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Riemann zeta function and arithmetic progression of higher order

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Abstract: Riemann zeta function has a great importance in number theory, constituting one of the most studied functions. The zeta function, being a series, has a close relationship with the arithmetic progressions (AP). AP of higher order allows the understanding of several probabilities involving sequences. In this paper, we will approach Riemann zeta function with an AP of higher order. We will deduce a formula from the progression that will allow to express of the zeta function for a natural number greater than or equal to 2. In this way, we will show that the study of an AP of higher order can be very useful in the study of Riemann zeta function, and it may open other possibilities for studying the value of this function for odd numbers.
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1 Introduction

An arithmetic sequence is defined by [8] as being a never-ending list of real numbers. An example would be $1, 4, 7, 10, \ldots$ Series are very important because they are used to represent a lot function of mathematics as sine, cosine, exponential and others. An important series to mathematics is defined below.

$$S(n) = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots = \sum_{i=1}^{\infty} \frac{1}{i^n} \quad (1)$$

It was proved that this series converges if $n > 1$. Due to its importance, the $S(n)$ series was studied for a long time by different mathematicians who tried to establish advanced knowledge about its properties. Euler (1707–1783) studied this series in the 18th century and it was subsequently named zeta function by Riemann (1826–1866). In 1859, Riemann extended the definition of the Euler's zeta function to complex variables [1]. In [7], the Riemann zeta function is defined through the following identity:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} . \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \, , R(s) > 1. \quad (2)$$

The Riemann zeta function has a fundamental application in mathematics, appearing in other areas of knowledge as in problems of regularization in physics, field theory, Stefan–Boltzmann law, Debye model for two dimensions and also in nuclear magnetic resonance and magnetic resonance by [4]. The Stefan–Boltzmann law that measures the total energy radiated by a blackbody is given by

$$u = \frac{48\pi k^4 T^4}{c^3 . h^3} . \zeta(4).$$

The zeta function, to be more precise $\zeta(5)$, also appears in the Bloch–Gruneissen approximation for resistance in a monovalent metal. Another utility of the zeta function is in the quantum theory of transport effects – thermal and electrical conductivity [2].

A proof of the following well-known formula can be found, for instance, in [6]:

$$\zeta(2n) = \frac{(-1)^{n+1} . (2\pi)^{2n} . B_{2n}}{2 . (2n)!} \quad (3)$$

In the above formula, the values of B_{2n} are known as Bernoulli numbers, whose first values are $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}$. It is known that for every odd n greater than 1, B_n is null. Based on the above formula, it is possible to write $\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}$ and so on. On the other hand, the problem of expressing the value of the Riemann zeta function for odd integers remains open. Thus, it is not known with certainty whether the value, for example of $\zeta(3)$would be a function of π or any other known constant, like for instance e or γ. The reciprocal of $\zeta(3)$ is the probability that three randomly chosen positive integers are prime to each other.

Although the algebraic representation of $\zeta(3), \zeta(5), \zeta(7)$ etc. is unknown, it is known that many of these values are irrational as shown in [5], which indicates that an infinity of numbers of the form $\zeta(2m + 1)$ for m integers are irrational. In [9] it is showed that at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11), \zeta(13), \zeta(15), \zeta(17), \zeta(19)$ and $\zeta(21)$ is irrational. The irrationality of $\zeta(3)$ has been proven by Apery in 1977.

The difference operator (Δ) is the difference between any term in a sequence from the second and its predecessor ($\Delta = a_{n+1} - a_n$). As it puts us [3], if the difference operator is constant, then the sequence is an arithmetic progression (AP). Thus, based on the difference operator, it is possible to define the order of an AP. A sequence would be an AP of order 2 if the difference between the terms is an AP of non-constant terms (if they are constants, we would have an AP of order 1). As an example, the sequence of the squares of the natural numbers $(0, 1, 4, 9, 16, 25, \dots)$ is a second-order AP since $(1 - 0, 4 - 1, 9 - 4, 16 - 9, 25 - 16, \dots) = (1, 3, 5, 7, 9, \dots)$ forms an AP. An example of third-order AP is the sequence $S = (0, 6, 24, 60, 120, \dots)$ because the differences operators are $\Delta 1$ $(6, 18, 36, 60, \dots)$ and $\Delta 2 = (12, 18, 24, \dots)$ and the latter sequence forms an AP of non-constant terms.

