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Square-full numbers with an even number of prime factors

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Abstract: In this article, we study the functions $\omega(n)$ and $\Omega(n)$, where n is an s-full number. For example, we prove that the square-full numbers with $\Omega(n)$ even are in greater proportion than the square-full numbers with $\Omega(n)$ odd. The methods used are elementary.
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1 Introduction and preliminary notes

Let us consider the prime factorization of a positive integer $n = q_1^{s_1} \cdots q_r^{s_r}$, where q_i $(i = 1, \ldots, r)$ $(r \geq 1)$ are the different primes in the prime factorization and s_i $(i = 1, \ldots, r)$ are the multiplicities or exponents. We need the following well-known arithmetical functions: $\omega(n) = r$ that is the number of different prime factors in the prime factorization of n, $\Omega(n) = s_1 + \cdots + s_r$ that is the total number of prime factors in the prime factorization of n, $u(n) = q_1 \cdots q_r$ that denotes the kernel of n and $w(n) = (q_1 + 1) \cdots (q_r + 1)$. Note that $w(n)$ is the sum of the positive divisors of the kernel of n.

The functions $\omega(n)$ and $\Omega(n)$ were studied by G. H. Hardy and S. Ramanujan in 1917 [6]. They obtained the following formulas

$$\sum_{n \leq x} \omega(n) = x \log \log x + Mx + o(x),$$

$$\sum_{n\leq x}\Omega(n) = x\log\log x + \left(M + \sum_p \frac{1}{p(p-1)}\right)x + o(x),$$

where M is Mertens's constant. In the same paper they define the normal order of an arithmetical function and they prove that the normal order of $\omega(n)$ and $\Omega(n)$ is $\log\log n$.

Let $\Omega_p(x)$ be the number of positive integers n not exceeding x such that $\Omega(n)$ is even and $\Omega_i(x)$ the number of positive integers n not exceeding x such that $\Omega(n)$ is odd. The following asymptotic formulas are well-known

$$\Omega_i(x) = \frac{1}{2}x + o(x), \qquad \Omega_p(x) = \frac{1}{2}x + o(x).$$

That is, these two sets of positive integers have density $1/2$.

Let $\omega_p(x)$ be the number of positive integers n not exceeding x such that $\omega(n)$ is even and $\omega_i(x)$ is the number of positive integers n not exceeding x such that $\omega(n)$ is odd. Recently, R. Jakimczuk [10] proved that also these two sets of positive integers have density $1/2$. That is

$$\omega_i(x) = \frac{1}{2}x + o(x), \qquad \omega_p(x) = \frac{1}{2}x + o(x).$$

A number is h-full if all the distinct primes in its prime factorization have multiplicity (or exponent) greater than or equal to h. If $h = 2$ the numbers are called square-full. The square-full numbers were studied by P. Erdős and G. Szekeres [3] and many other authors. For example, P. T. Bateman and E. Grosswald [1], A. Ivić and P. Shiu (see [8] and [9]), S. W. Golomb [5], etc. Also, recently, R. Jakimczuk [12] studied the kernel of h-full numbers. See also the reference [2]. An elementary proof on the distribution of h-full numbers is established here.

In this article, we study the functions $\Omega(n)$ and $\omega(n)$ on the h-full numbers. In particular, on the square-full numbers. For example, between other results, we prove that the square-full numbers n with $\Omega(n)$ even are in greater proportion than the square-full numbers n with $\Omega(n)$ odd.

We shall need the following theorems on the distribution of square-free numbers. In this note a square-free number will be denoted q_1.

Theorem 1.1. *Let $Q_1(x)$ be the number of square-free numbers not exceeding x, we have*

$$Q_1(x) = \sum_{q_1\leq x} 1 = \frac{6}{\pi^2}x + o(x).$$

Let $Q_p(x)$ be the number of square-free n not exceeding x such that $\Omega(n) = \omega(n)$ is even and let $Q_i(x)$ be the number of square-free n not exceeding x such that $\Omega(n) = \omega(n)$ is odd. We have (prime number theorem)

$$Q_p(x) = \frac{1}{2}\frac{6}{\pi^2}x + o(x),$$

$$Q_i(x) = \frac{1}{2}\frac{6}{\pi^2}x + o(x).$$

Proof. See [7, chapter XVIII]. □

In this note a square-free multiple of the different and fixed primes $q_1, \ldots, q_s$, that is multiple of the square-free $q_1 q_2 \cdots q_s$, will be denoted $q_{q_1 \cdots q_s}$.

