2k-1)n+1}, y_{(2k-1)n+1}), (x_{(2k-1)n+i+1}, y_{(2k-1)n+i+1}), (x_{(2k-1)n+k}, y_{(2k-1)n+k}) : n \geq 0\} \cup \{(x_{(2k-1)n}, y_{(2k-1)n}), (x_{(2k-1)n-i}, y_{(2k-1)n-i}) : n \geq 1\},$$

where

$$\begin{aligned}
(x_{(2k-1)n+1}, y_{(2k-1)n+1}) &= (2B_n + tC_n, 4tB_n + C_n)\\
(x_{(2k-1)n+i+1}, y_{(2k-1)n+i+1}) &= (2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n + t_{2i}C_n)\\
(x_{(2k-1)n+k}, y_{(2k-1)n+k}) &= (2hB_n + hC_n, 4hB_n + hC_n)\\
(x_{(2k-1)n}, y_{(2k-1)n}) &= (-2B_n + tC_n, 4tB_n - C_n)\\
(x_{(2k-1)n-i}, y_{(2k-1)n-i}) &= (-2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n - t_{2i}C_n).
\end{aligned}$$

2. *The general terms of t-cobalancers, t-cobalancing numbers and Lucas t-cobalancing numbers are*

$$\begin{aligned}
r^t_{(2k-1)n} &= \frac{2B_n + tC_n - t}{2}\\
r^t_{(2k-1)n-1} &= \frac{-2B_n + tC_n - t}{2}\\
r^t_{(2k-1)n-i-1} &= \frac{-2t_{2i}B_n + t_{2i-1}C_n - t}{2}\\
b^t_{(2k-1)n} &= \frac{2t(B_n + b_{n+1}) + 2B_n + C_n - 1}{2}\\
b^t_{(2k-1)n-1} &= \frac{2t(B_n + b_{n+1}) - 2B_n - C_n - 1}{2}\\
b^t_{(2k-1)n-i-1} &= \frac{(-2t_{2i} + 4t_{2i-1})B_n + (t_{2i-1} - t_{2i})C_n - t - 1}{2}\\
c^t_{(2k-1)n} &= \sqrt{8(b^t_{(2k-1)n})^2 + 8(t+1)b^t_{(2k-1)n} + (2t+1)^2}\\
c^t_{(2k-1)n-1} &= \sqrt{8(b^t_{(2k-1)n-1})^2 + 8(t+1)b^t_{(2k-1)n-1} + (2t+1)^2}\\
c^t_{(2k-1)n-i-1} &= \sqrt{8(b^t_{(2k-1)n-1-i})^2 + 8(t+1)b^t_{(2k-1)n-1-i} + (2t+1)^2}
\end{aligned}$$

for $n \geq 1$ and

$$\begin{aligned}
r^t_{(2k-1)n+i} &= \frac{2t_{2i}B_n + t_{2i-1}C_n - t}{2}\\
r^t_{(2k-1)n+k-1} &= \frac{2hB_n + hC_n - t}{2}\\
b^t_{(2k-1)n+i} &= \frac{(2t_{2i} + 4t_{2i-1})B_n + (t_{2i-1} + t_{2i})C_n - t - 1}{2}\\
b^t_{(2k-1)n+k-1} &= \frac{6hB_n + 2hC_n - t - 1}{2}\\
c^t_{(2k-1)n+i} &= \sqrt{8(b^t_{(2k-1)n+i})^2 + 8(t+1)b^t_{(2k-1)n+i} + (2t+1)^2}\\
c^t_{(2k-1)n+k-1} &= \sqrt{8(b^t_{(2k-1)n+k-1})^2 + 8(t+1)b^t_{(2k-1)n+k-1} + (2t+1)^2}
\end{aligned}$$

for $n \geq 0$,

where t_{2i-1} and t_{2i} are positive integers such that $2t^2_{2i-1} - t^2_{2i} = 2t^2 - 1$ for $1 \leq i \leq k-2, t < t_1 < t_3 < \cdots < t_{2k-5} < h$ and $1 < t_2 < t_4 < \cdots < t_{2k-4} < h$.

