

On generalized bicomplex k -Fibonacci numbers

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Abstract: In this paper, we introduce the generalized bicomplex k -Fibonacci numbers. We also give the generating function and Binet's formula for these numbers. In addition, we obtain some identities such as Honsberger, d'Ocagne's, Catalan's, and Cassini's identities involving the generalized bicomplex k -Fibonacci numbers.

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1 Introduction

Bicomplex numbers were first introduced by Corrado Segre [13] in 1892. Bicomplex numbers, just like quaternions, are a generalization of complex numbers by four real numbers. However, there are two differences between quaternions and bicomplex numbers: First one; quaternions form a division algebra, but bicomplex numbers do not form a division algebra. Secondly; quaternions are non-commutative, whereas bicomplex numbers are commutative.

A bicomplex number x is of the form

$$x = x_0 + x_1i + x_2j + x_3ij = (x_0 + x_1i) + (x_2 + x_3i)j,$$

where x_0, x_1, x_2 and x_3 are real numbers, i and j are imaginary units which satisfy the equalities

$$i^2 = -1, j^2 = -1, ij = ji. \quad (1)$$

The set of bicomplex numbers is denoted by \mathbb{C}_2 . For more details, one can see, for example [9].

In 2015, Karakus and Aksoyak [8] defined a generalized bicomplex number p as follows:

$$p = p_0 + p_1i + p_2j + p_3ij = (p_0 + p_1i) + (p_2 + p_3i)j,$$

where p_0, p_1, p_2 and p_3 are real numbers, i and j are imaginary units which satisfy the equalities

$$i^2 = -\alpha, j^2 = -\beta, (ij)^2 = \alpha\beta, ij = ji,$$

where α and β are real numbers.

The set of generalized bicomplex numbers is denoted by $\mathbb{C}_{\alpha\beta}$.

Let $p = p_0 + p_1i + p_2j + p_3ij$ and $q = q_0 + q_1i + q_2j + q_3ij$ be two generalized bicomplex numbers. Then the addition of two generalized bicomplex numbers is defined as

$$p + q = (p_0 + q_0) + (p_1 + q_1)i + (p_2 + q_2)j + (p_3 + q_3)ij.$$

The multiplication of a generalized bicomplex number by a real scalar λ is defined as

$$\lambda p = \lambda p_0 + \lambda p_1i + \lambda p_2j + \lambda p_3ij.$$

The multiplication of two generalized bicomplex numbers is defined as

$$\begin{aligned} p \times q = & (p_0q_0 - \alpha p_1q_1 - \beta p_2q_2 + \alpha\beta p_3q_3) + (p_0q_1 + p_1q_0 - \beta p_2q_3 - \beta p_3q_2)i \\ & + (p_0q_2 + p_2q_0 - \alpha p_1q_3 - \alpha p_3q_1)j + (p_0q_3 + p_3q_0 + p_1q_2 + p_2q_1)ij. \end{aligned}$$

Moreover, three different conjugations for generalized bicomplex numbers are given by

$$\begin{aligned} p^{\dagger i} &= (p_0 - p_1i) + (p_2 - p_3i)j, \\ p^{\dagger j} &= (p_0 + p_1i) - (p_2 + p_3i)j, \\ p^{\dagger ij} &= (p_0 - p_1i) - (p_2 - p_3i)j. \end{aligned}$$

Accordingly, we can write (cf. [8])

$$\begin{aligned} p \times p^{\dagger i} &= (p_0^2 + \alpha p_1^2 - \beta p_2^2 - \alpha\beta p_3^2) + 2(p_0p_2 + \alpha p_1p_3)j, \\ p \times p^{\dagger j} &= (p_0^2 - \alpha p_1^2 + \beta p_2^2 - \alpha\beta p_3^2) + 2(p_0p_1 + \beta p_2p_3)i, \\ p \times p^{\dagger ij} &= (p_0^2 + \alpha p_1^2 + \beta p_2^2 + \alpha\beta p_3^2) + 2(p_0p_3 - p_1p_2)ij. \end{aligned}$$

The sequence of Fibonacci numbers, denoted by $\{F_n\}_{n=0}^{\infty}$, is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

Fibonacci numbers have been widely used in science, engineering, art and architecture. In literature, Fibonacci numbers have been studied and also generalized by many authors in many ways. One of the generalization of these numbers is k -Fibonacci numbers introduced by Falcon and Plaza [6]. Additionally, in [7], the authors studied k -Fibonacci numbers.

