Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 26, 2020, No. 1, 31–39
DOI: 10.7546/nntdm.2020.26.1.31-39

Extension factor: Definition, properties and problems. Part 2

Krassimir T. Atanassov[1] and József Sándor[2]

[1] Department of Bioinformatics and Mathematical Modelling
IBPhBME – Bulgarian Academy of Sciences,
Acad. G. Bonchev Str. Bl. 105, Sofia-1113, Bulgaria
and
Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Bourgas-8010, Bulgaria
e-mail: krat@bas.bg

[2] Babes-Bolyai University of Cluj, Romania
e-mails: jjsandor@hotmail.com, jsandor@math.ubbcluj.ro

Received: 1 September 2019 **Accepted:** 8 January 2020

Abstract: Some new properties of the arithmetic function called "Extension Factor" and introduced in Part 1 (see [5]) are studied.
Keywords: Arithmetic function, Extension factor.
2010 Mathematics Subject Classification: 11A25.

1 Introduction

In Part 1 of the present research (see [5]), we introduced the following arithmetic functions

$$EF(n) = \prod_{i=1}^{k} p_i^{\alpha_i+1}$$

that we called *Extension Factor,* where for the natural number $n = \prod_{i=1}^{k} p_i^{\alpha_i}$: $k, \alpha_1, \alpha_2, ..., \alpha_k \geq 1$ are natural numbers and $p_1, p_2, ..., p_k$ are different prime numbers.

It is suitable to suppose that $p_1 < p_2 < ... < p_k$ and that $EF(1) = 1$.

In the present paper, we will discuss other properties of function EF. In the text, we will use also the definitions of the following three well-known arithmetic functions (see [8]):

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i - 1), \quad \varphi(1) = 1 \quad \text{(Euler's totient function)},$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i + 1), \quad \psi(1) = 1 \quad \text{(Dedekind's function)},$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}, \quad \sigma(1) = 1.$$

We will use also the arithmetic functions (see [1,8]):

$$\underline{\text{mult}}(n) = \prod_{i=1}^{k} p_i, \ \underline{\text{mult}}(1) = 1,$$

$$\Omega(n) = \sum_{i=1}^{k} \alpha_i, \ \Omega(1) = 1,$$

$$\delta(n) = \sum_{i=1}^{k} \alpha_i p_1^{\alpha_1} ... p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} ... p_k^{\alpha_k},$$

$$\underline{set}(n) = \{p_1, p_2, ..., p_k\},$$

$$\omega(n) = k - \text{ the cardinaliry of set } \underline{set}(n).$$

It is immediate that, $\Omega(n) \geq \omega(n)$, with equality only if n is a squarefree number, i.e. n is a prime or a product of distinct primes.

2 New properties of the arithmetic function EF

Here, we will discuss some new properties of the arithmetic function EF.

Theorem 1. *If p is a prime number and $s < p$ is a natural number, then $EF(s) < EF(p)$.*

Proof. Let $s < p$ be an arbitrary natural number. Then $EF(s) = s.\underline{\text{mult}}(s) \leq s^2 < p^2 = EF(p)$. □

Corollary 1. *Let $n > 1$ be squarefree, written as $n = \prod_{i=1}^{k} p_i$. Then $EF(\varphi(n)) \leq EF(n)$.*

Proof. We obtain sequentially that

$$EF(\varphi(n)) = EF(\varphi(\prod_{i=1}^{k} p_i)) = EF(\prod_{i=1}^{k}(p_i - 1))$$

$$\leq \prod_{i=1}^{k} EF(p_i - 1) \leq \prod_{i=1}^{k} EF(p_i) = EF(n).$$ □

Theorem 2. *For infinitely many n one has $EF(\varphi(n)) < EF(n)$, and for infinitely many m one has $EF(\varphi(m)) > EF(m)$.*

