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An application of exponential sums over the divisor function

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Abstract: We apply the rational exponential sums over the divisor function to estimate the average of some arithmetic functions. The method of proof relies on the classic Abel's summation formula.
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1 Introduction

Let $p > 2$ be prime number and $r \in \mathbb{Z}$. Let $\tau(n)$ denote the divisor function

$$\tau(n) = \sum_{d|n} 1.$$

Let $e_p(z)$ denote the exponential function $\exp(2\pi iz/p)$.

In this paper, we shall present an application of rational exponential sums of the form

$$T_{a,p}(N) = \sum_{n=1}^{N} e_p(a\tau(n)),$$

where N is sufficiently large. The problem of rational exponential sums over the divisor function is posed by Shparlinski [9]. The congruence properties of the divisor function have been considered in a number of works, see in [4, 6, 7]. The motivation of this paper follows from the distribution of values of arithmetic function in residue classes. Namely, for a fixed function $f(n)$,

$$S(x,r) := \sum_{\substack{n\le x \\ f(n)\equiv r \pmod{m}}} 1 \tag{1}$$

for $x > 0$ and $\gcd(r,m) = 1$. In 2016, Shirokov and Gromakovskaya [8] derived the sum (1) by using Perron's formula. The sum (1) can be considered as the exponential sum

$$\sum_{j=0}^{m-1} e_m(-jr) \sum_{n\le x} e_m(jf(n)).$$

Exponential sum over an arithmetic function is a topic in the number theory (see [2, 3]). In particularly, Kerr [5] considered the rational exponential sums over the divisor function

$$T_{a,m}(N) = \sum_{n=1}^{N} e_m(a\tau(n)), \tag{2}$$

for $(a,m) = 1$ and m odd. Kerr reduced the problem of bounding (2) to bounding the sums

$$S_m(r) = \sum_{n=1}^{t} e_m(r2^n),$$

where t denotes the order of $2 \pmod{m}$.

In this paper, we shall apply Kerr's results to derive the following sums.

$$\sum_{n\le N} \tau(n,r,p) := \sum_{n\le N} \sum_{\substack{d\mid n \\ \tau(d)\equiv r \pmod{p}}} 1,$$

$$\sum_{n\le N} \sigma_s(n,r,p) := \sum_{n\le N} \sum_{\substack{d\mid n \\ \tau(d)\equiv r \pmod{p}}} \left(\frac{n}{d}\right)^s, \qquad \text{for real } s > 0,$$

$$\sum_{n\le N} \phi(n,r,p) := \sum_{n\le N} \sum_{\substack{d\mid n \\ \tau(d)\equiv r \pmod{p}}} \mu(d)\frac{n}{d},$$

where μ denotes the Möbius function.

Here and in the sequel we shall use the notation

$$H(i,p) = \sum_{\substack{q\in Q_p \\ \tau(q)\equiv i \pmod{p}}} \frac{h(q)}{q}, \qquad h(q) = \prod_{p\mid q} \left(1+\frac{1}{p}\right)^{-1}$$

and Q_p denotes the set of integers n, such that if any prime $q^\theta || n$, then $2 \leq \theta \leq p-1$. Let t denote the order of $2 \pmod p$ and define $\alpha_t = 1 - \cos(2\pi/t)$. Denote

$$C(j,p) = \frac{6\zeta(p)}{\pi^2 p}\left(\sum_{i=0}^{p-1} H(i,p) S_p(ji)\right)$$

and

$$\delta(r,p) = \sum_{j=1}^{p-1} e_p\left(-jr\right) C(j,p).$$

We shall prove the following results.

Theorem 1.1. *Let $p \geq 3$ be prime, let $r \in \mathbb{Z}$. Then, for a positive integer N,*

$$\sum_{n \leq N} \tau(n,r,p) = \frac{N}{p}(\log N + 2\gamma - 1) + \delta(r,p)(\log N + \gamma - 1)\frac{N}{p} + O\left(\sqrt{N} + N(\log N)^{-\alpha_t}\right).$$

Theorem 1.2. *Let $p \geq 3$ be prime, let $r \in \mathbb{Z}$ and $s > 0$ be a real number. Then, for a positive integer N,*

$$\sum_{n \leq N} \sigma_s(n,r,p) = \frac{\zeta(s+1)N^{s+1}}{(s+1)p} + \frac{N^{s+1}}{sp}\delta(r,p) + \begin{cases} O(N\log N), & \text{if } s = 1, \\ O(N^z), & \text{if } s \neq 1, \end{cases}$$

where $z = \max(1,s)$.

