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Generalized Fibonacci and *k*-Pell matrix sequences: Another way of demonstrating their properties

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Abstract: Recently Wani, Artaf, A, Badshah, V., Rathore, G. P. & Catarino introduced commutative matrices derived from the generalized Fibonacci matrix sequence and the *k*-Pell matrix sequence. In the present work, through the identification of certain special matrices, we can identify other forms of demonstration and also the description of commutative matrix properties for negative indices.
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1 Introduction

Let us consider the following definitions introduced recently in the work [2]. We can find several properties resulting from Generalized Fibonacci sequences and the *k*-Pell sequence. On the other hand, from certain properties of matrices that generate the elements of the generalized Fibonacci sequences and the *k*-Pell sequence we can find other more immediate and new forms of the corresponding theorems that we find in the work [2]. Other interesting properties about the Fibonacci matrix sequences can be found at works [1, 3].

Definition 1. For $k \in IR^+$, the generalized Fibonacci sequence $\langle R_{k,n} \rangle$ is defined by

$$R_{k,n+1} = 2R_{k,n} + kR_{k,n-1}, \ \ n \geq 1, R_{k,0} = 2, R_{k,1} = 1.$$

Definition 2. For $k \in IR^+$, the k-Pell sequence $\langle P_{k,n} \rangle$ is defined by

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, n \geq 1, P_{k,0} = 0, P_{k,1} = 1.$$

Next, let us look at two mathematical definitions recently introduced in [2] related to the matrix sequence.

Definition 3. For $k \in IR^+$, the generalized Fibonacci matrix sequence $\langle S_{k,n} \rangle$ is defined by

$$S_{k,n} = 2S_{k,n-1} + kS_{k,n-2}, n \geq 2, \mathrm{S}_{k,0} = \begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix}, \mathrm{S}_{k,1} = \begin{pmatrix} 2+2k & k \\ 1 & 2k \end{pmatrix}.$$

Definition 4. For $k \in IR^+$, the k-Pell matrix sequence $\langle V_{k,n} \rangle$ is defined by

$$V_{k,n} = 2V_{k,n-1} + kV_{k,n-2}, n \geq 2, V_{k,0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, V_{k,1} = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}.$$

Before discussing other ways of demonstrating the results addressed in the work [2], we will consider the following matrix

$$V_{k,1} = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}.$$

In addition, we can also verify the behavior of the following powers:

$$V_{k,1}{}^2 = \begin{pmatrix} 4+k & k \cdot 2 \\ 2 & k \cdot 1 \end{pmatrix} = \begin{pmatrix} P_{k,3} & k \cdot P_{k,2} \\ P_{k,2} & k \cdot P_{k,1} \end{pmatrix},$$

$$V_{k,1}{}^3 = \begin{pmatrix} 8+4k & k \cdot (4+k) \\ 4+k & k \cdot 2 \end{pmatrix} = \begin{pmatrix} P_{k,4} & k \cdot P_{k,3} \\ P_{k,3} & k \cdot P_{k,2} \end{pmatrix},$$

$$V_{k,1}{}^4 = \begin{pmatrix} k^2+12k+16 & k \cdot (8+4k) \\ 8+4k & k \cdot (4+k) \end{pmatrix} = \begin{pmatrix} P_{k,5} & k \cdot P_{k,4} \\ P_{k,4} & k \cdot P_{k,3} \end{pmatrix},$$

$$V_{k,1}{}^5 = \begin{pmatrix} 6k^2+32k+32 & k \cdot (k^2+12k+16) \\ k^2+12k+16 & k \cdot (8+4k) \end{pmatrix} = \begin{pmatrix} P_{k,6} & k \cdot P_{k,5} \\ P_{k,5} & k \cdot P_{k,4} \end{pmatrix},$$

$$V_{k,1}{}^6 = \begin{pmatrix} k^3+24k^2+80k+64 & k \cdot (6k^2+32k+32) \\ 6k^2+32k+32 & k \cdot (k^2+12k+16) \end{pmatrix} = \begin{pmatrix} P_{k,7} & k \cdot P_{k,6} \\ P_{k,6} & k \cdot P_{k,5} \end{pmatrix},$$

$$V_{k,1}{}^7 = \begin{pmatrix} k^3+24k^2+80k+64 & k \cdot (6k^2+32k+32) \\ 6k^2+32k+32 & k \cdot (k^2+12k+16) \end{pmatrix} = \begin{pmatrix} P_{k,8} & k \cdot P_{k,7} \\ P_{k,7} & k \cdot P_{k,6} \end{pmatrix},$$

etc. In this way, we will state the following theorem.

Theorem 1. For any integer $n \geq 1$, we obtain $V_{k,1}{}^n = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} P_{k,\mathrm{n}+1} & k \cdot P_{k,\mathrm{n}} \\ P_{k,\mathrm{n}} & k \cdot P_{k,\mathrm{n}-1} \end{pmatrix}.$

Proof. The result holds for $n = 1$. By mathematical induction, we assume that

$$V_{k,1}{}^{n} = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n} = \begin{pmatrix} P_{k,\mathrm{n}+1} & k \cdot P_{k,\mathrm{n}} \\ P_{k,\mathrm{n}} & k \cdot P_{k,\mathrm{n}-1} \end{pmatrix}.$$

Next, consider the following matrix power:

$$\begin{aligned} V_{k,1}{}^{n+1} &= \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n+1} \\ &= \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} P_{k,\mathrm{n}+1} & k \cdot P_{k,\mathrm{n}} \\ P_{k,\mathrm{n}} & k \cdot P_{k,\mathrm{n}-1} \end{pmatrix} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2P_{k,\mathrm{n}+1} + k \cdot P_{k,\mathrm{n}} & k \cdot P_{k,\mathrm{n}+1} \\ P_{k,\mathrm{n}} + k \cdot P_{k,\mathrm{n}-1} & k \cdot P_{k,\mathrm{n}} \end{pmatrix} \\ &= \begin{pmatrix} P_{k,\mathrm{n}+2} & k \cdot P_{k,\mathrm{n}+1} \\ P_{k,\mathrm{n}+1} & k \cdot P_{k,\mathrm{n}} \end{pmatrix} \end{aligned}$$

from the Definition 2. □

Next, let us consider the following matrix

$$\mathrm{S}_{k,0} = \begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix}.$$

In addition, we will take the following matrix products indicated in the expression

$$\mathrm{S}_{k,0}\, V_{k,1}{}^{n} = \begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n}.$$

Easily, we can see that

$$\mathrm{S}_{k,0}\, V_{k,1} = V_{k,1}\, \mathrm{S}_{k,0} = \begin{pmatrix} 2+2k & k \cdot 1 \\ 1 & k \cdot 2 \end{pmatrix} = \begin{pmatrix} R_{k,2} & k \cdot R_{k,1} \\ R_{k,1} & k \cdot R_{k,0} \end{pmatrix}.$$