Theorem 1.2. *Let $Q_{q_1 \cdots q_s}(x)$ be the number of square-free not exceeding x multiple of the different and fixed primes $q_1, q_2, \ldots, q_s$, we have*

$$Q_{q_1 q_2 \cdots q_s}(x) = \sum_{q_{q_1 q_2 \cdots q_s} \leq x} 1 = \frac{6}{\pi^2} \prod_{i=1}^{s} \frac{1}{q_i + 1} x + o(x).$$

Proof. See [11]. □

Let $(MP)_{q_1 \cdots q_s}(x)$ be the number of square-free n not exceeding x multiple of $q_1 \cdots q_s$ such that $\Omega(n) = \omega(n)$ is even. On the other hand, let $(MI)_{q_1 \cdots q_s}(x)$ be the number of square-free n not exceeding x multiple of $q_1 \cdots q_s$ such that $\Omega(n) = \omega(n)$ is odd. We have the following theorem.

Theorem 1.3. *The following asymptotic formulas hold.*

$$(MP)_{q_1 \cdots q_s}(x) = \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{s} \frac{1}{q_i + 1} x + o(x),$$

$$(MI)_{q_1 \cdots q_s}(x) = \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{s} \frac{1}{q_i + 1} x + o(x).$$

Proof. See [10]. □

Theorem 1.4. *If $\alpha > 0$ the following two series of positive terms are convergent*

$$\sum_{n=1}^{\infty} \frac{1}{w(n) n^{\alpha}}, \qquad \sum_{n=1}^{\infty} \frac{1}{u(n) n^{\alpha}}$$

and besides the following two equations hold

$$\sum_{n=1}^{\infty} \frac{1}{w(n) n^{\alpha}} = \prod_{p} \left(1 + \frac{1}{(p+1)(p^{\alpha} - 1)} \right),$$

$$\sum_{n=1}^{\infty} \frac{1}{u(n) n^{\alpha}} = \prod_{p} \left(1 + \frac{1}{p(p^{\alpha} - 1)} \right),$$

where the notation $\prod_p$ means that the product runs over all positive primes p.

Proof. We have

$$\sum_{n=1}^{\infty} \frac{1}{w(n) n^{\alpha}} = \prod_{p} \left(1 + \frac{1}{(p+1)p^{\alpha}} + \frac{1}{(p+1)(p^{\alpha})^2} + \frac{1}{(p+1)(p^{\alpha})^3} + \cdots \right)$$
$$= \prod_{p} \left(1 + \frac{1}{(p+1)p^{\alpha}} \left(\frac{1}{1 - \frac{1}{p^{\alpha}}} \right) \right) = \prod_{p} \left(1 + \frac{1}{(p+1)(p^{\alpha} - 1} \right).$$

Now, the product

$$\prod_p \left(1 + \frac{1}{(p+1)(p^\alpha - 1)}\right)$$

converges to a positive number, since the series of positive terms

$$\sum_p \frac{1}{(p+1)(p^\alpha - 1)}$$

clearly converges. The theorem is proved. □

2 Main results

Let $h \geq 2$ be an arbitrary but fixed positive integer. A number is h-full if all the distinct primes in its prime factorization have multiplicity (or exponent) greater than or equal to h. That is, the number $q_1^{s_1} \cdots q_r^{s_r}$ is h-full if $s_i \geq h$ $(i = 1, \ldots, r)$ $(r \geq 1)$. We shall denote a general h-full number n_h. If $h = 2$, the numbers are called square-full. The h-kernel of the h-full number n_h we define in the form $(u(n_h))^h$ and the h-remainder in the form $\frac{n_h}{(u(n_h))^h}$. Note that the h-remainder is 1 if and only if the h-full number is of the form $(q_1 \cdots q_r)^h$.