Theorem 1.3. *Let $p \geq 3$ be prime, let $r \in \mathbb{Z}$. Then, for a positive integer N,*

$$\sum_{n \leq N} \phi(n,r,p) = \frac{3N^2}{p\pi^2} + \frac{N^2}{p}\sum_{j=1}^{p-1} e_p\left(-jr\right)\rho_j + O\left(N\log N\right),$$

where for $j = 1, 2, ..., p-1$

$$\rho_j = \int_1^\infty \sum_{d \leq u} \mu(d) e_p\left(j\tau(d)\right)\frac{du}{u^3}.$$

2 Lemmas

The following theorem and lemmas will be used in our results.

Theorem 2.1 ([5, Theorem 2])**.** *Suppose $p \geq 3$ is prime. Then we have*

$$\sum_{n=1}^{N} e_p(a\tau(n)) = C(a,p)N + O\left(pN(\log N)^{-\alpha_t - 1}\right).$$

Lemma 2.2 ([1, Theorem 3.2])**.** *If* $x \geq 1$ *we have:*

$$\sum_{n\leq x}\frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right), \ \gamma \text{ is Euler's constant}. \tag{3}$$

$$\sum_{n\leq x}\frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O\left(\frac{1}{x^s}\right) \ \text{ if } s>0, s\neq 1. \tag{4}$$

$$\sum_{n>x}\frac{1}{n^s} = O\left(\frac{1}{x^{1-s}}\right) \ \text{ if } s>1. \tag{5}$$

Lemma 2.3 ([1, Theorem 8.1])**.**

$$\frac{1}{m}\sum_{j=0}^{m-1} e\left(\frac{jn}{m}\right) = \begin{cases} 1 & \text{if } m|n, \\ 0 & \text{otherwise.}\end{cases} \tag{6}$$

3 Proof of Theorem 1.1

Proof. Given N large enough, we have

$$\begin{aligned}\sum_{n\leq N}\tau(n,r,p) &= \sum_{\substack{d,l \\ dl\leq N \\ \tau(d)\equiv r \pmod p}} 1 \\ &= S_1 + S_2 - S_3,\end{aligned} \tag{7}$$

where

$$S_1 = \sum_{\substack{d\leq\sqrt{N} \\ \tau(d)\equiv r \pmod p}} \sum_{l\leq N/d} 1, \qquad S_2 = \sum_{l\leq\sqrt{N}} \sum_{\substack{d\leq N/l \\ \tau(d)\equiv r \pmod p}} 1$$

and

$$S_3 = \sum_{\substack{d\leq\sqrt{N} \\ \tau(d)\equiv r \pmod p}} \sum_{l\leq\sqrt{N}} 1.$$

In view of the relation (6), we obtain

$$\begin{aligned} S_1 &= \sum_{\substack{d\leq\sqrt{N} \\ \tau(d)\equiv r \pmod p}} \left\lfloor \frac{N}{d}\right\rfloor \\ &= \frac{1}{p}\sum_{d\leq\sqrt{N}}\left\lfloor\frac{N}{d}\right\rfloor \sum_{0\leq j\leq p-1} e_p\left(j(\tau(d)-r)\right) \\ &= \frac{N}{p}\sum_{d\leq\sqrt{N}}\frac{1}{d} + \frac{N}{p}\sum_{1\leq j\leq p-1} e_p\left(-jr\right)A_j(N) + O(\sqrt{N}), \end{aligned} \tag{8}$$

where $j = 1, 2, \ldots, p-1$

$$A_j(N) = \sum_{d \le \sqrt{N}} \frac{e_p\left(j\tau(d)\right)}{d}.$$

From (3), the first sum of (8) is

$$\frac{1}{2p} N \log N + \frac{\gamma N}{p} + O\left(\sqrt{N}\right).$$

Now we use Abel's summation formula for $A_j(N)$ and obtain

$$\begin{aligned} A_j(N) &= \sum_{d \le \sqrt{N}} \frac{e_p\left(j\tau(d)\right)}{d} \\ &= \frac{1}{\sqrt{N}} \sum_{d \le \sqrt{N}} e_p\left(j\tau(d)\right) + \int_1^{\sqrt{N}} \left(\sum_{d \le u} e_p\left(j\tau(d)\right)\right) \frac{du}{u^2}, \end{aligned}$$