Moreover, we can observe that:

$$\mathrm{S}_{k,0}\, V_{k,1}{}^{2} = \begin{pmatrix} 4+5k & k \cdot (2+2k) \\ 2+2k & k \cdot 1 \end{pmatrix} = \begin{pmatrix} R_{k,3} & k \cdot R_{k,2} \\ R_{k,2} & k \cdot R_{k,1} \end{pmatrix},$$

$$\mathrm{S}_{k,0}\, V_{k,1}{}^{3} = \begin{pmatrix} 2k^2+12k+8 & k \cdot (4+5k) \\ 4+5k & k \cdot (2+2k) \end{pmatrix} = \begin{pmatrix} R_{k,4} & k \cdot R_{k,3} \\ R_{k,3} & k \cdot R_{k,2} \end{pmatrix},$$

$$\mathrm{S}_{k,0}\, V_{k,1}{}^{4} = \begin{pmatrix} 9k^2+28k+16 & k \cdot (2k^2+12k+8) \\ 2k^2+12k+8 & k \cdot (4+5k) \end{pmatrix} = \begin{pmatrix} R_{k,5} & k \cdot R_{k,4} \\ R_{k,4} & k \cdot R_{k,3} \end{pmatrix},$$

$$\mathrm{S}_{k,0}\, V_{k,1}{}^{5} = \begin{pmatrix} 2k^3+30k^2+64k+32 & k \cdot (9k^2+28k+16) \\ 9k^2+28k+16 & k \cdot (2k^2+12k+8) \end{pmatrix} = \begin{pmatrix} R_{k,6} & k \cdot R_{k,5} \\ R_{k,5} & k \cdot R_{k,4} \end{pmatrix},$$

$$\mathrm{S}_{k,0}\, V_{k,1}{}^{6} = \begin{pmatrix} 13k^3+88k^2+144k+64 & k \cdot (2k^3+30k^2+64k+32) \\ 2k^3+30k^2+64k+32 & k \cdot (9k^2+28k+16) \end{pmatrix} = \begin{pmatrix} R_{k,7} & k \cdot R_{k,6} \\ R_{k,6} & k \cdot R_{k,5} \end{pmatrix},$$

etc. From these examples, we consider the following theorem.

Theorem 2. For any integer $n \geq 1$, we obtain $S_{k,0} V_{k,1}{}^n = \begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} R_{k,n+1} & k \cdot R_{k,n} \\ R_{k,n} & k \cdot R_{k,n-1} \end{pmatrix}$.

Proof. Similar to the previous theorem, the result holds for $n = 1$ and by mathematical induction, it is enough to verify $S_{k,0} V_{k,1}{}^{n+1} = \begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} R_{k,n+1} & k \cdot R_{k,n} \\ R_{k,n} & k \cdot R_{k,n-1} \end{pmatrix} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}$ $= \begin{pmatrix} R_{k,n+2} & k \cdot R_{k,n+1} \\ R_{k,n+1} & k \cdot R_{k,n} \end{pmatrix}$ for any integer $n \geq 1$. □

In addition, we can also verify the behavior of the following determinants indicated by $\det(V_{k,1}{}^n) = \det \begin{pmatrix} P_{k,n+1} & k \cdot P_{k,n} \\ P_{k,n} & k \cdot P_{k,n-1} \end{pmatrix}$ and $\det(S_{k,0} \cdot V_{k,1}{}^n) = \det \begin{pmatrix} R_{k,n+1} & k \cdot R_{k,n} \\ R_{k,n} & k \cdot R_{k,n-1} \end{pmatrix}$.

Corollary 1. For any integer $n \geq 1$, we obtain:

(i) $\det V_{k,1}{}^n = \det \begin{pmatrix} P_{k,n+1} & k \cdot P_{k,n} \\ P_{k,n} & k \cdot P_{k,n-1} \end{pmatrix} = (-1)^n k^n$;

(ii) $\det(S_{k,0} V_{k,1}{}^n) = \det \begin{pmatrix} R_{k,n+1} & k \cdot R_{k,n} \\ R_{k,n} & k \cdot R_{k,n-1} \end{pmatrix} = (3+4k)(-1)^n k^{n+1}$.

Proof. We note that $\det V_{k,1} = -k = (-1)k$, $\det V_{k,1}{}^2 = k^2$, $\det V_{k,1}{}^3 = -k^3$, $\det V_{k,1}{}^4 = k^4$. By mathematical induction, let us write $\det V_{k,1}{}^n = \det \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^n = (-1)^n k^n$. Then, we immediately find that $\det V_{k,1}{}^{n+1} = \det \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n+1} = \det \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^n \det \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix} = (-1)^n k^n \cdot (-1)k = (-1)^{n+1} k^{n+1}$.

Similarly, let us admit that $\det(S_{k,0} V_{k,1}{}^n) = (3+4k)(-1)^n k^{n+1}$. Thus, we can see that $\det(S_{k,0} \cdot (V_{k,1})^{n+1}) = \det\left(S_{k,0} \cdot (V_{k,1})^n\right) \cdot \det(V_{k,1}) = (3+4k)(-1)^n k^{n+1}(-k) = (3+4k)(-1)^{n+1} k^{n+2}$. □

Let us consider the following matrices, from the work indicated in [2]. For any integer $n \geq 1$,

$$V_{k,n} = \begin{pmatrix} P_{k,n+1} & k \cdot P_{k,n} \\ P_{k,n} & k \cdot P_{k,n-1} \end{pmatrix}, \quad S_{k,n} = \begin{pmatrix} R_{k,n+1} & k \cdot R_{k,n} \\ R_{k,n} & k \cdot R_{k,n-1} \end{pmatrix}.$$

On the other hand, we can consider the process of extension to the field of negative integers, corresponding to the indices, when we define two new matrices.

Definition 5. For $k \in IR^+$, the *k*-Pell matrix for negative indices is indicated by

$$V_{k,-n} = \begin{pmatrix} P_{k,-n+1} & k \cdot P_{k,-n} \\ P_{k,-n} & k \cdot P_{k,-n-1} \end{pmatrix}$$

and the generalized Fibonacci matrix for negative indices is indicated by

$$S_{k,-n} = \begin{pmatrix} R_{k,-n+1} & k \cdot R_{k,-n} \\ R_{k,-n} & k \cdot R_{k,-n-1} \end{pmatrix},$$

for every integer $n \geq 1$.

2 Other demonstrations for the commutative properties

In this section, we will discuss other simpler and more immediate ways to demonstrate the matrix properties of the matrices previously defined.