Let $A_h(x)$ be the number of h-full numbers not exceeding x.

Theorem 2.1. *Let $h \geq 2$ be an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$A_h(x) = \sum_{n_h \leq x} 1 = \frac{6}{\pi^2} C_{0,h} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right), \tag{1}$$

where

$$C_{0,h} = \sum_{n=1}^{\infty} \frac{1}{w(n)} \frac{1}{n^{\frac{1}{h}}} = \prod_p \left(1 + \frac{1}{(p+1)(p^{\frac{1}{h}} - 1)}\right) \qquad (w(1) = 1). \tag{2}$$

Proof. Let us consider the prime factorization of a positive integer $a \geq 2$

$$a = q_1^{s_1} q_2^{s_2} \cdots q_t^{s_t},$$

where $q_1, q_2, \ldots, q_t$ are the different primes in the prime factorization of a. We put

$$a' = q_1 q_2 \cdots q_t$$

and

$$a'' = (q_1 + 1)(q_2 + 1) \cdots (q_t + 1).$$

If $a = 1$, then we put $a' = a'' = 1$.

Therefore, we have (see Theorem 1.1 and Theorem 1.2)

$$\sum_{q_{a'} \leq x} 1 = \frac{6}{\pi^2} \frac{1}{a''} x + o(x). \tag{3}$$

Let us consider the set H of all h-full numbers n_h not exceeding x. Now, let us consider the set T_a of all h-full numbers n_h not exceeding x with the same h-remainder a, that is, $T_a = \{n_h : n_h \leq x, v_h(n_h) = a\}$. Note that if $a_1 \neq a_2$ we have $T_{a_1} \cap T_{a_2} = \phi$, that is, the sets T_{a_1} and T_{a_2} are disjoint. Suppose that A_x (depending on x) is the greatest h-remainder among the numbers in the set H. Then

$$\bigcup_{a=1}^{A_x} T_a = H.$$

Therefore, the sets T_a are partitions of the set H. Note that some T_a can be empty.

The set of the h-kernel of the numbers in the set T_a will be denoted by S_a. Hence,

$$S_a = \left\{ q_{a'}^h : q_{a'}^h \leq \frac{x}{a} \right\} = \left\{ q_{a'}^h : q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}} \right\}. \tag{4}$$

The series $\sum_{a=1}^{\infty} \frac{1}{a''} \frac{1}{a^{\frac{1}{h}}}$ converges (see Theorem 1.4). Hence

$$\sum_{a=1}^{\infty} \frac{1}{a''} \frac{1}{a^{\frac{1}{h}}} = C_{0,h}. \tag{5}$$

We choose B such that (see Theorem 1.4)

$$\sum_{a=B+1}^{\infty} \frac{1}{a''} \frac{1}{a^{\frac{1}{h}}} < \epsilon \tag{6}$$

and

$$\frac{\pi^2}{6} \sum_{a=B+1}^{\infty} \frac{1}{a' a^{\frac{1}{h}}} < \epsilon. \tag{7}$$

Therefore, we have (see (3), (4), (5) and (6))

$$\begin{aligned} A_h(x) = \sum_{a=1}^{A(x)} \left(\sum_{q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}}} 1 \right) &= \sum_{a=1}^{B} \left(\sum_{q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}}} 1 \right) \\ + \sum_{a=B+1}^{A(x)} \left(\sum_{q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}}} 1 \right) &= \sum_{a=1}^{B} \left(\frac{1}{a''} \frac{6}{\pi^2} \frac{x^{\frac{1}{h}}}{a^{\frac{1}{h}}} \right) + o\left(x^{\frac{1}{h}} \right) \\ + \sum_{a=B+1}^{A(x)} \left(\sum_{q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}}} 1 \right) &= \frac{6}{\pi^2} x^{\frac{1}{h}} \left(\sum_{a=1}^{B} \frac{1}{a''} \frac{1}{a^{\frac{1}{h}}} \right) + o\left(x^{\frac{1}{h}} \right) \\ + \sum_{a=B+1}^{A(x)} \left(\sum_{q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}}} 1 \right) &= \frac{6}{\pi^2} x^{\frac{1}{h}} C_{0,h} - \frac{6}{\pi^2} x^{\frac{1}{h}} \left(\sum_{a=B+1}^{\infty} \frac{1}{a''} \frac{1}{a^{\frac{1}{h}}} \right) \\ + o(1) \frac{6}{\pi^2} x^{\frac{1}{h}} + \sum_{a=B+1}^{A(x)} &\left(\sum_{q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}}} 1 \right). \end{aligned} \tag{8}$$