In view of Theorem 2.1, we have, for $j = 1, 2, \ldots, p-1$

$$\begin{aligned} A_j(N) &= \frac{1}{\sqrt{N}} \left(C(j,p)\sqrt{N} + O\left(p\sqrt{N}(\log N)^{-\alpha_t - 1}\right)\right) \\ &\quad + \int_1^{\sqrt{N}} \left(C(j,p)u + O\left(pu(\log u)^{-\alpha_t - 1}\right)\right) \frac{du}{u^2}, \\ &= C(j,p)\frac{\log N}{2} + O\left(p(\log N)^{-\alpha_t}\right). \end{aligned}$$

Thus, the second sum is

$$\frac{1}{2p} \sum_{1 \le j \le p-1} e_p\left(-jr\right) C(j,p) N \log N + O\left(N(\log N)^{-\alpha_t}\right).$$

Thus, we have

$$S_1 = \frac{1}{2p} N \log N + \frac{\gamma N}{p} + \frac{\delta(r,p)}{2p} N \log N + O\left(N^{1/2} + N(\log N)^{-\alpha_t}\right).$$

For S_2, we use again the relation (6) and (3). Then

$$\begin{aligned} S_2 &= \frac{1}{p} \sum_{l \le \sqrt{N}} \sum_{d \le N/l} \sum_{j=0}^{p-1} e_p\left(j(\tau(d) - r)\right) \\ &= \frac{1}{p} \sum_{l \le \sqrt{N}} \left(\frac{N}{l} + O(1)\right) + \frac{1}{p} \sum_{l \le \sqrt{N}} \sum_{d \le N/l} \sum_{j=1}^{p-1} e_p\left(j(\tau(d) - r)\right) \\ &= \frac{1}{2p} N \log N + \frac{\gamma N}{p} + O\left(\sqrt{N}\right) + \frac{1}{p} \sum_{j=1}^{p-1} e_p\left(-jr\right) \sum_{l \le \sqrt{N}} \sum_{d \le N/l} e_p\left(j\tau(d)\right). \end{aligned}$$

We apply Theorem 2.1 to the last inner sum and (3) again. Thus, we have

$$\begin{aligned}
S_2 &= \frac{1}{2p} N \log N + \frac{\gamma N}{p} + O\left(\sqrt{N}\right) \\
&\quad + \frac{1}{p} \sum_{j=1}^{p-1} e_p(-jr) \sum_{l \le \sqrt{N}} \left(C(j,p) \frac{N}{l} + O\left(p \frac{N}{l} (\log \frac{N}{l})^{-\alpha_t - 1} \right) \right) \\
&= \frac{1}{2p} N \log N + \frac{\gamma N}{p} + \frac{\delta(r,p)}{p} N \left(\frac{\log N}{2} + \gamma \right) + O\left(\sqrt{N} + N (\log N)^{-\alpha_t} \right).
\end{aligned}$$

For S_3, we use the relation (6) and Theorem 2.1. We have

$$\begin{aligned}
S_3 &= \sum_{\substack{d \le \sqrt{N} \\ \tau(d) \equiv r \pmod p}} \sum_{l \le \sqrt{N}} 1 = \sum_{\substack{d \le \sqrt{N} \\ \tau(d) \equiv r \pmod p}} \left(\sqrt{N} + O(1) \right) \\
&= \frac{1}{p} \sum_{d \le \sqrt{N}} \left(\sqrt{N} + O(1) \right) + \frac{\sqrt{N} + O(1)}{p} \sum_{j=1}^{p-1} e_p(-jr) \sum_{d \le \sqrt{N}} e_p(j\tau(d)) \\
&= \frac{N}{p} + O\left(\sqrt{N}\right) + \frac{\sqrt{N} + O(1)}{p} \sum_{j=1}^{p-1} e_p(-jr) \left(C(j,p)\sqrt{N} + O\left(p\sqrt{N} (\log N)^{-\alpha_t - 1} \right) \right) \\
&= \frac{N}{p} + \frac{\delta(r,p) N}{p} + O\left(\sqrt{N} + N (\log N)^{-\alpha_t - 1} \right).
\end{aligned}$$