Theorem 3. For $m,n \geq 1$ the following results hold:

(i) $V_{k,m+n} = V_{k,m}V_{k,n} = V_{k,n}V_{k,m}$;

(ii) $S_{k,m}S_{k,n} = S_{k,n}S_{k,m}$;

(iii) $V_{k,m} \cdot S_{k,n} = S_{k,n} \cdot V_{k,m}$.

Proof. For the first item, we see that the following commutative properties occur $V_{k,m+n} = \begin{pmatrix} P_{k,m+n+1} & k \cdot P_{k,n} \\ P_{k,m+n} & k \cdot P_{k,n-1} \end{pmatrix} = V_{k,1}{}^{m+n} = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{m+n} = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{m} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n} = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{m} = V_{k,n+m}$. The second item of this theorem, let us also see that $S_{k,n}S_{k,m} = \begin{pmatrix} R_{k,n+1} & k \cdot R_{k,n} \\ R_{k,n} & k \cdot R_{k,n-1} \end{pmatrix} \begin{pmatrix} R_{k,m+1} & k \cdot R_{k,m} \\ R_{k,m} & k \cdot R_{k,m-1} \end{pmatrix} =$
$\begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{m} = \begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{m} \begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n} = \begin{pmatrix} R_{k,m+1} & k \cdot R_{k,m} \\ R_{k,m} & k \cdot R_{k,m-1} \end{pmatrix} \begin{pmatrix} R_{k,n+1} & k \cdot R_{k,n} \\ R_{k,n} & k \cdot R_{k,n-1} \end{pmatrix}$
$= S_{k,m}S_{k,n}$ and we record the commutative property of $S_{k,0}V_{k,1} = V_{k,1}S_{k,0}$. To conclude, let us easily see that $V_{k,m} \cdot S_{k,n} = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{m} \begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n} = \begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{m} = S_{k,n} \cdot V_{k,m}$. □

Theorem 4. For $m,n \geq 1$ the following results holds:

(i) $V_{k,n} + 2kV_{k,n-1} = S_{k,n}$;

(ii) $2V_{k,n+1} - 3V_{k,n} = S_{k,n}$;

(iii) $S_{k,n+1}{}^2 = S_{k,1}{}^2 \cdot V_{k,2n}$;

(iv) $S_{k,2n+1} = V_{k,n}S_{k,n+1}$;

(iv) $S_{k,2n} = V_{k,n}S_{k,n}$.

Proof. We can get $V_{k,n} + 2kV_{k,n-1} = V_{k,1}{}^{n} + 2k \cdot V_{k,1}{}^{n-1} = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n} + 2k \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n-1} = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n} \left(I + 2kV_{k,1}{}^{-1} \right)$.

On the other hand, we can determine that $I + 2kV_{k,1}{}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2k \begin{pmatrix} 0 & 1 \\ \frac{1}{k} & -\frac{2}{k} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2k \\ 2 & -4 \end{pmatrix} =$
$\begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix} = S_{k,0}$. Finally, we determine equality $V_{k,n} + 2kV_{k,n-1} = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n} \left(I + 2kV_{k,1}{}^{-1} \right) = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n} S_{k,0} =$
$S_{k,0} \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n} = S_{k,n}$. For the second item, we consider that $2V_{k,n+1} - 3V_{k,n} = 2\begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n+1} - 3\begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n}$

$$=\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^n\left(2\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}-3\cdot I\right)=\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^n\left(\begin{pmatrix}4 & 2k\\ 2 & 0\end{pmatrix}-\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}\right)=\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^n\begin{pmatrix}1 & 2k\\ 2 & -3\end{pmatrix}=S_{k,0}V_{k,1}{}^n=S_{k,n}.$$

Finally, we can determine directly from the definition that: $S_{k,\mathrm{n}+1}{}^2=\left(S_{k,\mathrm{n}+1}\cdot S_{k,\mathrm{n}+1}\right)=$

$$=\begin{pmatrix}1 & 2k\\ 2 & -3\end{pmatrix}\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^{n+1}\begin{pmatrix}1 & 2k\\ 2 & -3\end{pmatrix}\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^{n+1}=\begin{pmatrix}1 & 2k\\ 2 & -3\end{pmatrix}\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^n\begin{pmatrix}1 & 2k\\ 2 & -3\end{pmatrix}\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^n=$$

$$=(S_{k,0}V_{k,1})\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^n(S_{k,0}V_{k,1})\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^n=S_{k,1}\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^n S_{k,1}\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^n=S_{k,1}{}^2\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^{2n}=S_{k,1}{}^2V_{k,2\mathrm{n}}.$$

Similarly, we see that $S_{k,2\mathrm{n}+1}=\mathrm{S}_{k,0}V_{k,1}{}^{2n+1}=\mathrm{S}_{k,0}V_{k,1}{}^{2n}V_{k,1}=\mathrm{S}_{k,0}V_{k,1}{}^n\cdot V_{k,1}{}^n\cdot V_{k,1}=V_{k,1}{}^n\left(\mathrm{S}_{k,0}V_{k,1}{}^{n+1}\right)$ $=V_{k,n}S_{k,n+1}$. In addition, we can see that $S_{k,2n}=S_{k,0}V_{k,1}{}^{2n}=(S_{k,0}V_{k,1}{}^n)\cdot V_{k,1}{}^n=V_{k,n}(S_{k,n})$. □

We observe that the authors indicated in the work [2] establish the following theorem.

Theorem 5. For any integer $n\geq 1$, we obtain

$$S_{k,n}=\begin{pmatrix}R_{k,\mathrm{n}+1} & k\cdot R_{k,\mathrm{n}}\\ R_{k,\mathrm{n}} & k\cdot R_{k,\mathrm{n}-1}\end{pmatrix}=S_{k,0}V_{k,n}.$$

Proof. Just use Theorem 2. □

We shall now see another demonstration for the following theorem discussed in [1].

Theorem 6. For any integers $m, n\geq 1$, we obtain $S_{k,n+m}=S_{k,\mathrm{n}}V_{k,\mathrm{m}}=V_{k,\mathrm{n}}S_{k,\mathrm{m}}$.

Proof. Let us consider that $S_{k,n+m}=\begin{pmatrix}1 & 2k\\ 2 & -3\end{pmatrix}\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^{n+m}=S_{k,0}\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^n\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^m=S_{k,0}V_{k,1}{}^n\cdot V_{k,1}{}^m=S_{k,n}\cdot V_{k,m}$.