Thus, the objective of this work is to deepen the notion of higher order AP and to associate it with the Riemann zeta function.

2 Results

Consider the sequence $1^x, 2^x, 3^x, 4^x, 5^x, \dots$ Let us study its behavior for some positive integer values x. The justification for not working with 0 is obvious: we would have the sequence $(1, 1, 1, 1, 1, \dots)$ that is not interesting to us. Initially, we will study the behaviour of this sequence for positive integers $(1, 2, 3, \dots)$.

- $x = 1$: $(1, 2, 3, 4, 5, \dots)$. We can observe that this is an AP of ratio $r = 1$ and first term $a_1 = 1$. It is a first-order AP.
- $x = 2$: $(1^2, 2^2, 3^2, 4^2, 5^2, \dots) \equiv (1, 4, 9, 16, 25, \dots)$. If we calculate the successive differences between the terms of this sequence, we will obtain a new sequence $(4 - 1, 9 - 4, 16 - 9, 25 - 16, \dots) \equiv (3, 5, 7, 9, \dots)$, which is a AP of ratio $r = 2$ and first term $a_1 = 3$. Note that after a successive difference, we found an AP and, therefore, it is a second-order AP.
- $x = 3$: $(1^3, 2^3, 3^3, 4^3, 5^3, \dots) \equiv (1, 8, 27, 64, 125, \dots)$. Let us calculate the successive differences, obtaining $(7, 19, 37, 61, \dots)$. This sequence is not an AP. But let us again make the successive differences and obtain $(12, 18, 24, \dots)$. After two successive differences, we found an AP of ratio $r = 6$ and first term $a_1 = 12$. Therefore, it is a third-order AP.
- $x = 4$: $(1^4, 2^4, 3^4, 4^4, 5^4, 6^4 \dots) \equiv (1, 16, 81, 256, 625, 1296 \dots)$. When we calculate the first successive differences, we obtain the sequence $(15, 65, 175, 369, 671 \dots)$, which is not an AP. So let us calculate the successive differences the second time to get $(50, 110, 194, 302 \dots)$, which is not yet an AP. We will calculate the successive differences for the third time to find the sequence $(60, 84, 108, 132, \dots)$, and finally we can find an AP of ratio $r = 24$ and first term $a_1 = 60$. This AP is of fourth order.

As can be seen, the ratio of AP associated with the series $1^x, 2^x, 3^x, 4^x, 5^x, \dots$ is equal to the factorial of x $(r = x!)$. On the other hand, the first terms were $1, 3, 12, 60, \dots$ If we multiply each of these values by 2, we obtain the series $(2, 6, 24, 120, \dots)$, which are the factorials $(2!, 3!, 4!, 5!, \dots)$. Thus, we conclude that the term $a_1 = \frac{(x+1)!}{2}$.

What will happen if we use some negative integer values x? First, let us see what happens when we replace $x = -2$ and calculate the successive differences.

$$x = -2: \ 1/1, 1/4, 1/9, 1/16, \dots$$

Successive differences: $(-3/4, -5/36, -7/144, \dots) \equiv \left(-\frac{3}{(1.2)^2}, -\frac{5}{(2.3)^2}, -\frac{7}{(3.4)^2}, \dots\right)$. Calling up the successive differences by $d_1 = -\frac{3}{(1.2)^2}$, $d_2 = -\frac{5}{(2.3)^2}$, $d_3 = -\frac{7}{(3.4)^2}$ and so on, we note that:

$$a_1 = 1$$
$$a_2 = \frac{1}{4} = 1 - \frac{3}{(1.2)^2}$$
$$a_3 = \frac{1}{9} = \frac{1}{4} - \frac{5}{(2.3)^2} = 1 - \frac{3}{(1.2)^2} - \frac{5}{(2.3)^2}$$
$$a_4 = \frac{1}{16} = \frac{1}{9} - \frac{7}{(3.4)^2} = 1 - \frac{3}{(1.2)^2} - \frac{5}{(2.3)^2} - \frac{7}{(3.4)^2}$$

Continuing this infinitely, we can observe that the numbers $3, 5, 7, \ldots$ are the same terms of the AP associated to the sequence $(1^2, 2^2, 3^2, 4^2, 5^2, \ldots)$ as we saw previously. We can note that $\zeta(2) = a_1 + a_2 + a_3 + a_4 + \cdots$ Let us define a formula for the determination of the value of the zeta function of 2. For this, we calculate partial sums S_n.