For any positive real number k , the sequence of k -Fibonacci numbers, denoted by $\{F_{k,n}\}_{n=0}^{\infty}$, is defined by the recurrence relation

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad n \geq 2$$

with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$.

For $k = 1$, we obtain the classical Fibonacci numbers.

The n -th term of the sequence $\{F_{k,n}\}_{n=0}^{\infty}$ is given by

$$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad (2)$$

where $\alpha = \frac{k + \sqrt{k^2 + 4}}{2}$, $\beta = \frac{k - \sqrt{k^2 + 4}}{2}$ are roots of the equation $t^2 - kt - 1 = 0$.

Moreover, the generating function for the sequence $\{F_{k,n}\}_{n=0}^{\infty}$ is given by

$$f_k(t) = \frac{t}{1 - kt - t^2}.$$

There are several studies on k -Fibonacci numbers in literature. For example, Bolat and Kose [4] investigated some properties of k -Fibonacci numbers. In addition, Catarino [5] gave some identities involving k -Fibonacci numbers. Moreover, Ramirez [12] defined and studied k -Fibonacci and k -Lucas quaternions. Thereafter, Polatli et al. [11] defined split k -Fibonacci and k -Lucas quaternions. Furthermore, Bilgici et al. [3] introduced k -Fibonacci and k -Lucas generalized quaternions. Additionally, Aydin [2] defined k -Fibonacci dual quaternions.

In [10], Nurkan and Guven introduced the bicomplex Fibonacci and Lucas numbers, respectively, as follows:

$$\begin{aligned} BF_n &= F_n + F_{n+1}i + F_{n+2}j + F_{n+3}ij, \\ BL_n &= L_n + L_{n+1}i + L_{n+2}j + L_{n+3}ij, \end{aligned}$$

where F_n is the n -th Fibonacci number, L_n is the n -th Lucas number, i and j are imaginary units which satisfy the Eq. (1).

They gave the Binet's formulas for bicomplex Fibonacci and Lucas numbers, respectively, as

$$\begin{aligned} BF_n &= \frac{\bar{\alpha}\alpha^n - \bar{\beta}\beta^n}{\alpha - \beta}, \\ BL_n &= \bar{\alpha}\alpha^n + \bar{\beta}\beta^n, \end{aligned}$$

where $\bar{\alpha} = 1 + i\alpha + j\alpha^2 + ij\alpha^3$, $\bar{\beta} = 1 + i\beta + j\beta^2 + ij\beta^3$, α and β are roots of the equation $t^2 - t - 1 = 0$.

They also obtained some properties of these numbers in the same paper. Moreover, Aydin [1] introduced the bicomplex Pell and Pell–Lucas numbers.

The main objective of this paper is to introduce the generalized bicomplex k -Fibonacci numbers. For this purpose, we first define the generalized bicomplex k -Fibonacci numbers. We then give the generating function and Binet's formula for these numbers. We also obtain Honsberger identity, d'Ocagne's identity, Cassini's identity and Catalan's identity involving the generalized bicomplex k -Fibonacci numbers.

2 The generalized bicomplex k -Fibonacci numbers

The n -th generalized bicomplex k -Fibonacci number is defined for $n \geq 0$ by

$$GBF_{k,n} = F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + ijF_{k,n+3}, \quad (3)$$

where $F_{k,n}$ is the n -th k -Fibonacci number, and i, j are imaginary units which satisfy the equalities

$$i^2 = -\alpha, j^2 = -\beta, (ij)^2 = \alpha\beta, ij = ji, \quad (4)$$

where α and β are real numbers.