In addition, we can write immediately that $S_{k,n+m}=S_{k,0}V_{k,n+m}=(S_{k,0}V_{k,n})V_{k,m}=S_{k,\mathrm{n}}V_{k,\mathrm{m}}$. Moreover, we see that $S_{k,n+m}=S_{k,0}V_{k,n+m}=S_{k,0}V_{k,n}V_{k,m}=(S_{k,0}V_{k,\mathrm{m}})V_{k,\mathrm{n}}=V_{k,\mathrm{n}}(S_{k,0}V_{k,\mathrm{m}})=V_{k,\mathrm{n}}S_{k,\mathrm{m}}$ since, we know the commutativity of the matrix product $S_{k,0}V_{k,1}=V_{k,1}S_{k,0}$. □

3 Matrix sequence properties for negative indices

Now, we will develop the study of certain properties determined by the following inverse k-Pell matrix indicated by $\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^{-n}$. We can immediately determine some particular cases

$$V_{k,1}{}^{-1}=\begin{pmatrix}0 & 1\\ \frac{1}{k} & -\frac{2}{k}\end{pmatrix}=\begin{pmatrix}0 & k\left(\frac{1}{k}\right)\\ \frac{1}{k} & k\left(-\frac{2}{k^2}\right)\end{pmatrix}=\begin{pmatrix}P_{k,0} & k\cdot P_{k,-1}\\ P_{k,-1} & k\cdot P_{k,-2}\end{pmatrix},$$

$$V_{k,1}{}^{-2}=\begin{pmatrix}\frac{1}{k} & -\frac{2}{k}\\ -\frac{2}{k^2} & \frac{4+k}{k^2}\end{pmatrix}=\begin{pmatrix}\frac{1}{k} & k\left(-\frac{2}{k}\right)\\ -\frac{2}{k^2} & \frac{4+k}{k^2}\end{pmatrix}=\begin{pmatrix}P_{k,-1} & k\cdot P_{k,-2}\\ P_{k,-2} & k\cdot P_{k,-3}\end{pmatrix},$$

$$V_{k,1}^{-3}=\begin{pmatrix}-\frac{2}{k^2} & \frac{4+k}{k^2}\\ \frac{4+k}{k^3} & -\frac{4(2+k)}{k^3}\end{pmatrix}=\begin{pmatrix}-\frac{2}{k^2} & k\left(\frac{4+k}{k^3}\right)\\ \frac{4+k}{k^3} & k\left(-\frac{4(2+k)}{k^4}\right)\end{pmatrix}=\begin{pmatrix}P_{k,-2} & k\cdot P_{k,-3}\\ P_{k,-3} & k\cdot P_{k,-4}\end{pmatrix},$$

$$V_{k,1}^{-4}=\begin{pmatrix}\frac{4+k}{k^3} & -\frac{4(2+k)}{k^3}\\ -\frac{4(2+k)}{k^4} & \frac{k^2+12k+16}{k^4}\end{pmatrix}=\begin{pmatrix}\frac{4+k}{k^3} & k\left(-\frac{4(2+k)}{k^4}\right)\\ -\frac{4(2+k)}{k^4} & k\left(\frac{k^2+12k+16}{k^5}\right)\end{pmatrix}=\begin{pmatrix}P_{k,-3} & k\cdot P_{k,-4}\\ P_{k,-4} & k\cdot P_{k,-5}\end{pmatrix},$$

$$V_{k,1}^{-5}=\begin{pmatrix}-\frac{4(2+k)}{k^4} & \frac{k^2+12k+16}{k^4}\\ \frac{k^2+12k+16}{k^5} & -\frac{2(3k^2+16k+16)}{k^5}\end{pmatrix}=\begin{pmatrix}-\frac{4(2+k)}{k^4} & k\left(\frac{k^2+12k+16}{k^5}\right)\\ \frac{k^2+12k+16}{k^5} & k\left(-\frac{2(3k^2+16k+16)}{k^6}\right)\end{pmatrix}=\begin{pmatrix}P_{k,-4} & k\cdot P_{k,-5}\\ P_{k,-5} & k\cdot P_{k,-6}\end{pmatrix},$$

$$V_{k,1}^{-6}=\begin{pmatrix}-\frac{k^2+12k+16}{k^5} & -\frac{2(3k^2+16k+16)}{k^5}\\ -\frac{2(3k^2+16k+16)}{k^6} & \frac{k^3+24k^2+80k+64}{k^6}\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{k^2+12k+16}{k^5} & k\left(-\frac{2(3k^2+16k+16)}{k^6}\right)\\ -\frac{2(3k^2+16k+16)}{k^6} & k\left(\frac{k^3+24k^2+80k+64}{k^7}\right)\end{pmatrix}$$

$$=\begin{pmatrix}P_{k,-5} & k\cdot P_{k,-6}\\ P_{k,-6} & k\cdot P_{k,-7}\end{pmatrix},$$

etc. We have observed that the elements of the type $P_{k,-n}$, for a positive integer $n\geq 0$ can be determined directly from the recurrence relation indicated by $P_{k,n+1}=2P_{k,n}+kP_{k,n-1}$.
From these preliminary examples, we will state the following theorem.

Theorem 7. For any integer $n\geq 1$, we obtain

$$V_{k,-n}=\begin{pmatrix}P_{k,-n+1} & k\cdot P_{k,-n}\\ P_{k,-n} & k\cdot P_{k,-n-1}\end{pmatrix}=\begin{pmatrix}P_{k,-(n-1)} & k\cdot P_{k,-n}\\ P_{k,-n} & k\cdot P_{k,-(n+1)}\end{pmatrix},$$

$$\left(V_{k,1}{}^{n}\right)^{-1}=\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^{-n}=\left(V_{k,1}{}^{-1}\right)^{n}.$$

Proof. The authors of the work [2] consider the following matrix

$$V_{k,n}=\begin{pmatrix}P_{k,\mathrm{n}+1} & k\cdot P_{k,\mathrm{n}}\\ P_{k,\mathrm{n}} & k\cdot P_{k,\mathrm{n}-1}\end{pmatrix}.$$

By Definition 5, we consider the *k*-Pell matrix for negative indices

$$V_{k,-n}=\begin{pmatrix}P_{k,-(n-1)} & k\cdot P_{k,-\mathrm{n}}\\ P_{k,-\mathrm{n}} & k\cdot P_{k,-(\mathrm{n}+1)}\end{pmatrix}.$$