$S_1 = 1 = a_1 = 1.$

$S_2 = 1 + \frac{1}{4} = a_1 + a_2 = 1 + 1 - \frac{3}{(1.2)^2} = 2 - 1.\frac{3}{(1.2)^2}$

$S_3 = 1 + \frac{1}{4} + \frac{1}{9} = a_1 + a_2 + a_3 = 1 + 1 - \frac{3}{(1.2)^2} + 1 - \frac{3}{(1.2)^2} - \frac{5}{(2.3)^2} = 3 - 2.\frac{3}{(1.2)^2} - 1.\frac{5}{(2.3)^2}$

$S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = a_1 + a_2 + a_3 + a_4 = 1 + 1 - \frac{3}{(1.2)^2} + 1 - \frac{3}{(1.2)^2} - \frac{5}{(2.3)^2} + 1 - \frac{3}{(1.2)^2} - \frac{5}{(2.3)^2} - \frac{7}{(3.4)^2} = 4 - 3.\frac{3}{(1.2)^2} - 2.\frac{5}{(2.3)^2} - 1.\frac{7}{(3.4)^2}$

If we do this to infinity, we have $S_n = 1 + 1/4 + 1/9 + 1/16 + 1/25 + \cdots 1/n^2$, that is, S_n will match the function $\zeta(2)$ for large n enough. We can observe from the above results that the sum S_n always starts with n and the other terms are negative. The negative terms have as coefficient the decreasing numbers from $-1,\ n-2, \ldots, 1$. In addition, the numerators are the terms of the AP $(3,5,7,\ldots)$. Therefore, we define a general formula for S_n and, consequently, for $\zeta(2)$.

$$S_n = n - (n-1).\frac{3}{(1.2)^2} - (n-2).\frac{5}{(2.3)^2} - (n-3).\frac{7}{(3.4)^2} - \cdots - 1.\frac{(2n-1)}{[n(n-1)]^2} \qquad (4)$$

For the AP $(3,5,7,\ldots)$, we see that 3 is the first term, 5 is the second term, 7 is the third term, etc., its general term is $c_k = 1 + 2k$.

$$S_n = n - (n-1).\frac{c_1}{(1.2)^2} - (n-2).\frac{c_2}{(2.3)^2} - (n-3).\frac{c_3}{(3.4)^2} - \cdots - 1.\frac{c_{n-1}}{[n(n-1)]^2} \qquad (5)$$

$$S_n = n - \sum_{k=1}^{n-1} \frac{(n-k).c_k}{[k.(k+1)]^2} = n - \sum_{k=1}^{n-1} \frac{(n-k).(1+2k)}{k^2(k+1)^2} \qquad (6)$$

For S_n to be numerically equal to the function $\zeta(2)$, it is necessary that n is large enough, that is, S_n approaches $\zeta(2)$ when n tends to infinity. Therefore:

$$\zeta(2) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[n - \sum_{k=1}^{n-1} \frac{(n-k)}{k^2(k+1)^2}.(1+2k)\right] \qquad (7)$$

Proceeding from the previous form for $x = -3$ we have:

$$x = -3:\ 1/1, 1/8, 1/27, 1/64, \ldots$$

First successive differences: $\left(-\frac{7}{(1.2)^3}, -\frac{19}{(2.3)^3}, -\frac{37}{(3.4)^3}, \ldots\right)$. Calling the successive differences by $d_1 = -\frac{7}{(1.2)^3}$, $d_2 = -\frac{19}{(2.3)^3}$, $d_3 = -\frac{37}{(3.4)^3}$ and so on, we note that:

$$a_1 = 1$$
$$a_2 = \frac{1}{8} = 1 - \frac{7}{(1.2)^3}$$
$$a_3 = \frac{1}{27} = \frac{1}{8} - \frac{19}{(2.3)^3} = 1 - \frac{7}{(1.2)^3} - \frac{19}{(2.3)^3}$$
$$a_4 = \frac{1}{64} = \frac{1}{27} - \frac{37}{(3.4)^3} = 1 - \frac{7}{(1.2)^3} - \frac{19}{(2.3)^3} - \frac{37}{(3.4)^3}$$