Let $GBF_{k,n}$ be the generalized bicomplex k -Fibonacci number. Then $GBF_{k,n}$ can be expressed as

$$GBF_{k,n} = (F_{k,n} + iF_{k,n+1}) + j(F_{k,n+2} + iF_{k,n+3}). \quad (5)$$

After some necessary calculations, one can obtain the following recurrence relation:

$$GBF_{k,n} = kGBF_{k,n-1} + GBF_{k,n-2}, \quad n \geq 2 \quad (6)$$

with initial conditions

$$GBF_{k,0} = i + jk + ij(k^2 + 1), \quad (7)$$

$$GBF_{k,1} = 1 + ik + j(k^2 + 1) + ij(k^3 + 2k). \quad (8)$$

Let

$$GBF_{k,n} = F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + ijF_{k,n+3}$$

and

$$GBF_{k,m} = F_{k,m} + iF_{k,m+1} + jF_{k,m+2} + ijF_{k,m+3}$$

be two generalized bicomplex k -Fibonacci numbers. Then the addition and subtraction of two generalized bicomplex k -Fibonacci numbers are defined by

$$\begin{aligned} GBF_{k,n} \pm GBF_{k,m} &= (F_{k,n} \pm F_{k,m}) + i(F_{k,n+1} \pm F_{k,m+1}) + j(F_{k,n+2} \pm F_{k,m+2}) \\ &\quad + ij(F_{k,n+3} \pm F_{k,m+3}). \end{aligned} \quad (9)$$

The multiplication of a generalized bicomplex k -Fibonacci number by a real scalar λ is defined by

$$\lambda GBF_{k,n} = \lambda F_{k,n} + i\lambda F_{k,n+1} + j\lambda F_{k,n+2} + ij\lambda F_{k,n+3}. \quad (10)$$

The multiplication of two generalized bicomplex k -Fibonacci numbers is defined by

$$\begin{aligned} &GBF_{k,n} \times GBF_{k,m} \\ &= (F_{k,n}F_{k,m} - \alpha F_{k,n+1}F_{k,m+1} - \beta F_{k,n+2}F_{k,m+2} + \alpha\beta F_{k,n+3}F_{k,m+3}) \\ &\quad + i(F_{k,n}F_{k,m+1} + F_{k,n+1}F_{k,m} - \beta F_{k,n+2}F_{k,m+3} - \beta F_{k,n+3}F_{k,m+2}) \\ &\quad + j(F_{k,n}F_{k,m+2} - \alpha F_{k,n+1}F_{k,m+3} + F_{k,n+2}F_{k,m} - \alpha F_{k,n+3}F_{k,m+1}) \\ &\quad + ij(F_{k,n}F_{k,m+3} + F_{k,n+1}F_{k,m+2} + F_{k,n+2}F_{k,m+1} + F_{k,n+3}F_{k,m}) \\ &= GBF_{k,m} \times GBF_{k,n}. \end{aligned} \quad (11)$$

Any generalized bicomplex k -Fibonacci number $GBF_{k,n}$ has three different conjugations which are

$$GBF_{k,n}^{\dagger i} = (F_{k,n} - iF_{k,n+1}) + j(F_{k,n+2} - iF_{k,n+3}), \quad (12)$$

$$GBF_{k,n}^{\dagger j} = (F_{k,n} + iF_{k,n+1}) - j(F_{k,n+2} + iF_{k,n+3}), \quad (13)$$

$$GBF_{k,n}^{\dagger ij} = (F_{k,n} - iF_{k,n+1}) - j(F_{k,n+2} - iF_{k,n+3}). \quad (14)$$

Accordingly, we can give the followings:

$$GBF_{k,n} \times GBF_{k,n}^{\dagger i} = (F_{k,n}^2 + \alpha F_{k,n+1}^2 - \beta F_{k,n+2}^2 - \alpha\beta F_{k,n+3}^2) + 2j(F_{k,n}F_{k,n+2} + \alpha F_{k,n+1}F_{k,n+3}),$$

$$GBF_{k,n} \times GBF_{k,n}^{\dagger j} = (F_{k,n}^2 - \alpha F_{k,n+1}^2 + \beta F_{k,n+2}^2 - \alpha\beta F_{k,n+3}^2) + 2i(F_{k,n}F_{k,n+1} + \beta F_{k,n+2}F_{k,n+3}),$$

$$GBF_{k,n} \times GBF_{k,n}^{\dagger ij} = (F_{k,n}^2 + \alpha F_{k,n+1}^2 + \beta F_{k,n+2}^2 + \alpha\beta F_{k,n+3}^2) + 2ijk(-1)^{n+1},$$

where $k(-1)^{n+1} = F_{k,n}F_{k,n+3} - F_{k,n+1}F_{k,n+2}$ (see [7]).