On the other hand, the authors of the work [1] use the following Binet formula $P_{k,n}=\dfrac{a^n-b^n}{a-b}$, where they use the properties of the characteristic equation defined by $x^2-2x-k=0$, with the roots indicated by $a=1+\sqrt{1+k}$ and $b=1-\sqrt{1+k}$. From this, we can easily verify that $P_{k,-n}=\dfrac{(-1)^{n+1}}{k^n}P_{k,n}$. Now, let us make the corresponding substitutions in the matrix

$$V_{k,-n}=\begin{pmatrix} P_{k,-n+1} & k\cdot P_{k,-n} \\ P_{k,-n} & k\cdot P_{k,-n-1}\end{pmatrix}=\begin{pmatrix} P_{k,-(n-1)} & k\cdot P_{k,-n} \\ P_{k,-n} & k\cdot P_{k,-(n+1)}\end{pmatrix}$$

$$=\begin{pmatrix} \dfrac{(-1)^n}{k^{n-1}}P_{k,n-1} & k\cdot\dfrac{(-1)^{n+1}}{k^n}P_{k,n} \\ \dfrac{(-1)^{n+1}}{k^n}P_{k,n} & k\cdot\dfrac{(-1)^{n+2}}{k^{n+1}}P_{k,n+1}\end{pmatrix}=\begin{pmatrix} k\dfrac{(-1)^n}{k^n}P_{k,n-1} & -k\cdot\dfrac{(-1)^n}{k^n}P_{k,n} \\ -\dfrac{(-1)^n}{k^n}P_{k,n} & \dfrac{(-1)^n}{k^n}P_{k,n+1}\end{pmatrix}$$

$$=(-1)^n\begin{pmatrix} \dfrac{1}{k^n}kP_{k,n-1} & -\dfrac{1}{k^n}kP_{k,n} \\ -\dfrac{1}{k^n}P_{k,n} & \dfrac{1}{k^n}P_{k,n+1}\end{pmatrix}=\frac{(-1)^n}{k^n}\begin{pmatrix} k\cdot P_{k,n-1} & -k\cdot P_{k,n} \\ -P_{k,n} & P_{k,n+1}\end{pmatrix}$$

On the other hand, we know that $\det V_{k,1}{}^n=\det\begin{pmatrix} P_{k,\mathrm{n}+1} & k\cdot P_{k,\mathrm{n}} \\ P_{k,\mathrm{n}} & k\cdot P_{k,\mathrm{n}-1}\end{pmatrix}=(-1)^n k^n$. In this way, we will write $V_{k,-n}=\dfrac{(-1)^n(-1)^n}{(-1)^n k^n}\begin{pmatrix} k\cdot P_{k,n-1} & -k\cdot P_{k,n} \\ -P_{k,n} & P_{k,n+1}\end{pmatrix}=\dfrac{1}{\det V_{k,n}}\begin{pmatrix} k\cdot P_{k,n-1} & -P_{k,n} \\ -k\cdot P_{k,n} & P_{k,n+1}\end{pmatrix}^T=(V_{k,n})^{-1}=\left(V_{k,1}{}^n\right)^{-1}=\begin{pmatrix} 2 & k \\ 1 & 0\end{pmatrix}^{-n}$. ☐

Corollary 2. For any integers $m,n\geq 1$, we have $V_{k,-(\mathrm{m}+n)}=V_{k,-\mathrm{m}}V_{k,-n}=V_{k,-n}V_{k,-\mathrm{m}}$.

Proof. From the previous theorem, we can write

$$V_{k,-(\mathrm{m}+n)}=\left(V_{k,1}{}^{m+n}\right)^{-1}=\begin{pmatrix} 2 & k \\ 1 & 0\end{pmatrix}^{-(m+n)}=\begin{pmatrix} 2 & k \\ 1 & 0\end{pmatrix}^{-m}\begin{pmatrix} 2 & k \\ 1 & 0\end{pmatrix}^{-n}$$

$$=\left(V_{k,1}{}^{m}\right)^{-1}\left(V_{k,1}{}^{n}\right)^{-1}=V_{k,-\mathrm{m}}V_{k,-n}$$

In an analogous way, we have determined that the result hold $V_{k,-(\mathrm{m}+n)}=V_{k,-n}V_{k,-\mathrm{m}}$, for any integers $m,n\geq 1$. ☐

In the following theorem, we indicate the Binet´s formula discussed at work [2].

Theorem 8. For any integer $n\geq 1$, $R_{k,n}=\left(\dfrac{1-2b}{a-b}\right)a^n+\left(\dfrac{2a-1}{a-b}\right)b^n$, where a,b are the roots of the characteristic equation $x^2-2x-k=0$.

Proof. It can be consulted at work [2]. ☐

In the following theorem, we will reduce the Binet formula corresponding to the terms of negative indices and not discussed in [2].

Theorem 9. For any integer $n \geq 1$, $R_{k,-n} = \frac{(-1)^n}{k^n}\left[\left(\frac{1-2b}{a-b}\right)b^n + \left(\frac{2a-1}{a-b}\right)a^n\right]$

Proof. Easily, we can see that $R_{k,-n} = \left(\frac{1-2b}{a-b}\right)a^{-n} + \left(\frac{2a-1}{a-b}\right)b^{-n} = \left(\frac{1-2b}{a-b}\right)\left(\frac{1}{a}\right)^n + \left(\frac{2a-1}{a-b}\right)\left(\frac{1}{b}\right)^n$. But from Theorem 8, we know that $a \cdot b = -k \therefore \frac{1}{a} = -\frac{b}{k}$ and $\frac{1}{b} = -\frac{a}{k}$. In this way, we will make the following substitutions $R_{k,-n} = \left(\frac{1-2b}{a-b}\right)\left(\frac{1}{a}\right)^n + \left(\frac{2a-1}{a-b}\right)\left(\frac{1}{b}\right)^n = \left(\frac{1-2b}{a-b}\right)\left(-\frac{b}{k}\right)^n + \left(\frac{2a-1}{a-b}\right)\left(-\frac{a}{k}\right)^n =$

$$= \frac{1}{k^n}\left[\left(\frac{1-2b}{a-b}\right)(-b)^n + \left(\frac{2a-1}{a-b}\right)(-a)^n\right] = \frac{(-1)^n}{k^n}\left[\left(\frac{1-2b}{a-b}\right)b^n + \left(\frac{2a-1}{a-b}\right)a^n\right]. \qquad \square$$

Let us consider the following equation $R_{k,n} = 2P_{k,n+1} - 3P_{k,n}$. This relation can be determined from Theorem 4 or from work [2]. We will reduce the corresponding identity to the terms of negative indices and not discussed in [2].

Theorem 10. For any integer $n \geq 1$, we obtain that $R_{k,-n} = 2P_{k,-n+1} - 3P_{k,-n}$.