Proof. When $n = 2^k$ for arbitrary natural number k, we obtain for $k = 1$:

$$EF(\varphi(n)) = EF(\varphi(2)) = EF(1) = 1 < 4 = EF(2) = EF(n),$$

and for $k \geq 2$ we obtain:

$$EF(\varphi(n)) = EF(\varphi(2^k)) = EF(2^{k-1}) = 2^k < 2^{k+1} = EF(2^k) = EF(n),$$

but, when $n = 3^k$, for $k = 1$:

$$EF(\varphi(n)) = EF(\varphi(3)) = EF(2) = 4 < 9 = EF(3) = EF(n),$$

and for $k \geq 2$ we obtain:

$$EF(\varphi(n)) = EF(\varphi(3^k)) = EF(2.3^{k-1}) = 4.3^k > 3^{k+1} = EF(3^k) = EF(n). \qquad \square$$

More generally, we can prove the following:

Theorem 3. *For any odd prime p, and $k \geq 2$, for $m = p^k$, one has $EF(\varphi(m)) > EF(m)$.*

Proof. First, we see that for $k = 1$ from Theorem 1:

$$EF(\varphi(m)) = EF(\varphi(p)) = EF(p-1) < EF(p) = EF(m),$$

and for $k \geq 2$ we obtain: $\varphi(m) = p^{k-1}.(p-1)$, so $EF(\varphi(m)) = p^k.EF(p-1)$. Now,

$$EF(p-1) = (p-1).\underline{\text{mult}}(p-1) \geq 2.(p-1),$$

as $\underline{\text{mult}}(p-1) \geq 2$, because $p \geq 3$. Now, as $2(p-1) > p$, inequality (1) follows. $\square$

Theorem 4. *Let $k, s \geq 1$ and $p \geq 3$ be an odd prime, satisfying $EF(p-1) \leq 2p$. Let $n = 2^k.p^s$. Then one has*

$$E(\varphi(n)) < EF(n). \tag{1}$$

Proof. One has $\varphi(n) = 2^{k-1}.p^{s-1}.(p-1)$, so using the inequality $EF(u.v) \leq EF(u).EF(v)$, we can write

$$EF(\varphi(n)) \leq 2^k.p^s.EF(p-1) \leq 2^{k+1}.p^{s+1} = EF(2^k.p^s) = EF(n),$$

by using the assumption $EF(p-1) \leq 2p$.

The cases $k = 1$ and/or $s = 1$ are checked as above. So, the inequality (2) holds. $\square$

We must mention that examples of odd primes p, for which $EF(p-1) \leq 2p$ are the following: $p = 3, 5, 17$, and generally any Fermat prime $p = 2^r + 1$.

On the other hand, in the general case, between $EF(p)$ and, e.g., $EF(p+1)$ there is not a fixed relation, because, for example

$$EF(8) = 8 \times 2 = 16 < 49 = 7^2 = EF(7),$$

while

$$EF(13) = 13^2 = 169 < 196 = 14 \times 14 = EF(14).$$

For this case, the following assertion is valid.

Theorem 5. *If p is a prime number and $p+1$ is squarefree, then*

$$EF(p) < EF(p+1).$$

If $p \geq 3$ is a prime number and $p+1$ is not squarefree, then

$$EF(p) > EF(p+1).$$

Proof. First suppose that for the prime number p, $p+1$ is squarefree. If $p = 2$, then

$$EF(2) = 4 < 9 = EF(3) = EF(2+1).$$

If $p \geq 3$, then

$$EF(p+1) = (p+1)^2 > p^2 = EF(p),$$

i.e., the first case is valid.

If $p+1$ is not squarefree, then

$$p + 1 = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where k, α_1, α_2, ..., $\alpha_k \geq 1$ are natural numbers and $p_1, p_2, ..., p_k$ are different prime numbers and there is at least one i for which $\alpha_i > 1$. Let $\alpha_1 = 2, p_1 = 2$ for $\alpha_2 = \cdots = \alpha_k = 1$. Then

$$EF(p+1) = (p+1)\underline{\text{mult}}(p+1) = \frac{(p+1)^2}{2} < p^2 = EF(p),$$

for $p \geq 3$. Obviously, if there is more than one $\alpha_i > 1$, or if the smallest $p_i > 2$, then the inequality will be more powerful. $\square$

Corollary 2. *For every prime number $p \geq 3$ with $p+1$ squarefree,*

$$EF(\psi(n)) > EF(n),$$

while, otherwise, we have

$$EF(\psi(n)) < EF(n).$$

From Theorem 1 it follows that for each prime number p:

$$EF(p) > EF(p-1),$$

but there is not a fixed relation between $EF(q)$ and $EF(p-1)$, where q is the greatest prime number smaller than p, because, for example for $p = 7, q = 5$: $EF(5) = 25 < 36 = EF(6)$, while for $p = 17, q = 13$: $EF(16) = 32 < 169 = EF(13)$.