Theorem 2.1. *Let $GBF_{k,n}$ and $GBF_{k,m}$ be two generalized bicomplex k -Fibonacci numbers. Then we have the followings:*

$$GBF_{k,n} + GBF_{k,n}^{\dagger i} = 2(F_{k,n} + jF_{k,n+2}), \quad (15)$$

$$GBF_{k,n} + GBF_{k,n}^{\dagger j} = 2(F_{k,n} + iF_{k,n+1}), \quad (16)$$

$$GBF_{k,n} + GBF_{k,n}^{\dagger ij} = 2(F_{k,n} + ijF_{k,n+3}), \quad (17)$$

$$(GBF_{k,n} + GBF_{k,m})^{\dagger i} = GBF_{k,n}^{\dagger i} + GBF_{k,m}^{\dagger i}, \quad (18)$$

$$(GBF_{k,n} + GBF_{k,m})^{\dagger j} = GBF_{k,n}^{\dagger j} + GBF_{k,m}^{\dagger j}, \quad (19)$$

$$(GBF_{k,n} + GBF_{k,m})^{\dagger ij} = GBF_{k,n}^{\dagger ij} + GBF_{k,m}^{\dagger ij}, \quad (20)$$

$$(GBF_{k,n}GBF_{k,m})^{\dagger i} = GBF_{k,n}^{\dagger i}GBF_{k,m}^{\dagger i}, \quad (21)$$

$$(GBF_{k,n}GBF_{k,m})^{\dagger j} = GBF_{k,n}^{\dagger j}GBF_{k,m}^{\dagger j}, \quad (22)$$

$$(GBF_{k,n}GBF_{k,m})^{\dagger ij} = GBF_{k,n}^{\dagger ij}GBF_{k,m}^{\dagger ij}. \quad (23)$$

Proof. The theorem can be proved easily using the Eqs. (9)–(14). □

We now give some properties related to the generalized bicomplex k -Fibonacci numbers in the following theorem.

Theorem 2.2. *Let $GBF_{k,n}$ be the n -th generalized bicomplex k -Fibonacci number. Then we have the following:*

$$GBF_{k,n}^2 + GBF_{k,n+1}^2 = 2GBF_{k,2n+1} - F_{k,2n+1} - \alpha F_{k,2n+3} - \beta F_{k,2n+5} + \alpha\beta F_{k,2n+7} + 2(-\beta iF_{k,2n+6} - \alpha jF_{k,2n+5} + ijF_{k,2n+4}), \quad (24)$$

$$GBF_{k,n+1}^2 - GBF_{k,n-1}^2 = k(2GBF_{k,2n} - F_{k,2n} - \alpha F_{k,2n+2} - \beta F_{k,2n+4} + \alpha\beta F_{k,2n+6}) + 2k(-\beta iF_{k,2n+5} - \alpha jF_{k,2n+4} + ijF_{k,2n+3}), \quad (25)$$

$$\begin{aligned}
& GBF_{k,n} - iGBF_{k,n+1} + jGBF_{k,n+2} - ijGBF_{k,n+3} \\
&= F_{k,n} + \alpha F_{k,n+2} - \beta F_{k,n+4} - \alpha\beta F_{k,n+6} + 2j(F_{k,n+2} + \alpha F_{k,n+4}), \tag{26}
\end{aligned}$$

$$\begin{aligned}
& GBF_{k,n} - iGBF_{k,n+1} - jGBF_{k,n+2} - ijGBF_{k,n+3} \\
&= F_{k,n} + \alpha F_{k,n+2} + \beta F_{k,n+4} - \alpha\beta F_{k,n+6} + 2\beta i F_{k,n+5} + 2\alpha j F_{k,n+4} - 2ij F_{k,n+3}. \tag{27}
\end{aligned}$$

Proof. (24): Using the Eqs. (3) and (4), we get

$$\begin{aligned}
GBF_{k,n}^2 + GBF_{k,n+1}^2 &= (F_{k,n}^2 + F_{k,n+1}^2) - \alpha(F_{k,n+1}^2 + F_{k,n+2}^2) \\
&\quad - \beta(F_{k,n+2}^2 + F_{k,n+3}^2) + \alpha\beta(F_{k,n+3}^2 + F_{k,n+4}^2) \\
&\quad + 2i(F_{k,n}F_{k,n+1} + F_{k,n+1}F_{k,n+2} - \beta F_{k,n+2}F_{k,n+3} \\
&\quad\quad - \beta F_{k,n+3}F_{k,n+4}) \\
&\quad + 2j(F_{k,n}F_{k,n+2} + F_{k,n+1}F_{k,n+3} - \alpha F_{k,n+1}F_{k,n+3} \\
&\quad\quad - \alpha F_{k,n+2}F_{k,n+4}) \\
&\quad + 2ij(F_{k,n}F_{k,n+3} + F_{k,n+1}F_{k,n+4} + F_{k,n+1}F_{k,n+2} \\
&\quad\quad + F_{k,n+2}F_{k,n+3}).
\end{aligned}$$