Substituting S_1, S_2 and S_3 in (7), then the proof is complete. □

4 Proof of Theorem 1.2

Proof. Throughout the proof, we detect the congruences by using the relation (6). Let $s > 0$ be a real number. Given N large enough, in view of (4), we have

$$\begin{aligned}
\sum_{n \le N} \sigma_s(n,r,p) &= \sum_{\substack{d,l \\ dl \le N \\ \tau(d) \equiv r \pmod p}} \left(\frac{n}{d} \right)^s = \sum_{\substack{d \le N \\ \tau(d) \equiv r \pmod p}} \sum_{l \le N/d} l^s \\
&= \sum_{\substack{d \le N \\ \tau(d) \equiv r \pmod p}} \left(\frac{(N/d)^{s+1}}{s+1} + O\left(\frac{N^s}{d^s} \right) \right).
\end{aligned}$$

Here, we shall consider separately the sum into two subcases.

If $s = 1$,

$$\begin{aligned}
\sum_{n \le N} \sigma_1(n,r,p) &= \frac{N^2}{2} \sum_{\substack{d \le N \\ \tau(d) \equiv r \pmod p}} \frac{1}{d^2} + O\left(N \sum_{d \le N} \frac{1}{d} \right) \\
&= \frac{N^2}{2p} \sum_{j=0}^{p-1} e_p(-jr) \sum_{d \le N} \frac{e_p(j\tau(d))}{d^2} + O(N \log N) \\
&= \frac{N^2}{2p} \sum_{d \le N} \frac{1}{d^2} + \frac{N^2}{2p} \sum_{j=1}^{p-1} e_p(-jr) \sum_{d \le N} \frac{e_p(j\tau(d))}{d^2} + O(N \log N).
\end{aligned}$$

The big-oh term follows from (3). In view of (4), the first term is

$$\frac{N^2}{2p}\left(\frac{-1}{N}+\zeta(2)+O(N^{-2})\right)=\frac{\zeta(2)N^2-N}{2p}+O(1).$$

We apply Abel's formula and Theorem 2.1. The second term is

$$\begin{aligned}
&\frac{N^2}{2p}\sum_{j=1}^{p-1}e_p\left(-jr\right)\Big(\frac{1}{N^2}\sum_{d\le N}e_p\left(j\tau(d)\right)+2\int_1^N\Big(\sum_{d\le u}e_p\left(j\tau(d)\right)\frac{du}{u^3}\Big)\Big)\\
&=\frac{N^2}{2p}\sum_{j=1}^{p-1}e_p\left(-jr\right)\Big(\frac{1}{N^2}\left(C(j,p)N+O(N(\log N)^{-\alpha-1})\right)\\
&\quad+2\int_1^N\Big(\left(C(j,p)u+O(u(\log u)^{-\alpha-1})\right)\frac{du}{u^3}\Big)\Big)\\
&=\frac{2N^2-N}{2p}\sum_{j=1}^{p-1}e_p\left(-jr\right)C(j,p)+O(N(\log N)^{-\alpha-1}).
\end{aligned}$$

Collecting all, we have

$$\sum_{n\le N}\sigma_1(n,r,p)=\frac{\zeta(2)N^2}{2p}+\delta(r,p)\frac{N^2}{p}+O(N\log N).$$

Next, if $s\neq 1$, we set $z=\max(1,s)$. In view of (4), we have

$$\begin{aligned}
\sum_{n\le N}\sigma_s(n,r,p)&=\frac{N^{s+1}}{s+1}\sum_{\substack{d\le N\\ \tau(d)\equiv r \pmod p}}\frac{1}{d^{s+1}}+O\left(N^s\sum_{d\le N}\frac{1}{d^s}\right)\\
&=\frac{N^{s+1}}{(s+1)p}\sum_{j=0}^{p-1}e_p\left(-jr\right)\sum_{d\le N}\frac{e_p\left(j\tau(d)\right)}{d^{s+1}}+O\bigg(N^s\left(\frac{N^{1-s}}{1-s}+\zeta(s)+O(N^{-s})\right)\bigg)\\
&=\frac{N^{s+1}}{(s+1)p}\sum_{d\le N}\frac{1}{d^s}+\frac{N^2}{2p}\sum_{j=1}^{p-1}e_p\left(-jr\right)\sum_{d\le N}\frac{e_p\left(j\tau(d)\right)}{d^2}+O(N^z).
\end{aligned}$$