It can be directly seen that for $n = 1, 2$,

$$EF(n)^n = n^{EF(n)},$$

because $EF(1)^1 = 1^1 = 1 = 1^1 = 1^{EF(1)}$ and $EF(2)^2 = 4^2 = 16 = 2^4 = 2^{EF(2)}$.

Theorem 6. *For the natural number* $n \geq 3$*:* $EF(n)^n < n^{EF(n)}$.

Proof. The proof follows directly from D. Mitrinović's inequality $(n+r)^r < n^{n+r}$ for the natural numbers r and $n \geq 3$ [7]. □

3 Function EF and other arithmetic functions

First, for a fixed natural number n with the above canonical representation, we introduce the arithmetic functions

$$IF(n) = \prod_{i=1}^{k} p_i^{\frac{1}{\alpha_i}}$$

(see [2]),

$$CF(n) = \prod_{i=1}^{k} \alpha_i^{p_i}$$

(see [3]),

$$RF(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}$$

(see [4]), and well-known functions π, defined by $\pi(n)$ being the number of primes smaller or equal to n, and Möbius function μ (see [8]).

Obviously,

$$\mu(EF(n)) = 0$$

for each natural number $n > 1$.

It is seen directly that for each natural number n:

$$\frac{EF(n)}{RF(n)} = \prod_{i=1}^{k} p_i^{1+\text{sg}(\alpha_i - 1)} = \underline{\text{mult}}(n). \prod_{i=1}^{k} p_i^{\text{sg}(\alpha_i - 1)},$$

where for each real number x:

$$\text{sg}(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}.$$

Now, we prove the following assertions.

Theorem 7. *For every natural number* n*:* $IF(EF(n)) < IF(n)$.

Proof. Let n be a fixed natural number. Then

$$IF(EF(n)) = IF(\prod_{i=1}^{k} p_i^{\alpha_i+1}) = \prod_{i=1}^{k} p_i^{\frac{1}{\alpha_i+1}} < \prod_{i=1}^{k} p_i^{\frac{1}{\alpha_i}} = IF(n). \qquad \square$$

Theorem 8. *For every natural number* n*:* $CF(EF(n)) > CF(n)$.

Proof. Let n be a fixed natural number. Then

$$CF(EF(n)) = CF(\prod_{i=1}^{k} p_i^{\alpha_i+1}) = \prod_{i=1}^{k} (\alpha_i+1)^{p_i} > \prod_{i=1}^{k} p_i^{\alpha_i+1} = CF(n).$$

We check directly that

$$EF(1)^{RF(1)} = 1^1 = 1 = 1^1 = RF(1)^{EF(1)},$$
$$EF(2)^{RF(2)} = 4^1 = 4 > 1^4 = RF(2)^{EF(2)},$$
$$EF(4)^{RF(4)} = 8^2 = 64 < 256 = 2^8 = RF(4)^{EF(4)}. \qquad \square$$

More generally, the following assertion holds.

Theorem 9. *For every squarefree natural number* $n > 1$ *one has:* $EF(n)^{RF(n)} > RF(n)^{EF(n)}$*; and if* n *is not squarefree, then* $EF(n)^{RF(n)} < RF(n)^{EF(n)}$.

Proof. In the first case, we get $EF(n)^{RF(n)} > 1 = RF(n)^{EF(n)}$.