Since $F_{k,n}^2 + F_{k,n+1}^2 = F_{k,2n+1}$ and $F_{k,n}F_{k,m-1} + F_{k,n+1}F_{k,m} = F_{k,m+n}$ (see [6]), we get

$$\begin{aligned}
GBF_{k,n}^2 + GBF_{k,n+1}^2 &= 2GBF_{k,2n+1} - F_{k,2n+1} - \alpha F_{k,2n+3} - \beta F_{k,2n+5} + \alpha\beta F_{k,2n+7} \\
&\quad + 2(-\beta i F_{k,2n+6} - \alpha j F_{k,2n+5} + ij F_{k,2n+4}).
\end{aligned}$$

(25): Using the Eqs. (3) and (4), we get

$$\begin{aligned}
GBF_{k,n+1}^2 - GBF_{k,n-1}^2 &= (F_{k,n+1}^2 - F_{k,n-1}^2) - \alpha(F_{k,n+2}^2 - F_{k,n}^2) \\
&\quad - \beta(F_{k,n+3}^2 - F_{k,n+1}^2) + \alpha\beta(F_{k,n+4}^2 - F_{k,n+2}^2) \\
&\quad + 2i(F_{k,n+1}F_{k,n+2} - F_{k,n-1}F_{k,n} - \beta F_{k,n+3}F_{k,n+4} \\
&\quad\quad + \beta F_{k,n+1}F_{k,n+2}) \\
&\quad + 2j(F_{k,n+1}F_{k,n+3} - F_{k,n-1}F_{k,n+1} - \alpha F_{k,n+2}F_{k,n+4} \\
&\quad\quad + \alpha F_{k,n}F_{k,n+2}) \\
&\quad + 2ij(F_{k,n+1}F_{k,n+4} - F_{k,n-1}F_{k,n+2} + F_{k,n+2}F_{k,n+3} \\
&\quad\quad - F_{k,n}F_{k,n+1}).
\end{aligned}$$

Using the relations $F_{k,n+1}^2 - F_{k,n-1}^2 = kF_{k,2n}$ (see [6]) and $F_{k,n}F_{k,m} - F_{k,n-2}F_{k,m-2} = kF_{k,m+n-2}$ (see [4]), the desired result can be obtained.

Using the Eqs. (3) and (4), the identities (26) and (27) can be proved in a similar manner. \square

The generating function for the generalized bicomplex k -Fibonacci numbers is given in the following theorem.

Theorem 2.3. *The generating function for the generalized bicomplex k -Fibonacci numbers is given by*

$$G_k(t) = \frac{i + jk + ij(k^2 + 1) + (1 + j + ijk)t}{1 - kt - t^2}.$$

Proof. Let $G_k(t)$ be the generating function for the generalized bicomplex k -Fibonacci numbers. Then we write

$$G_k(t) = \sum_{n=0}^{\infty} GBF_{k,n} t^n = GBF_{k,0} + GBF_{k,1}t + \cdots + GBF_{k,n}t^n + \cdots \quad (28)$$

Multiplying the Eq. (28) with kt and t^2 , respectively, we get

$$ktG_k(t) = kGBF_{k,0}t + kGBF_{k,1}t^2 + \cdots + kGBF_{k,n-1}t^n + \cdots$$

and

$$t^2G_k(t) = GBF_{k,0}t^2 + GBF_{k,1}t^3 + \cdots + GBF_{k,n-2}t^n + \cdots$$

Then we have

$$\begin{aligned} (1 - kt - t^2)G_k(t) &= GBF_{k,0} + (GBF_{k,1} - kGBF_{k,0})t \\ &\quad + \sum_{n=2}^{\infty} (GBF_{k,n} - kGBF_{k,n-1} - GBF_{k,n-2})t^n \\ &= GBF_{k,0} + (GBF_{k,1} - kGBF_{k,0})t. \end{aligned}$$

By the Eqs. (7) and (8), we get

$$(1 - kt - t^2)G_k(t) = i + jk + ij(k^2 + 1) + (1 + j + ijk)t$$

which is the desired result. \square

The following theorem gives the Binet's formula for the generalized bicomplex k -Fibonacci numbers.