Proof. **(1)** Let $\#\text{Rep} = 2k > 4$, then the set of representatives is

$$\text{Rep} = \{[\pm t \quad 1], [\pm t_{2i-1} \quad t_{2i}], [\pm h \quad h]\},$$

where t_{2i-1} and t_{2i} are positive integers such that $2t_{2i-1}^2 - t_{2i}^2 = 2t^2 - 1$ for $1 \leq i \leq k-2, t < t_1 < t_3 < \cdots < t_{2k-5} < h$ and $1 < t_2 < t_4 < \cdots < t_{2k-4} < h$. In this case

1. $[t \quad 1]M^n$ generates all integer solutions $(x_{(2k-1)n+1}, y_{(2k-1)n+1})$ for $n \geq 0$,
2. $[t \quad -1]M^n$ generates all integer solutions $(x_{(2k-1)n}, y_{(2k-1)n})$ for $n \geq 1$,
3. $[h \quad h]M^n$ generates all integer solutions $(x_{(2k-1)n+k}, y_{(2k-1)n+k})$ for $n \geq 0$,
4. $[t_{2i-1} \quad t_{2i}]M^n$ generates all integer solutions $(x_{(2k-1)n+i+1}, y_{(2k-1)n+i+1})$ for $n \geq 0$,
5. $[t_{2i-1} \quad -t_{2i}]M^n$ generates all integer solutions $(x_{(2k-1)n-i}, y_{(2k-1)n-i})$ for $n \geq 1$.

So the set of all integer solutions is $\{(2B_n + tC_n, 4tB_n + C_n), (2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n + t_{2i} C_n), (2hB_n+hC_n, 4hB_n+hC_n) : n \geq 0\} \cup \{(-2B_n+tC_n, 4tB_n-C_n), (-2t_{2i}B_n+t_{2i-1}C_n, 4t_{2i-1} B_n - t_{2i}C_n) : n \geq 1\}$.

(2) It can be proved in the same way that Theorem 2.3 was proved. □

In Table 1, the set of representatives is given for some values of t. As we can see in Table 1, when $\#\text{Rep} = 2k > 4$, it is impossible to determine the set of representatives and #Rep in terms of t. That is why we assume that $\text{Rep} = \{[\pm t \quad 1], [\pm t_{2i-1} \quad t_{2i}], [\pm h \quad h]\}$, where t_{2i-1} and t_{2i} are positive integers such that $2t_{2i-1}^2 - t_{2i}^2 = 2t^2 - 1$ for $1 \leq i \leq k-2, t < t_1 < t_3 < \cdots < t_{2k-5} < h$ and $1 < t_2 < t_4 < \cdots < t_{2k-4} < h$.

t	**Set of representatives**
985	$\{[\pm 985 \quad 1], [\pm 995 \quad 199], [\pm 1025 \quad 401], [\pm 1267 \quad 1127], [\pm 1393 \quad 1393]\}$
5741	$\{[\pm 5741 \quad 1], [\pm 6001 \quad 2471], [\pm 6739 \quad 4991], [\pm 6805 \quad 5167], [\pm 8119 \quad 8119]\}$
33461	$\{[\pm 33461 \quad 1], [\pm 35155 \quad 15247], [\pm 38935 \quad 28153], [\pm 40409 \quad 32039], [\pm 47321 \quad 47321]\}$
195025	$\{[\pm 195025 \quad 1], [\pm 195083 \quad 6767], [\pm 195257 \quad 13457], [\pm 197005 \quad 39401], [\pm 197743 \quad 46207], [\pm 199547 \quad 59737], [\pm 202985 \quad 79601], [\pm 205933 \quad 93527], [\pm 205973 \quad 93703], [\pm 207607 \quad 100657], [\pm 209405 \quad 107849], [\pm 211327 \quad 115103], [\pm 219883 \quad 143623], [\pm 222425 \quad 151249], [\pm 227837 \quad 166583], [\pm 236623 \quad 189503], [\pm 243355 \quad 205849], [\pm 243443 \quad 206057], [\pm 246977 \quad 214303], [\pm 250747 \quad 222887], [\pm 254665 \quad 231601], [\pm 271133 \quad 266377], [\pm 275807 \quad 275807]\}$

Table 1. $2t^2 - 1$ is a perfect square

2.3 Case 3: $2t^2 - 1$ is not a perfect square

When $2t^2 - 1$ is not a perfect square for $t \geq 2$, we have two cases: $\#\text{Rep} = 2$ or $\#\text{Rep} > 2$.