In the second one, we will discuss the simplest case, when $n = 2^k$ for $k \geq 2$. The case $k = 2$ was checked above, so, let $k \geq 3$. Then:

$$RF(n)^{EF(n)} - EF(n)^{RF(n)} = RF(2^k)^{EF(2^k)} - EF(2^k)^{RF(2^k)} = \left(2^{k-1}\right)^{2^{k+1}} - \left(2^{k+1}\right)^{2^{k-1}}$$
$$= 2^{(k-1)2^{k+1}} - 2^{(k+1)2^{k-1}} = 2^{4(k-1)2^{k-1}} - 2^{(k+1)2^{k-1}} > 0,$$

because $4(k-1) - (k+1) = 3k - 5 > 0$ for each natural number $k \geq 1$. Obviously, in all other cases for n the inequality will be more powerful. $\square$

Theorem 10. *For every natural number* n *not squarefree and with* $\underline{\text{mult}}(n) > 6$*, it follows that* $RF(n)^{EF(n)} > EF(n)^n$.

Proof. Let the natural number n not squarefree be given, and let for brevity $r = \underline{\text{mult}}(n) > 6$.

We check sequentially that

$$RF(n)^{EF(n)} - EF(n)^n = \left(\frac{n}{r}\right)^{nr} - (nr)^n > 0,$$

if and only if $n^{nr} - (nr)^n.r^{nr} = n^{nr} - r^{n+nr}.n^n = n^{nr} - r^{n(r+1)}.n^n > 0$,
if and only if $n^{n(r-1)} - r^{n(r+1)} > 0$.

Since n is not squarefree, it follows that $n \geq 2r$. Let us assume that $n = 2r$. Then

$$(2r)^{2r(r-1)} - r^{2r(r+1)} = 2^{2r(r-1)}.r^{2r(r-1)} - r^{2r(r+1)}$$
$$= 4^{r(r-1)}.r^{2r(r-1)} - r^{2r(r+1)} = r^{2r(r-1)}.(4^{r(r-1)} - r^{4r}) > 0,$$

that is obvious valid for $r > 6$. $\square$

Theorem 11. *For every odd not squarefree number n one has:* $RF(n)^{EF(n)} > EF(n)^n$.

Proof. Let the odd number n be not squarefree. Then, as above, we must prove that $n^{n(r-1)} - r^{n(r+1)} > 0$.

Now, clearly $n \geq 3r$. Let us assume that $n = 3r$. Then

$$(3r)^{3r(r-1)} - r^{3r(r+1)} = 3^{3r(r-1)}.r^{3r(r-1)} - r^{3r(r+1)}$$

$$= r^{3r(r-1)}.(3^{3r(r-1)}. - r^{6r}) > 0,$$

because $3^{3r(r-1)}. - r^{6r} > 0$ for $r \geq 3$ that is valid, because $n \geq 3$. □

Corollary 3. *For every squarefree natural number n*

$$n^{EF(n)} > EF(n)^n > EF(n)^{RF(n)} > RF(n)^{EF(n)};$$

If n is not squarefree, and n is even, and $\underline{\text{mult}}(n) > 6$, or n is an odd number, then

$$n^{EF(n)} > RF(n)^{EF(n)} > EF(n)^n > EF(n)^{RF(n)}.$$

Finally, we will discuss the relations between arithmetic functions EF and π. In Table 1 we give the values for the first 20 natural numbers.

n	$\pi(n)$	$\underline{\text{mult}}(n)$	$\pi(\underline{\text{mult}}(n))$	$EF(n)$	$\pi(EF(n))$
1	0	1	0	1	0
2	1	2	1	4	2
3	2	3	2	9	4
4	2	2	1	8	4
5	3	5	3	25	9
6	3	6	3	36	11
7	4	7	4	49	15
8	4	2	1	16	6
9	4	3	2	27	9
10	4	10	4	100	25
11	5	11	5	121	31
12	5	6	3	72	20
13	6	13	6	169	39
14	6	13	6	196	44
15	6	15	6	225	48
16	6	2	1	32	11
17	7	17	7	289	61
18	7	6	3	108	30
19	8	19	8	361	72
20	8	10	4	200	46

Table 1. Values of the functions for $n = 1, \ldots, 20$.

We are ready to formulate the following theorem.

Theorem 12. *For each natural number $n \geq 8$:*

$$\pi(EF(n)) \geq \pi(n).\pi(\underline{mult}(n)).$$

Proof. In 1998, in [9], L. Panaitopol proved the inequality

$$\pi(x).\pi(y) > \pi(xy)$$

for any integers $x, y \geq 2$, excepting the following pairs: $(x, y) = (5, 7), (7, 5), (7, 7)$.