Theorem 2.4. *The n -th term of the generalized bicomplex k -Fibonacci number is given by*

$$GBF_{k,n} = \frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta},$$

where $\alpha^* = 1 + i\alpha + j\alpha^2 + ij\alpha^3$, $\alpha = \frac{k + \sqrt{k^2 + 4}}{2}$ and $\beta^* = 1 + i\beta + j\beta^2 + ij\beta^3$, $\beta = \frac{k - \sqrt{k^2 + 4}}{2}$.

Proof. Using the Eqs. (2) and (3), we get

$$\begin{aligned} GBF_{k,n} &= F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + ijF_{k,n+3} \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} + i \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + j \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} + ij \frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta} \\ &= \frac{\alpha^n(1 + i\alpha + j\alpha^2 + ij\alpha^3) - \beta^n(1 + i\beta + j\beta^2 + ij\beta^3)}{\alpha - \beta}. \end{aligned}$$

If we take $\alpha^* = 1 + i\alpha + j\alpha^2 + ij\alpha^3$ and $\beta^* = 1 + i\beta + j\beta^2 + ij\beta^3$, we obtain the desired result. \square

The Honsberger identity involving the generalized bicomplex k -Fibonacci numbers is given in the following theorem.

Theorem 2.5. *Let m and n be two positive integers. Then we have*

$$\begin{aligned} & GBF_{k,m}GBF_{k,n} + GBF_{k,m+1}GBF_{k,n+1} \\ &= 2GBF_{k,m+n+1} - F_{k,m+n+1} - \alpha F_{k,m+n+3} - \beta F_{k,m+n+5} + \alpha\beta F_{k,m+n+7} \\ &\quad - 2\beta i F_{k,m+n+6} - 2\alpha j F_{k,m+n+5} + 2ij F_{k,m+n+4}. \end{aligned}$$

Proof. Using the Eqs. (3) and (4), we get

$$\begin{aligned} & GBF_{k,m}GBF_{k,n} + GBF_{k,m+1}GBF_{k,n+1} \\ &= F_{k,m}F_{k,n} + F_{k,m+1}F_{k,n+1} - \alpha(F_{k,m+1}F_{k,n+1} + F_{k,m+2}F_{k,n+2}) \\ &\quad - \beta(F_{k,m+2}F_{k,n+2} + F_{k,m+3}F_{k,n+3}) + \alpha\beta(F_{k,m+3}F_{k,n+3} + F_{k,m+4}F_{k,n+4}) \\ &\quad + i[F_{k,m}F_{k,n+1} + F_{k,m+1}F_{k,n+2} + F_{k,m+1}F_{k,n} + F_{k,m+2}F_{k,n+1} \\ &\quad\quad - \beta(F_{k,m+2}F_{k,n+3} + F_{k,m+3}F_{k,n+4} + F_{k,m+3}F_{k,n+2} + F_{k,m+4}F_{k,n+3})] \\ &\quad + j[F_{k,m}F_{k,n+2} + F_{k,m+1}F_{k,n+3} + F_{k,m+2}F_{k,n} + F_{k,m+3}F_{k,n+1} \\ &\quad\quad - \alpha(F_{k,m+1}F_{k,n+3} + F_{k,m+2}F_{k,n+4} + F_{k,m+3}F_{k,n+1} + F_{k,m+4}F_{k,n+2})] \\ &\quad + ij[F_{k,m}F_{k,n+3} + F_{k,m+1}F_{k,n+4} + F_{k,m+1}F_{k,n+2} + F_{k,m+2}F_{k,n+3} \\ &\quad\quad + F_{k,m+2}F_{k,n+1} + F_{k,m+3}F_{k,n+2} + F_{k,m+3}F_{k,n} + F_{k,m+4}F_{k,n+1}]. \end{aligned}$$