Equation (8) can be written in the form

$$\frac{A_h(x)}{\frac{6}{\pi^2}x^{\frac{1}{h}}} - C_{0,h} = -\left(\sum_{a=B+1}^{\infty} \frac{1}{a''}\frac{1}{a^{\frac{1}{h}}}\right) + o(1)$$

$$+ \quad \frac{\sum_{a=B+1}^{A(x)}\left(\sum_{q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}}} 1\right)}{\frac{6}{\pi^2}x^{\frac{1}{h}}}. \tag{9}$$

We have (see (8) and (7))

$$\begin{aligned} 0 &\leq \sum_{a=B+1}^{A(x)}\left(\sum_{q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}}} 1\right) \leq \sum_{a=B+1}^{A(x)}\left(\sum_{q_1 \leq \frac{x^{(1/h)}}{a'a^{(1/h)}}} 1\right) \\ &\leq \sum_{a=B+1}^{A(x)}\left(\sum_{n \leq \frac{x^{(1/h)}}{a'a^{(1/h)}}} 1\right) \leq \sum_{a=B+1}^{A(x)}\left(\frac{x^{(1/h)}}{a'a^{(1/h)}}\right) \\ &= x^{\frac{1}{h}}\sum_{a=B+1}^{A(x)} \frac{1}{a'a^{\frac{1}{h}}} \leq \frac{6}{\pi^2}x^{\frac{1}{h}}\frac{\pi^2}{6}\sum_{a=B+1}^{\infty}\frac{1}{a'a^{\frac{1}{h}}} \\ &\leq \epsilon\frac{6}{\pi^2}x^{\frac{1}{h}}. \end{aligned} \tag{10}$$

We choose x_0 such that if $x \geq x_0$ then $|o(1)| < \epsilon$ in equation (9). Equations (9), (6) and (10) give

$$\left|\frac{A_h(x)}{\frac{6}{\pi^2}x^{\frac{1}{h}}} - C_{0,h}\right| \leq 3\epsilon.$$

Therefore, since ϵ is arbitrarily small, we have

$$\lim_{x\to\infty} \frac{A_h(x)}{\frac{6}{\pi^2}x^{\frac{1}{h}}} = C_{0,h}.$$

That is (1). The theorem is proved. □

Remark 2.2. *If $h = 2$ then it is well-known that the constant can be written in terms of the Riemann zeta function $\zeta(s)$, that is, the value of the constant is $\frac{\zeta(3/2)}{\zeta(3)}$. This can be obtained from our formulas (16) and (17), since*

$$\begin{aligned} \frac{6}{\pi^2}C_{0,2} &= \prod_p \left(\left(1-\frac{1}{p^2}\right)\left(1+\frac{1}{(p+1)(p^{1/2}-1)}\right)\right) \\ &= \prod_p \left(1-\frac{1}{p^2}+\frac{p^{1/2}+1}{p^2}\right) = \prod_p\left(1+\frac{1}{p^{3/2}}\right) = \prod_p\left(\frac{\frac{1}{1-p^{-3/2}}}{\frac{1}{1-p^{-3}}}\right) \\ &= \frac{\zeta(3/2)}{\zeta(3)} = 2.1732543125... \end{aligned}$$

See [4, page 112].

Let $\omega_{p,h}(x)$ be the number of h-full numbers n_h not exceeding x such that $\omega(n_h)$ is even and let $\omega_{i,h}(x)$ be the number of h-full numbers n_h not exceeding x such that $\omega(n_h)$ is odd. We have the following theorem.