When we calculate the successive differences of the sequence $(1^3, 2^3, 3^3, 4^3, 5^3, \ldots)$ we obtain $(7, 19, 37, 61, \ldots)$ in the first difference and this is already well evidenced that the numerators present in the fractions of terms a_n above are just the same terms as the first successive difference of $(1^3, 2^3, 3^3, 4^3, 5^3, \ldots)$. In addition, we obtain the AP $(12, 18, 24, \ \ldots)$ as being associated with $(1^3, 2^3, 3^3, 4^3, 5^3, \ldots)$, so that

$$19 = 7 + 12$$
$$37 = 19 + 18 = 7 + 12 + 18$$
$$61 = 37 + 24 = 7 + 12 + 18 + 24$$

This shows that the numerators of the fractions of the terms above are expressed by $c_k = 7 + A_{k-1}$, where A_{k-1} is the sum of the $k-1$ terms of the AP $(12, 18, 24, \ \ldots)$. Given all of this, we conclude, using the formula of the sum of the terms of an AP, that $A_{k-1} = 3(k+2)(k-1)$ and $c_k = 7 + 3(k+2)(k-1) = 1 + 3k + 3k^2$. Calculating the partial sums, we have:

$S_1 = 1 = a_1 = 1$

$S_2 = 1 + \frac{1}{8} = a_1 + a_2 = 1 + 1 - \frac{7}{(1.2)^3} = 2 - 1.\frac{7}{(1.2)^3}$

$S_3 = 1 + \frac{1}{8} + \frac{1}{27} = a_1 + a_2 + a_3 = 1 + 1 - \frac{7}{(1.2)^3} + 1 - \frac{7}{(1.2)^3} - \frac{19}{(2.3)^3} = 3 - 2.\frac{7}{(1.2)^3} - 1.\frac{19}{(2.3)^3}$

$S_4 = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} = a_1 + a_2 + a_3 + a_4$

$= 1 + 1 - \frac{7}{(1.2)^3} + 1 - \frac{7}{(1.2)^3} - \frac{19}{(2.3)^3} + 1 - \frac{7}{(1.2)^3} - \frac{19}{(2.3)^3} - \frac{37}{(3.4)^3} = 4 - 3.\frac{7}{(1.2)^3} - 2.\frac{19}{(2.3)^3} - 1.\frac{37}{(3.4)^3}$

Using the same previous reasoning for $\zeta(2)$, we have:

$$S_n = n - (n-1).\frac{7}{(1.2)^3} - (n-2).\frac{19}{(2.3)^3} - (n-3).\frac{37}{(3.4)^3} - \cdots - 1.\frac{(1-3n+3n^2)}{[n(n-1)]^3} \qquad (8)$$

$$S_n = n - (n-1).\frac{c_1}{(1.2)^3} - (n-2).\frac{c_2}{(2.3)^3} - (n-3).\frac{c_3}{(3.4)^3} - \cdots - 1.\frac{c_{n-1}}{[n(n-1)]^3} \qquad (9)$$

$$S_n = n - \sum_{k=1}^{n-1} \frac{(n-k).c_k}{[k.(k+1)]^3} = n - \sum_{k=1}^{n-1} \frac{(n-k).(1+3k+3k^2)}{k^3(k+1)^3}. \qquad (10)$$

For S_n to be numerically equal to the function $\zeta(3)$, it is necessary that n is large enough, that is, S_n approaches $\zeta(3)$ when n tends to infinity. Therefore:

$$\zeta(3) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[n - \sum_{k=1}^{n-1} \frac{(n-k)}{k^3(k+1)^3}.(1 + 3k + 3k^2) \right]. \qquad (11)$$

It should be immediately noted that the exponents of the variable k in $1 + 2k$ and $1 + 3k + 3k^2$ grow from 0 to $n-1$. It is observed that $(1, 2)$ and $(1, 3, 3)$ are the coefficients of the triangle of Pascal except for the last number (1).