Theorem 2.5. *If $\#Rep = 2$, then*

1. *The set of all integer solutions of (11) is $\{(x_{2n+1}, y_{2n+1}) : n \geq 0\} \cup \{(x_{2n}, y_{2n}) : n \geq 1\}$, where*

$$(x_{2n+1}, y_{2n+1}) = (2B_n + tC_n, 4tB_n + C_n)$$
$$(x_{2n}, y_{2n}) = (-2B_n + tC_n, 4tB_n - C_n).$$

2. *The general terms of t-cobalancers, t-cobalancing numbers and Lucas t-cobalancing numbers are*

$$\begin{aligned}
r^t_{2n} &= \frac{2B_n + tC_n - t}{2} \\
r^t_{2n-1} &= \frac{-2B_n + tC_n - t}{2} \\
b^t_{2n} &= \frac{2t(B_n + b_{n+1}) + 2B_n + C_n - 1}{2} \\
b^t_{2n-1} &= \frac{2t(B_n + b_{n+1}) - 2B_n - C_n - 1}{2} \\
c^t_{2n} &= \sqrt{8(b^t_{2n})^2 + 8(t+1)b^t_{2n} + (2t+1)^2} \\
c^t_{2n-1} &= \sqrt{8(b^t_{2n-1})^2 + 8(t+1)b^t_{2n-1} + (2t+1)^2}
\end{aligned}$$

for $n \geq 1$.

Proof. **(1)** If $\#\text{Rep} = 2$, then the set of representatives is $\text{Rep} = \{[\pm t \quad 1]\}$ and in this case, $[t \quad 1]M^n$ generates all integer solutions (x_{2n+1}, y_{2n+1}) for $n \geq 0$ and $[t \quad -1]\ M^n$ generates all integer solutions (x_{2n}, y_{2n}) for $n \geq 1$. Thus the set of all integer solutions is $\{(2B_n + tC_n, 4tB_n + C_n) : n \geq 0\} \cup \{(-2B_n + tC_n, 4tB_n - C_n) : n \geq 1\}$.

(2) From (1), we get

$$r^t_{2n} = \frac{2B_n + tC_n - t}{2}.$$

Hence we get from (9)

$$b^t_{2n} = \frac{2t(B_n + b_{n+1}) + 2B_n + C_n - 1}{2}$$

and from (10)

$$c^t_{2n} = \sqrt{8(b^t_{2n})^2 + 8(t+1)b^t_{2n} + (2t+1)^2}$$

for $n \geq 1$. □

Theorem 2.6. *If $\#Rep = 2k > 2$, then*

1. *The set of all integer solutions of (11) is*

$$\{(x_{2kn+1}, y_{2kn+1}), (x_{2kn+i+1}, y_{2kn+i+1}) : n \geq 0\} \cup \{(x_{2kn}, y_{2kn}), (x_{2kn-i}, y_{2kn-i}) : n \geq 1\},$$

where

$$(x_{2kn+1}, y_{2kn+1}) = (2B_n + tC_n, 4tB_n + C_n)$$
$$(x_{2kn+i+1}, y_{2kn+i+1}) = (2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n + t_{2i}C_n)$$
$$(x_{2kn}, y_{2kn}) = (-2B_n + tC_n, 4tB_n - C_n)$$
$$(x_{2kn-i}, y_{2kn-i}) = (-2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n - t_{2i}C_n).$$

2. *The general terms of t-cobalancers, t-cobalancing numbers and Lucas t-cobalancing numbers are*

$$r^t_{2kn} = \frac{2B_n + tC_n - t}{2}$$
$$r^t_{2kn-1} = \frac{-2B_n + tC_n - t}{2}$$
$$r^t_{2kn-i-1} = \frac{-2t_{2i}B_n + t_{2i-1}C_n - t}{2}$$
$$b^t_{2kn} = \frac{2t(B_n + b_{n+1}) + 2B_n + C_n - 1}{2}$$
$$b^t_{2kn-1} = \frac{2t(B_n + b_{n+1}) - 2B_n - C_n - 1}{2}$$
$$b^t_{2kn-i-1} = \frac{(-2t_{2i} + 4t_{2i-1})B_n + (t_{2i-1} - t_{2i})C_n - t - 1}{2}$$
$$c^t_{2kn} = \sqrt{8(b^t_{2kn})^2 + 8(t+1)b^t_{2kn} + (2t+1)^2}$$
$$c^t_{2kn-1} = \sqrt{8(b^t_{2kn-1})^2 + 8(t+1)b^t_{2kn-1} + (2t+1)^2}$$
$$c^t_{2kn-i-1} = \sqrt{8(b^t_{2kn-i-1})^2 + 8(t+1)b^t_{2kn-i-1} + (2t+1)^2}$$

for $n \geq 1$ and

$$r^t_{2kn+i} = \frac{2t_{2i}B_n + t_{2i-1}C_n - t}{2}$$
$$b^t_{2kn+i} = \frac{(2t_{2i} + 4t_{2i-1})B_n + (t_{2i-1} + t_{2i})C_n - t - 1}{2}$$
$$c^t_{2kn+i} = \sqrt{8(b^t_{2kn+i})^2 + 8(t+1)b^t_{2kn+i} + (2t+1)^2}$$

for $n \geq 0$,

where t_{2i-1} and t_{2i} are positive integers such that $2t^2_{2i-1} - t^2_{2i} = 2t^2 - 1$ for $1 \leq i \leq k-1, t < t_1 < t_3 < \cdots < t_{2k-3}$ and $1 < t_2 < t_4 < \cdots < t_{2k-2}$.