Proof. We can observe that $2P_{k,-n+1} - 3P_{k,-n} = 2P_{k,-(n-1)} - 3P_{k,-n} = 2\left(\frac{(-1)^n}{k^{n-1}}P_{k,n-1}\right) - 3\left(\frac{(-1)^{n+1}}{k^n}P_{k,n}\right)$

$$= 2\frac{(-1)^n}{k^{n-1}}P_{k,n-1} - 3\frac{(-1)^{n+1}}{k^n}P_{k,n} = 2\frac{(-1)^n}{k^{n-1}}P_{k,n-1} + 3\frac{(-1)^n}{k^n}P_{k,n} = 2\frac{(-1)^n}{k^{n-1}}\frac{a^{n-1}-b^{n-1}}{a-b} + 3\frac{(-1)^n}{k^n}\frac{a^n-b^n}{a-b}$$

$$= \left[2k\frac{(-1)^n}{k^n}\left(\frac{a^{n-1}-b^{n-1}}{a-b}\right) + 3\frac{(-1)^n}{k^n}\left(\frac{a^n-b^n}{a-b}\right)\right] = \frac{(-1)^n}{k^n}\left[\frac{2ka^{n-1}-2kb^{n-1}+3a^n-3b^n}{a-b}\right] =$$

$$= \frac{(-1)^n}{k^n}\left[\frac{2ka^{n-1}-2kb^{n-1}+3a^n-3b^n}{a-b}\right] = \frac{(-1)^n}{k^n}\left[\frac{(2ka^{n-1}+3a^n)-3b^n-2kb^{n-1}}{a-b}\right] =$$

$$= \frac{(-1)^n}{k^n}\left[\frac{2(-ab)a^{n-1}+3a^n-3b^n-2(-ab)b^{n-1}}{a-b}\right] = \frac{(-1)^n}{k^n}\left[\frac{3a^n-2ba^n-3b^n+2ab^n}{a-b}\right]$$

$= \frac{(-1)^n}{k^n}\left[\frac{(2a-3)}{a-b}b^n + \frac{(3-2b)}{a-b}a^n\right]$. On the other hand, we can directly verify that $2a-3 = 2(2-b)-3 = 4-2b-3 = 1-2b$ and $3-2b = 3-2(2-a) = -1+2a = 2a-1$. Finally, we deduce $2P_{k,-n+1} - 3P_{k,-n} = \frac{(-1)^n}{k^n}\left[\frac{(2a-3)}{a-b}b^n + \frac{(3-2b)}{a-b}a^n\right] = \frac{(-1)^n}{k^n}\left[\frac{(1-2b)}{a-b}b^n + \frac{(2a-1)}{a-b}a^n\right]$

$= R_{k,-n}$. $\square$

Before demonstrating the following theorem, let us see the following particular examples that determine elements of the generalized sequence with negative index.

$$(V_{k,1})^{-1}\cdot S_{k,0}=\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^{-1}\begin{pmatrix}1 & 2k\\ 2 & -3\end{pmatrix}=\begin{pmatrix}2 & k\left(-\frac{3}{k}\right)\\ -\frac{3}{k} & k\left(\frac{2k+6}{k^2}\right)\end{pmatrix}=\begin{pmatrix}R_{k,0} & kR_{k,-1}\\ R_{k,-1} & kR_{k,-2}\end{pmatrix},$$

$$(V_{k,1})^{-2}S_{k,0}=\begin{pmatrix}-\frac{3}{k} & k\left(\frac{2k+6}{k^2}\right)\\ \frac{2k+6}{k^2} & k\left(-\frac{7k+12}{k^3}\right)\end{pmatrix}=\begin{pmatrix}R_{k,-1} & kR_{k,-2}\\ R_{k,-2} & kR_{k,-3}\end{pmatrix},$$

$$(V_{k,1})^{-3}S_{k,0}=\begin{pmatrix}\frac{2k+6}{k^2} & k\left(-\frac{7k+12}{k^3}\right)\\ -\frac{7k+12}{k^3} & k\left(\frac{2(k^2+10k+12)}{k^4}\right)\end{pmatrix}=\begin{pmatrix}R_{k,-2} & kR_{k,-3}\\ R_{k,-3} & kR_{k,-4}\end{pmatrix},$$

$$(V_{k,1})^{-4}S_{k,0}=\begin{pmatrix}-\frac{7k+12}{k^3} & k\left(\frac{2(k^2+10k+12)}{k^4}\right)\\ \frac{2(k^2+10k+12)}{k^4} & k\left(-\frac{11k^2+52k+48}{k^5}\right)\end{pmatrix}=\begin{pmatrix}R_{k,-3} & kR_{k,-4}\\ R_{k,-4} & kR_{k,-5}\end{pmatrix},$$

$$(V_{k,1})^{-5}S_{k,0}=\begin{pmatrix}\frac{2(k^2+10k+12)}{k^4} & k\left(-\frac{11k^2+52k+48}{k^5}\right)\\ -\frac{11k^2+52k+48}{k^4} & k\left(\frac{2(k^3+21k^2+64k+48)}{k^5}\right)\end{pmatrix}=\begin{pmatrix}R_{k,-4} & kR_{k,-5}\\ R_{k,-5} & kR_{k,-6}\end{pmatrix},$$

etc. From these particular examples, we see the following theorem that allows determining the generating matrices for the family of matrices $\langle S_{k,-n}\rangle_{n\in IN}$, which we have preliminarily defined by

$$S_{k,-n}=\begin{pmatrix}R_{k,-n+1} & k\cdot R_{k,-n}\\ R_{k,-n} & k\cdot R_{k,-n-1}\end{pmatrix}=\begin{pmatrix}R_{k,-(n-1)} & k\cdot R_{k,-n}\\ R_{k,-n} & k\cdot R_{k,-(n+1)}\end{pmatrix}.$$

Theorem 11. For any integer $n\geq 1$, we obtain that $S_{k,-n}=(V_{k,1})^{-n}S_{k,0}=S_{k,0}(V_{k,1})^{-n}$.

Proof. From the previous theorem we will consider $S_{k,-n}=\begin{pmatrix}R_{k,-\mathrm{n}+1} & k\cdot R_{k,-\mathrm{n}}\\ R_{k,-\mathrm{n}} & k\cdot R_{k,-\mathrm{n}-1}\end{pmatrix}=\begin{pmatrix}R_{k,-(n-1)} & k\cdot R_{k,-\mathrm{n}}\\ R_{k,-\mathrm{n}} & k\cdot R_{k,-(n+1)}\end{pmatrix}$

$$=\begin{pmatrix}2P_{k,-(n-2)}-3P_{k,-(n-1)} & k\cdot(2P_{k,-n+1}-3P_{k,-n})\\ 2P_{k,-n+1}-3P_{k,-n} & k\cdot(2P_{k,-n}-3P_{k,-(n+1)})\end{pmatrix}=2\begin{pmatrix}P_{k,-(n-2)} & kP_{k,-n+1}\\ P_{k,-n+1} & kP_{k,-n}\end{pmatrix}-3\begin{pmatrix}P_{k,-(n-1)} & kP_{k,-n}\\ P_{k,-n} & kP_{k,-(n+1)}\end{pmatrix}.$$