Theorem 2.3. *The following asymptotic formulas hold.*

$$\omega_{p,h}(x) = \frac{1}{2}\frac{6}{\pi^2}C_{0,h}x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right), \tag{11}$$

$$\omega_{i,h}(x) = \frac{1}{2}\frac{6}{\pi^2}C_{0,h}x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right). \tag{12}$$

Proof. The proof of (11) is the same as the proof of Theorem 2.1. Equation (3) is replaced by (Theorem 1.1 and Theorem 1.3)

$$\sum_{\substack{q_{a'}\leq x\\ \omega(q_{a'})\equiv 0 \pmod 2}} 1 = \frac{1}{2}\frac{6}{\pi^2}\frac{1}{a''}x + o(x).$$

If $a = 1$ we put $a' = a'' = 1$. The proof of (12) is by difference using (11) and Theorem 2.1 or using the equation

$$\sum_{\substack{q_{a'}\leq x\\ \omega(q_{a'})\equiv 1 \pmod 2}} 1 = \frac{1}{2}\frac{6}{\pi^2}\frac{1}{a''}x + o(x).$$

The theorem is proved. □

Let $\Omega_{h,r}(x)$ be the number of h-full numbers n_h not exceeding x such that $\Omega(n_h) \equiv r \pmod h$ $(r = 0, \ldots, h-1)$. We have the following theorem.

Theorem 2.4. *The following asymptotic formulas hold.*

$$\Omega_{h,r}(x) = \frac{6}{\pi^2}C_{0,h,r}x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right) \qquad (r = 0, \ldots, h-1),$$

where the constants $C_{0,h,r}$ are given by the series

$$C_{0,h,r} = \sum_{\Omega(n)\equiv r \pmod h} \frac{1}{w(n)}\frac{1}{n^{\frac{1}{h}}} \qquad (r = 0, \ldots, h-1)$$

and

$$\sum_{r=0}^{h-1} C_{0,h,r} = C_{0,h}.$$

Proof. Since the total number of prime factors in the h-kernel is multiple of h, the proof is the same as the proof of Theorem 2.1, where we consider only the h-remainder a such that $\Omega(a) \equiv r \pmod h$. If $a = 1$ we put $a' = a'' = 1$ and $\Omega(a) = \Omega(1) = 0$, therefore $\Omega(1) \equiv 0 \pmod h$. The theorem is proved. □

Let $\Omega_{p,h}(x)$ be the number of h-full numbers n_h not exceeding x such that $\Omega(n_h)$ is even and let $\Omega_{i,h}(x)$ be the number of h-full numbers n_h not exceeding x such that $\Omega(n_h)$ is odd. We have the following theorem.

Theorem 2.5. *If h is even, then*

$$\Omega_{p,h}(x) = \frac{6}{\pi^2} D_{h,0} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right),$$

$$\Omega_{i,h}(x) = \frac{6}{\pi^2} D_{h,1} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right),$$

where the constants are given by the series

$$D_{h,0} = \sum_{\Omega(n)\equiv 0 \pmod 2} \frac{1}{w(n)} \frac{1}{n^{\frac{1}{h}}} = 1 + \sum_{n>1,\ \Omega(n)\equiv 0 \pmod 2} \frac{1}{w(n)} \frac{1}{n^{\frac{1}{h}}},$$

$$D_{h,1} = \sum_{\Omega(n)\equiv 1 \pmod 2} \frac{1}{w(n)} \frac{1}{n^{\frac{1}{h}}},$$

and

$$D_{h,0} + D_{h,1} = C_{0,h}.$$

If h is odd, then

$$\Omega_{p,h}(x) = \frac{1}{2}\frac{6}{\pi^2} C_{0,h} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right), \tag{13}$$

$$\Omega_{i,h}(x) = \frac{1}{2}\frac{6}{\pi^2} C_{0,h} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right),$$

Proof. If h is even, then the total number of prime factors in the h-kernel is even, therefore, the proof is the same as the proof of Theorem 2.4. If h is odd, in the proof of equation (13) we consider two cases.
Case 1. $\omega(q_{a'}) \equiv 0 \pmod 2$ and $\Omega(a) \equiv 0 \pmod 2$.
Case 2. $\omega(q_{a'}) \equiv 1 \pmod 2$ and $\Omega(a) \equiv 1 \pmod 2$.
Hence, the theorem is proved. □

If $h = 2$ (square-full numbers), we shall prove in the next theorem that $D_{2,0} > D_{2,1}$ and consequently the proportion of square-full numbers not exceeding x with a total even number of prime factors is greater than the proportion of square-full numbers not exceeding x with a total odd number of prime factors.