Proof. **(1)** If $\#\text{Rep} = 2k > 2$, then the set of representatives is $\text{Rep} = \{[\pm t \quad 1], [\pm t_{2i-1} \quad t_{2i}]\}$, where t_{2i-1} and t_{2i} are positive integers such that $2t^2_{2i-1} - t^2_{2i} = 2t^2 - 1$ for $1 \leq i \leq k-1, t < t_1 < t_3 < \cdots < t_{2k-3}$ and $1 < t_2 < t_4 < \cdots < t_{2k-2}$. Here,

1. $[t \quad 1]M^n$ generates all integer solutions (x_{2kn+1}, y_{2kn+1}) for $n \geq 0$,
2. $[t \quad -1]\ M^n$ generates all integer solutions (x_{2kn}, y_{2kn}) for $n \geq 1$,
3. $[t_{2i-1} \quad t_{2i}]M^n$ generates all integer solutions $(x_{2kn+i+1}, y_{2kn+i+1})$ for $n \geq 0$,
4. $[t_{2i-1} \quad -t_{2i}]\ M^n$ generates all integer solutions (x_{2kn-i}, y_{2kn-i}) for $n \geq 1$.

So the set of all integer solutions is $\{(2B_n+tC_n, 4tB_n+C_n), (2t_{2i}B_n+t_{2i-1}C_n, 4t_{2i-1}B_n+t_{2i}C_n) : n \geq 0\} \cup \{(-2B_n + tC_n, 4tB_n - C_n), (-2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n - t_{2i}C_n) : n \geq 1\}$.

(2) It can be proved in the same way that Theorem 2.5 was proved. □

Again when $\#\text{Rep} = 2k > 2$, it is impossible to determine the set of representatives and $\#\text{Rep}$ in terms of t. For example in Table 2, the set of representatives is given for some values of t. That is why we assume that $\text{Rep} = \{[\pm t \quad 1], [\pm t_{2i-1} \quad t_{2i}]\}$, where t_{2i-1} and t_{2i} are positive integers such that $2t_{2i-1}^2 - t_{2i}^2 = 2t^2 - 1$ for $1 \leq i \leq k-1, t < t_1 < t_3 < \cdots < t_{2k-3}$ and $1 < t_2 < t_4 < \cdots < t_{2k-2}$.

t	**Set of representatives**
58	$\{[\pm 58 \quad 1], [\pm 62 \quad 31], [\pm 74 \quad 65]\}$
142	$\{[\pm 142 \quad 1], [\pm 148 \quad 59], [\pm 182 \quad 161]\}$
54	$\{[\pm 54 \quad 1], [\pm 56 \quad 21], [\pm 60 \quad 37], [\pm 70 \quad 63]\}$
135	$\{[\pm 135 \quad 1], [\pm 137 \quad 33], [\pm 173 \quad 153], [\pm 187 \quad 183]\}$
152	$\{[\pm 152 \quad 1], [\pm 154 \quad 35], [\pm 158 \quad 61], [\pm 178 \quad 131], [\pm 196 \quad 175], [\pm 212 \quad 209]\}$
299	$\{[\pm 299 \quad 1], [\pm 301 \quad 49], [\pm 311 \quad 121], [\pm 359 \quad 281], [\pm 385 \quad 343], [\pm 415 \quad 407]\}$
275	$\{[\pm 275 \quad 1], [\pm 277 \quad 47], [\pm 293 \quad 143], [\pm 295 \quad 151], [\pm 307 \quad 193],$ $[\pm 317 \quad 223], [\pm 353 \quad 313], [\pm 383 \quad 377]\}$

Table 2. $2t^2 - 1$ is not a perfect square

3 Concluding remark

In this paper, we determine the general terms of all t-cobalancers, t-cobalancing numbers and Lucas t-cobalancing numbers by solving the Pell equation $2x^2 - y^2 = 2t^2 - 1$ for some fixed integer $t \geq 1$ in three cases: $t = 1$, $2t^2 - 1$ is a perfect square or not a perfect square for $t \geq 2$. But in all cases, we are able to determine the general terms of them.

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