Since $F_{k,n}F_{k,m-1} + F_{k,n+1}F_{k,m} = F_{k,n+m}$ (see [6]), we get

$$\begin{aligned} & GBF_{k,m}GBF_{k,n} + GBF_{k,m+1}GBF_{k,n+1} \\ &= F_{k,m+n+1} - \alpha F_{k,m+n+3} - \beta F_{k,m+n+5} + \alpha\beta F_{k,m+n+7} \\ &\quad + 2i(F_{k,m+n+2} - \beta F_{k,m+n+6}) + 2j(F_{k,m+n+3} - \alpha F_{k,m+n+5}) + 4ij F_{k,m+n+4} \\ &= 2GBF_{k,m+n+1} - F_{k,m+n+1} - \alpha F_{k,m+n+3} - \beta F_{k,m+n+5} + \alpha\beta F_{k,m+n+7} \\ &\quad - 2\beta i F_{k,m+n+6} - 2\alpha j F_{k,m+n+5} + 2ij F_{k,m+n+4}. \quad \square \end{aligned}$$

The following theorem gives the d'Ocagne's identity involving the generalized bicomplex k -Fibonacci numbers.

Theorem 2.6. *Let m and n be two positive integers. Then we have*

$$\begin{aligned} & GBF_{k,m}GBF_{k,n+1} - GBF_{k,m+1}GBF_{k,n} \\ &= (-1)^n [GBF_{k,m-n} + (\alpha - \beta + \alpha\beta)F_{k,m-n}] + (-1)^{n+1} [i(F_{k,m-n-1} + \beta k F_{k,m-n}) \\ &\quad - j((1 + \alpha)F_{k,m-n-2} + \alpha F_{k,m-n+2}) + ij(k^2 + 1)F_{k,m-n-1}]. \end{aligned}$$

Proof. Using the Eqs. (3) and (4), we get

$$\begin{aligned}
& GBF_{k,m}GBF_{k,n+1} - GBF_{k,m+1}GBF_{k,n} \\
&= (F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n}) - \alpha(F_{k,m+1}F_{k,n+2} - F_{k,m+2}F_{k,n+1}) \\
&\quad - \beta(F_{k,m+2}F_{k,n+3} - F_{k,m+3}F_{k,n+2}) + \alpha\beta(F_{k,m+3}F_{k,n+4} - F_{k,m+4}F_{k,n+3}) \\
&\quad + i[(F_{k,m}F_{k,n+2} - F_{k,m+1}F_{k,n+1}) + (F_{k,m+1}F_{k,n+1} - F_{k,m+2}F_{k,n}) \\
&\quad\quad - \beta(F_{k,m+2}F_{k,n+4} - F_{k,m+3}F_{k,n+3}) - \beta(F_{k,m+3}F_{k,n+3} - F_{k,m+4}F_{k,n+2})] \\
&\quad + j[(F_{k,m}F_{k,n+3} - F_{k,m+1}F_{k,n+2}) + (F_{k,m+2}F_{k,n+1} - F_{k,m+3}F_{k,n}) \\
&\quad\quad - \alpha(F_{k,m+1}F_{k,n+4} - F_{k,m+2}F_{k,n+3}) - \alpha(F_{k,m+3}F_{k,n+2} - F_{k,m+4}F_{k,n+1})] \\
&\quad + ij[(F_{k,m}F_{k,n+4} - F_{k,m+1}F_{k,n+3}) + (F_{k,m+1}F_{k,n+3} - F_{k,m+2}F_{k,n+2}) \\
&\quad\quad + (F_{k,m+2}F_{k,n+2} - F_{k,m+3}F_{k,n+1}) + (F_{k,m+3}F_{k,n+1} - F_{k,m+4}F_{k,n})].
\end{aligned}$$

Since $F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n} = (-1)^n F_{k,m-n}$ (see [7]), we get