Theorem 2.6. *The following inequality holds.*

$$D_{2,0} > D_{2,1}. \tag{14}$$

Proof. We have

$$\sum_{n=1}^{\infty} \frac{1}{w(n)} \frac{1}{n} = \prod_{p} \left(1 + \frac{1}{(p+1)p} + \frac{1}{(p+1)p^2} + \cdots \right) = \prod_{p} \left(\frac{1}{1 - \frac{1}{p^2}} \right) = \frac{\pi^2}{6}. \tag{15}$$

Let us consider the pairs (a, b): $(1, 1)$, $(2, 3)$, $(2, 5)$, $(2, 7)$, $(3, 5)$, $(2, 11)$, $(3, 7)$, $(2, 13)$. Note that by Remark 2.2 we have

$$\frac{6}{\pi^2} D_{2,0} + \frac{6}{\pi^2} D_{2,1} = \frac{6}{\pi^2} C_{0,2} = 2.1732543125... \tag{16}$$

Now (see (16))

$$\begin{aligned} \frac{6}{\pi^2} D_{2,0} &> \frac{6}{\pi^2} \sum_{(a,b)} \left(\sum_{n=1}^{\infty} \frac{1}{w(abn^2)} \frac{1}{\sqrt{abn^2}} \right) \\ &> \left(\frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{w(n)} \frac{1}{n} \right) \left(1 + \sum_{(a,b) \neq (1,1)} \frac{1}{(a+1)(b+1)} \frac{1}{\sqrt{ab}} \right) \\ &> \frac{1}{2} \frac{6}{\pi^2} C_{0,2} = 1.086627... \end{aligned}$$

since by (15) we have

$$\frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{w(n)} \frac{1}{n} = 1.$$

Therefore (14) holds. The theorem is proved. □

3 Conclusion

In this article we have studied the distribution of h-full numbers by use of an elementary method. By use of the same elementary method we have proved theorems on the functions $\omega(n)$ and $\Omega(n)$ defined on the sequence of h-full numbers. In particular, if $h = 2$ then we have obtained that the square-full numbers with $\Omega(n)$ even are in greater proportion than the square-full numbers with $\Omega(n)$ odd.

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References

[1] Bateman, P. T., & Grosswald, E. (1958). On a theorem of Erdős and Szekeres, *Illinois J. Math.*, 2, 88–98.

[2] Djamel, B., Abdelmadjid, B., & Ōzer, Ō.(2019). On a sequence formed by iterating a divisor operator, *Czechoslovak Mathematical Journal*, 69, 1177–1196 .

[3] Erdős, P., & Szekeres, G. (1934-1935). Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem, *Acta Sci. Math. (Szeged)*, 7, 95–102.

[4] Finch, S. (2003). *Mathematical Constants*, Cambridge University Press.

[5] Golomb, S. W. (1970). Powerful numbers, *American Mathematical Monthly*, 77, 848–852.

[6] Hardy, G. H., & Ramanujan, S. (1917). The normal number of prime factors of a number n, *Quart. J. Math.*, 48, 76–92.

[7] Hardy, G. H., & Wright, E. M (1960). *An Introduction to the Theory of Numbers*, Oxford.

[8] Ivić, A. (2003). *The Riemann Zeta-Function*, Dover.

[9] Ivić, A., & Shiu, P. (1982). The distribution of powerful numbers, *Illinois J. Math.*, 26, 576–690.

[10] Jakimczuk, R. (2018). On the function $\omega(n)$, *International Mathematical Forum*, 13, 107–116.

[11] Jakimczuk, R. (2017). On the distribution of certain subsets of quadratfrei numbers, *International Mathematical Forum*, 12, 185–194.

[12] Jakimczuk, R.(2017). The kernel of powerful numbers, *International Mathematical Forum* 12, 721–730.