By same calculation as in the case $s=1$, we have

$$\sum_{n\le N}\sigma_s(n,r,p)=\frac{\zeta(s+1)N^{s+1}}{(s+1)p}+\delta(r,p)\frac{N^{s+1}}{sp}+O(N^z).$$

The complete proof follows from the summarizing both cases. □

5 Proof of Theorem 1.3

Proof. Throughout the proof, we detect the congruences by using the relation (6). Given N large enough, we have

$$\begin{aligned}\sum_{n\le N}\phi(n,r,p) &= \sum_{\substack{d,l\\ dl\le N\\ \tau(d)\equiv r \pmod p}} \mu(d)\frac{n}{d} = \sum_{\substack{d\le N\\ \tau(d)\equiv r \pmod p}} \mu(d)\sum_{l\le N/d} l\\ &= \sum_{\substack{d\le N\\ \tau(d)\equiv r \pmod p}} \mu(d)\left(\frac{N^2}{2d^2}+O\left(\frac{N}{d}\right)\right)\\ &= \frac{N^2}{2p}\sum_{j=0}^{p-1} e_p\left(-jr\right)\sum_{d\le N}\frac{\mu(d)e_p\left(j\tau(d)\right)}{d^2}+O\left(N\log N\right)\\ &= \frac{N^2}{2p}\sum_{d\le N}\frac{\mu(d)}{d^2}+\frac{N^2}{2p}\sum_{j=1}^{p-1}e_p\left(-jr\right)\sum_{d\le N}\mu(d)\frac{e_p\left(j\tau(d)\right)}{d^2}+O\left(N\log N\right).\end{aligned}$$

We apply the well-known identity

$$\sum_{d\le N}\frac{\mu(d)}{d^2}=\frac{6}{\pi^2}+O(N)$$

and Abel's summation formula and obtain

$$\begin{aligned}\sum_{n\le N}\phi(n,r,p) = \frac{3N^2}{p\pi^2}+\frac{N^2}{2p}\sum_{j=1}^{p-1}e_p\left(-jr\right)\left(\frac{1}{N^2}\sum_{d\le N}\mu(d)e_p\left(j\tau(d)\right)+2\int_1^N\sum_{d\le u}\mu(d)e_p\left(j\tau(d)\right)\frac{du}{u^3}\right)\\ +O\left(N\log N\right).\end{aligned}$$

Since

$$\sum_{d\le N}\mu(d)e_p\left(j\tau(d)\right)=O(N),$$

and (5) the integral is trivially convergent. It follows

$$\int_N^\infty\sum_{d\le u}\mu(d)e_p\left(j\tau(d)\right)\frac{du}{u^3}=O(N^{-1}).$$

Thus

$$\int_1^N\sum_{d\le u}\mu(d)e_p\left(j\tau(d)\right)\frac{du}{u^3}=\int_1^\infty\sum_{d\le u}\mu(d)e_p\left(j\tau(d)\right)\frac{du}{u^3}+O(N^{-1})$$

and

$$\left(\frac{1}{N^2}\sum_{d\le N}\mu(d)e_p\left(j\tau(d)\right)\right)=O(N^{-1}).$$

Collecting all, we have

$$\sum_{n\le N}\phi(n,r,p)=\frac{3N^2}{p\pi^2}+\frac{N^2}{p}\sum_{j=1}^{p-1}e_p\left(-jr\right)\rho_j+O\left(N\log N\right),$$

where, for $j = 1, 2, ..., p-1$

$$\rho_j = \int_1^{\infty} \sum_{d \le u} \mu(d) e_p\left(j\tau(d)\right) \frac{du}{u^3}. \qquad \square$$

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