On the other hand, we know that $V_{k,-\mathrm{n}}=\begin{pmatrix}P_{k,-(n-1)} & k\cdot P_{k,-n}\\ P_{k,-n} & k\cdot P_{k,-(n+1)}\end{pmatrix}=\left(V_{k,1}{}^{n}\right)^{-1}=\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^{-n}$ and $V_{k,-(n-1)}=\begin{pmatrix}P_{k,-(n-1-1)} & k\cdot P_{k,-(n-1)}\\ P_{k,-(n-1)} & k\cdot P_{k,-(n-1+1)}\end{pmatrix}=\begin{pmatrix}P_{k,-(n-2)} & k\cdot P_{k,-(n-1)}\\ P_{k,-(n-1)} & k\cdot P_{k,-n}\end{pmatrix}=\left(V_{k,1}{}^{n-1}\right)^{-1}=\begin{pmatrix}2 & k\\ 1 & 0\end{pmatrix}^{-(n-1)}$. In this way, we will make the corresponding substitutions to determine that $S_{k,-n}=\begin{pmatrix}R_{k,-(n-1)} & k\cdot R_{k,-\mathrm{n}}\\ R_{k,-\mathrm{n}} & k\cdot R_{k,-(n+1)}\end{pmatrix}=$

$= \left(2\begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{-n+1} - 3\begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{-n} \right) = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{-n} \left(2\begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix} - 3I \right)$. We can determine that $2\begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix} - 3I = \begin{pmatrix} 4 & 2k \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix}$. Thus, we still determine that $S_{k,-n} = \begin{pmatrix} R_{k,-(n-1)} & k \cdot R_{k,-\mathrm{n}} \\ R_{k,-\mathrm{n}} & k \cdot R_{k,-(n+1)} \end{pmatrix} = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{-n} \begin{pmatrix} 1 & 2k \\ 2 & -3 \end{pmatrix} = (V_{k,1})^{-n} S_{k,0} = S_{k,0}(V_{k,1})^{-n}$. In addition, we can see that $V_{k,1}{}^{-1} S_{k,0} = S_{k,0} V_{k,1}{}^{-1}$ and $V_{k,1}{}^{-1} S_{k,0}{}^{-1} = S_{k,0}{}^{-1} V_{k,1}{}^{-1}$. □

Theorem 12. For any integer $n \geq 1$, we obtain that $\left(\mathrm{S}_{k,\mathrm{n}}\right)^{-1} = S_{k,0}{}^{-1} S_{k,-n} S_{k,0}{}^{-1}$.

Proof. From Theorem 11, we know that $S_{k,-n} = (V_{k,1})^{-n} S_{k,0} \therefore (V_{k,1})^{-n} = S_{k,0}{}^{-1} S_{k,-n}$. On the other hand, since $\mathrm{S}_{k,\mathrm{n}} = \mathrm{S}_{k,0}\, V_{k,1}{}^{n}$ it follows that $\left(\mathrm{S}_{k,\mathrm{n}}\right)^{-1} = V_{k,1}{}^{-n}\, \mathrm{S}_{k,0}{}^{-1} = \left(S_{k,0}{}^{-1} S_{k,-n}\right) \mathrm{S}_{k,0}{}^{-1} = S_{k,0}{}^{-1} S_{k,-n}\, \mathrm{S}_{k,0}{}^{-1}$. □

Corallary 3. For any integers $m, n \geq 1$, we obtain that:

(i) $\left(\mathrm{S}_{k,\mathrm{n}}\right)^{-m} = \left(\mathrm{S}_{k,\mathrm{n}}{}^{-1}\right)^{m} = S_{k,0}{}^{-m} (V_{k,1})^{-(m \cdot n)}$;

(ii) $\left(\mathrm{S}_{k,\mathrm{n}}\right)^{-n} = S_{k,0}{}^{-n} (V_{k,1})^{-n^2}$.

Proof. Initially, we will consider that $(S_{k,n})^{-1} = S_{k,0}{}^{-1} S_{k,-n} S_{k,0}{}^{-1}$. In the next step, we will see the behavior of the matrix $\left(\mathrm{S}_{k,\mathrm{n}}\right)^{-2} = S_{k,0}{}^{-1} S_{k,-n} S_{k,0}{}^{-1} S_{k,0}{}^{-1} S_{k,-n} S_{k,0}{}^{-1} = S_{k,0}{}^{-1} (V_{k,1})^{-n} S_{k,0} S_{k,0}{}^{-1} S_{k,0}{}^{-1} (V_{k,1})^{-n} S_{k,0} S_{k,0}{}^{-1}$ $= S_{k,0}{}^{-1} (V_{k,1})^{-n} I S_{k,0}{}^{-1} (V_{k,1})^{-n} I = S_{k,0}{}^{-1} S_{k,0}{}^{-1} (V_{k,1})^{-n} (V_{k,1})^{-n} = S_{k,0}{}^{-2} (V_{k,1})^{-2n}$, since $V_{k,1}{}^{-1} S_{k,0}{}^{-1} = S_{k,0}{}^{-1} V_{k,1}{}^{-1}$. Now, let us see what $\left(\mathrm{S}_{k,\mathrm{n}}\right)^{-3} = S_{k,0}{}^{-1} S_{k,-n} S_{k,0}{}^{-1} S_{k,0}{}^{-1} S_{k,-n} S_{k,0}{}^{-1} S_{k,0}{}^{-1} S_{k,-n} S_{k,0}{}^{-1}$. Repeating the previous argument, we will see that $\left(\mathrm{S}_{k,\mathrm{n}}\right)^{-3} = S_{k,0}{}^{-1} (V_{k,1})^{-n} S_{k,0}{}^{-1} (V_{k,1})^{-n} S_{k,0}{}^{-1} (V_{k,1})^{-n} = S_{k,0}{}^{-3} (V_{k,1})^{-3n}$. Similarly, we can also determine that $\left(\mathrm{S}_{k,\mathrm{n}}\right)^{-4} = S_{k,0}{}^{-1} (V_{k,1})^{-n} S_{k,0}{}^{-1} (V_{k,1})^{-n} S_{k,0}{}^{-1} (V_{k,1})^{-n} S_{k,0}{}^{-1} (V_{k,1})^{-n}$ $= S_{k,0}{}^{-4} (V_{k,1})^{-4n}$. By mathematical induction on *m*, let us admit that $\left(S_{k,n}\right)^{-m} = \left(S_{k,n}{}^{-1}\right)^{m}$ $= S_{k,0}{}^{-m} (V_{k,1})^{-(m \cdot n)}$ and, for the next step, we determine that $\left(S_{k,n}\right)^{-(m+1)} = \left(S_{k,n}{}^{-1}\right)^{m+1}$ $= \left(S_{k,n}{}^{-1}\right)^{m} \left(S_{k,n}{}^{-1}\right) = S_{k,0}{}^{-m} (V_{k,1})^{-(m \cdot n)} \left(S_{k,0}{}^{-1} S_{k,-n} S_{k,0}{}^{-1}\right) = S_{k,0}{}^{-m} (V_{k,1})^{-(m \cdot n)} S_{k,0}{}^{-1} S_{k,-n} S_{k,0}{}^{-1}$ $= S_{k,0}{}^{-m} (V_{k,1})^{-(m \cdot n)} S_{k,0}{}^{-1} (V_{k,1})^{-n} S_{k,0} S_{k,0}{}^{-1} = S_{k,0}{}^{-m} (V_{k,1})^{-(m \cdot n)} S_{k,0}{}^{-1} (V_{k,1})^{-n} = S_{k,0}{}^{-(m+1)} (V_{k,1})^{-(m \cdot n + n)}$ $= S_{k,0}{}^{-(m+1)} (V_{k,1})^{-((m \cdot +1)n)}$. Finally, if we take $m = n \therefore \left(\mathrm{S}_{k,\mathrm{n}}\right)^{-n} = S_{k,0}{}^{-n} (V_{k,1})^{-n^2}$. □