In view of this, we conclude the following theorem:

Theorem. If $s \in \mathbb{N}$, with $s \geq 2$, then the value of $\zeta(s)$ is given by the relation:

$$\zeta(s) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[n - \sum_{k=1}^{n-1} \left\{\frac{(n-k)}{k^s(k+1)^s} . \sum_{i=0}^{s-1} \binom{s}{i} . k^i\right\}\right]. \tag{12}$$

Proof:

$$\zeta(s) = \lim_{n\to\infty} \left[n - \sum_{k=1}^{n-1} \left\{\frac{(n-k)}{k^s(k+1)^s} . \sum_{i=0}^{s-1} \binom{s}{i} . k^i\right\}\right]. \tag{13}$$

The expression below is known:

$$\sum_{i=0}^{s-1} \binom{s}{i} . k^i = (k+1)^s - k^s. \tag{14}$$

Replacing (14) in (13), we have:

$$\zeta(s) = \lim_{n\to\infty} \left[n - \sum_{k=1}^{n-1} \left\{\frac{(n-k)}{k^s(k+1)^s} . ((k+1)^s - k^s)\right\}\right] \tag{15}$$

$$\zeta(s) = \lim_{n\to\infty} \left[n - \sum_{k=1}^{n-1} \left\{(n-k) . \left(\frac{1}{k^s} - \frac{1}{(k+1)^s}\right)\right\}\right] \tag{16}$$

$$\zeta(s) = \lim_{n\to\infty} \left[n - \left(n . \sum_{k=1}^{n-1} \frac{1}{k^s} - n . \sum_{k=1}^{n-1} \frac{1}{(k+1)^s} - \sum_{k=1}^{n-1} \frac{1}{k^{s-1}} + \sum_{k=1}^{n-1} \frac{k}{(k+1)^s}\right)\right] \tag{17}$$

$$\zeta(s) = \lim_{n\to\infty} \left[n - \left(n . \zeta(s) - n . (\zeta(s) - 1) - \zeta(s-1) + \zeta(s-1) - \zeta(s)\right)\right] \tag{18}$$

$$\zeta(s) = \lim_{n\to\infty} [n - n . \zeta(s) + n . \zeta(s) - n + \zeta(s)]. \tag{19}$$

Thus, we conclude that:

$$\zeta(s) = \lim_{n\to\infty} \zeta(s) = \zeta(s). \tag{20}$$

This completes the proof. □

3 Conclusion

Riemann's zeta function is of fundamental importance in number theory and is still the subject of many studies because it still provides several unsolved problems. The problem of calculating the value of this function for an odd integer is still open. In this work, we show that the arithmetic progression of high order allows to define a formula for the effective calculation of Riemann zeta function for a natural number $s \geq 2$.

References

[1] Ahlam, A., & Stromberg, F. (2017). The Riemann Zeta Function and its Analytic Continuation, *British Journal of Mathematics Computer Science*. 22 (5), 1–47.

[2] Arfken, G. B., & Weber, H. J. (2007). *Física Matemática: Métodos Matemáticos para Engenharia e Física*. 6ª edição, Campus. Trad. Arlete Simille Marques, Elsevier, Rio de Janeiro.

[3] Escada, F. A. L. (2012). *Teoria de Ramsey em progressões e recorrências de ordem superior, planificação de subunidade relativa ao tema III-sucessões reais*. Ph.D. dissertation, Universidade da Beira Interior.

[4] Mishra, D. K., Chandra, N., & Vaibhav, V. (2017). Equilibrium properties of blackbody radiation with an ultraviolet energy cut-off. *Annals of Physics*, 385, 605–622.

[5] Rivoal, T. (2000). La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 331 (4), 267–270.

[6] Simmons, G. (1987). *Cálculo com Geometria Analítica*, Vol. 2, S. Paulo: McGrawHill.

[7] Titchmarsh, E. C., Titchmarsh, E. C. T., & Heath-Brown, D. R. (1986). *The Theory of the Riemann Zeta-function*. Oxford University Press.

[8] Wassell, S. R. (2001). Arithmetic, geometric and harmonic sequences. *Nexus Network Journal*, 3 (2), 151–155.

[9] Zudilin, W. (2001). *Irrationality of values of zeta-function*. arXiv preprint math/0104249.