Now, put $x = n, y = \underline{\text{mult}}(n)$ in Panaitopol's inequality, to prove Theorem 12. ☐

In fact, from the above, a slightly stronger form can be deduced, namely, it holds true for any $n \geq 2$ distinct from 7.

Theorem 13. *For each natural number $n \geq 2$: $\pi(EF(n)) \geq \pi(n) + \pi(\underline{mult}(n))$.*

Proof. In 1934 in [6], H. Ishikawa proved the inequality

$$\pi(xy) \geq \pi(x) + \pi(y)$$

for any integers $x, y \geq 2$. Now put $x = n, y = \underline{\text{mult}}(n)$ in Ishikawa's inequality, to prove Theorem 13. ☐

We can also state two results similar to Theorems 12 and 13, but involving the function RF:

Theorem 14. *For each natural number $n \geq 2$:*

$$\pi(RF(n)) \leq \pi(n) - \pi(\underline{mult}(n)) \tag{2}$$

and

$$\pi(RF(n)) \leq \frac{\pi(n)}{\pi(\underline{mult}(n))}. \tag{3}$$

Proof. Apply the Ishikawa inequality $\pi(xy) \geq \pi(x) + \pi(y)$ for $x, y \geq 2$ for

$$x = \underline{\text{mult}}(n) \geq 2 \quad \text{and} \quad y = \frac{\pi(n)}{\pi(\underline{\text{mult}}(n))} \geq 1.$$

Then we get the inequality (3).

We will remark, that (3) holds also for $x, y \geq 1$, since $\pi(1) = 0$.

Now, for the proof of (3) apply the Panaitopol inequality $\pi(xy) \geq \pi(x)\pi(y)$ for

$$x = \underline{\text{mult}}(n) \quad \text{and} \quad y = \frac{\pi(n)}{\pi(\underline{\text{mult}}(n))}.$$

Now the pairs

$$\left(\underline{\text{mult}}(n), \frac{\pi(n)}{\pi(\underline{\text{mult}}(n))}\right)$$

should be distinct from (5,7), (7,5) and (7,7). This is possible only if we do not have $\underline{\text{mult}}(n) = 7$ and $\frac{\pi(n)}{\pi(\underline{\text{mult}}(n))} = 7$, i.e., when $n = 49$. Therefore, (4) follows. ☐

4 Conclusion

In conclusion, we will mention, that in future part we will study some extensions of the introduced new arithmetic function and some other functions.

References

[1] Atanassov K. (1987). New integer functions, related to φ and σ functions, *Bulletin of Number Theory and Related Topics*, XI (1), 3–26.

[2] Atanassov, K. (1996). Irrational factor: Definition, properties and problems. *Notes on Number Theory and Discrete Mathematics*, 2 (3), 42–44.

[3] Atanassov K. (2002). Converse factor: Definition, properties and problems, *Notes on Number Theory and Discrete Mathematics*, 8 (1), 37– 38.

[4] Atanassov K. (2002). Restrictive factor: Definition, properties and problems. *Notes on Number Theory and Discrete Mathematics*, 8 (4), 117–119.

[5] Atanassov, K. & Sándor, J. (2019). Extension factor: Definition, properties and problems. Part 1. *Notes on Number Theory and Discrete Mathematics*, 25 (3), 36–43.

[6] Ishikawa, I. (1934). Über die Verteileung der Primzahlen, *Sci. Rep. Tokyo Univ.*, 2, 21–44.

[7] Mitrinović, & D., Popadić, M. (1978). *Inequalities in Number Theory*. Niś, Univ. of Niś.

[8] Mitrinović, D., Sándor, J. (in coop. with B. Crstici). (1995). *Handbook of Number Theory*, Kluwer Acad. Publ.

[9] Panaitopol, L. (1998). On the inequality $\pi(a).\pi(b) > \pi(ab)$. *Bull. Math. Soc. Sci. Math. Roumanie*, 41 (89), 2, 135–139.