$$\begin{aligned}
& GBF_{k,m}GBF_{k,n+1} - GBF_{k,m+1}GBF_{k,n} \\
&= (-1)^n [F_{k,m-n} + \alpha F_{k,m-n} - \beta F_{k,m-n} + \alpha\beta F_{k,m-n}] \\
&\quad + (-1)^n i[-F_{k,m-n-1} + F_{k,m-n+1} + \beta(F_{k,m-n-1} - F_{k,m-n+1})] \\
&\quad + (-1)^n j[F_{k,m-n-2} + F_{k,m-n+2} + \alpha(F_{k,m-n-2} + F_{k,m-n+2})] \\
&\quad + (-1)^n ij[-F_{k,m-n-3} + F_{k,m-n-1} - F_{k,m-n+1} + F_{k,m-n+3}] \\
&= (-1)^n [GBF_{k,m-n} + (\alpha - \beta + \alpha\beta)F_{k,m-n}] \\
&\quad + (-1)^{n+1} [i(F_{k,m-n-1} + \beta k F_{k,m-n}) - j((1 + \alpha)F_{k,m-n-2} + \alpha F_{k,m-n+2}) \\
&\quad\quad + ij(k^2 + 1)F_{k,m-n-1}]. \quad \square
\end{aligned}$$

The following theorem gives the Catalan's identity involving the generalized bicomplex k -Fibonacci numbers.

Theorem 2.7. *Let n and r be two positive integers. Then we have $GBF_{k,n+r-1}GBF_{k,n+r+1} - GBF_{k,n+r}^2 = (-1)^{n+r} [1 + \alpha - \beta - \alpha\beta + i(1 - \beta)k + j(1 + \alpha)(k^2 + 2) + ij(k^3 + 2k)]$.*

Proof. Using the Eqs. (3) and (4), we get

$$\begin{aligned}
& GBF_{k,n+r-1}GBF_{k,n+r+1} - GBF_{k,n+r}^2 \\
&= (F_{k,n+r-1}F_{k,n+r+1} - F_{k,n+r}^2) - \alpha(F_{k,n+r}F_{k,n+r+2} - F_{k,n+r+1}^2) \\
&\quad - \beta(F_{k,n+r+1}F_{k,n+r+3} - F_{k,n+r+2}^2) + \alpha\beta(F_{k,n+r+2}F_{k,n+r+4} - F_{k,n+r+3}^2) \\
&\quad + i[(F_{k,n+r-1}F_{k,n+r+2} - F_{k,n+r}F_{k,n+r+1}) \\
&\quad\quad - \beta(F_{k,n+r+1}F_{k,n+r+4} - F_{k,n+r+2}F_{k,n+r+3})] \\
&\quad + j[(F_{k,n+r-1}F_{k,n+r+3} - F_{k,n+r}F_{k,n+r+2}) - (F_{k,n+r}F_{k,n+r+2} - F_{k,n+r+1}^2) \\
&\quad\quad + \alpha(F_{k,n+r+1}F_{k,n+r+3} - F_{k,n+r+2}^2) - \alpha(F_{k,n+r}F_{k,n+r+4} - F_{k,n+r+1}F_{k,n+r+3})] \\
&\quad + ij[F_{k,n+r-1}F_{k,n+r+4} - F_{k,n+r}F_{k,n+r+3}].
\end{aligned}$$

Using the relations $F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n} = (-1)^n F_{k,m-n}$ (see [7]), $F_{k,n+r-1}F_{k,n+r+1} - F_{k,n+r}^2 = (-1)^{n+r}$ (see [6]) and also $F_{k,-n} = (-1)^{n+1}F_{k,n}$ (see [4]), the desired result can be obtained. \square

Setting $r = 0$ in Theorem 2.7, we obtain the Cassini's identity involving the generalized bicomplex k -Fibonacci numbers which is given in the following corollary.

Corollary 2.8. *Let n be positive integer. Then we have*

$$\begin{aligned} GBF_{k,n-1}GBF_{k,n+1} - GBF_{k,n}^2 \\ = (-1)^n [1 + \alpha - \beta - \alpha\beta + i(1 - \beta)k + j(1 + \alpha)(k^2 + 2) + ij(k^3 + 2k)]. \end{aligned}$$

3 Conclusion

In this study, the generalized bicomplex k -Fibonacci numbers were introduced. Some properties of these numbers, including generating function and Binet's formula, were given. Furthermore, some well-known identities, including Honsberger, d'Ocagne's, Catalan's, Cassini's identities, involving these numbers were obtained.

It must be noted that for $k = 1$, $\alpha = 1$ and $\beta = 1$, the generalized bicomplex k -Fibonacci number becomes the bicomplex Fibonacci number [10]. Moreover, for $k = 2$, $\alpha = 1$ and $\beta = 1$, the generalized bicomplex k -Fibonacci number becomes the bicomplex Pell number [1].

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