Corallary 4. For any integers $m, n \geq 1$, we obtain that:

(i) $S_{k,-(n+m)} S_{k,0} = S_{k,-n} S_{k,-m} = S_{k,-m} S_{k,-n}$;

(ii) $S_{k,-(n+1)}{}^{2} = S_{k,-1}{}^{2} V_{k,-2n}$;

(iii) $S_{k,-n}{}^{2} = S_{k,0}{}^{2} V_{k,-2n}$.

Proof. Recalling the following commutativity of the matrix product ${V_{k,1}}^{-1}S_{k,0} = S_{k,0}{V_{k,1}}^{-1}$, we can verify $S_{k,-(n+m)}S_{k,0} = S_{k,0}S_{k,0}(V_{k,1})^{-(n+m)} = S_{k,0}(V_{k,1})^{-n}S_{k,0}(V_{k,1})^{-m} = S_{k,-n}S_{k,-m} = S_{k,-m}S_{k,-n}$. For the second item, we know that $S_{k,-n} = (V_{k,1})^{-n}S_{k,0} = S_{k,0}(V_{k,1})^{-n}$. Thus, we can see directly that ${S_{k,-(n+1)}}^2 = S_{k,-(n+1)}S_{k,-(n+1)} = (V_{k,1})^{-(n+1)}S_{k,0}(V_{k,1})^{-(n+1)}S_{k,0} = (V_{k,1})^{-n}(V_{k,1})^{-1}S_{k,0}(V_{k,1})^{-n}(V_{k,1})^{-1}S_{k,0}$ $= (V_{k,1})^{-2n}\left((V_{k,1})^{-1}S_{k,0}\right)\left((V_{k,1})^{-1}S_{k,0}\right) = {S_{k,-1}}^2(V_{k,1})^{-2n} = {S_{k,-1}}^2V_{k,-2n}$. Finally, we can easily see that ${S_{k,-n}}^2 = S_{k,-n}S_{k,-n} = S_{k,0}(V_{k,1})^{-n}S_{k,0}(V_{k,1})^{-n} = {S_{k,0}}^2(V_{k,1})^{-2n} = {S_{k,0}}^2V_{k,-2n}$, from the Theorem 7. □

4 Concluding remarks and future research

For our future research, we will consider the introduction of quaternions and octonions representations for the matrices discussed in the previous sections. For example, we can define the following and new matrix

$$QV_{k,n} = \begin{pmatrix} QP_{k,\mathrm{n}+1} & k\cdot QP_{k,\mathrm{n}} \\ QP_{k,\mathrm{n}} & k\cdot QP_{k,\mathrm{n}-1} \end{pmatrix},$$

where we considered $QP_{k,n} = P_{k,n} + P_{k,n-1}\vec{i} + P_{k,n-2}\vec{j} + P_{k,n-3}\vec{k}$, the canonical basis $\{1, \vec{i}, \vec{j}, \vec{k}\}$. From this, we take the following matrix decomposition indicated by

$$\begin{aligned}
QV_{k,n} &= \begin{pmatrix} QP_{k,n+1} & k\cdot QP_{k,n} \\ QP_{k,n} & k\cdot QP_{k,n-1} \end{pmatrix} \\
&= \begin{pmatrix} P_{k,n+1} + P_{k,n}\vec{i} + P_{k,n-1}\vec{j} + P_{k,n-2}\vec{k} & k\cdot(P_{k,n} + P_{k,n-1}\vec{i} + P_{k,n-2}\vec{j} + P_{k,n-3}\vec{k}) \\ P_{k,n} + P_{k,n-1}\vec{i} + P_{k,n-2}\vec{j} + P_{k,n-3}\vec{k} & k\cdot(P_{k,n-1} + P_{k,n-2}\vec{i} + P_{k,n-3}\vec{j} + P_{k,n-4}\vec{k}) \end{pmatrix} \\
&= \begin{pmatrix} P_{k,n+1} & k\cdot P_{k,n} \\ P_{k,n} & k\cdot P_{k,n-1} \end{pmatrix} + \begin{pmatrix} P_{k,n} & k\cdot P_{k,n-1} \\ P_{k,n-1} & k\cdot P_{k,n-2} \end{pmatrix}\vec{i} + \begin{pmatrix} P_{k,n-1} & k\cdot P_{k,n-2} \\ P_{k,n-2} & k\cdot P_{k,n-3} \end{pmatrix}\vec{j} + \begin{pmatrix} P_{k,n-2} & k\cdot P_{k,n-3} \\ P_{k,n-3} & k\cdot P_{k,n-4} \end{pmatrix}\vec{k} \\
&= \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^n + \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n-1}\vec{i} + \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n-2}\vec{j} + \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{n-3}\vec{k}
\end{aligned}$$

since we have

$${V_{k,1}}^n = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} P_{k,\mathrm{n}+1} & k\cdot P_{k,\mathrm{n}} \\ P_{k,\mathrm{n}} & k\cdot P_{k,\mathrm{n}-1} \end{pmatrix}.$$

Or, we can still consider

$$QV_{k,n} = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^n \left[I + \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{-1}\vec{i} + \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{-2}\vec{j} + \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{-3}\vec{k} \right].$$

From the previous results, we can now determine that

$$\begin{aligned}
&\left[I + \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{-1}\vec{i} + \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{-2}\vec{j} + \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^{-3}\vec{k} \right] \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ \frac{1}{k} & -\frac{2}{k} \end{pmatrix}\vec{i} + \begin{pmatrix} \frac{1}{k} & -\frac{2}{k} \\ -\frac{2}{k^2} & \frac{k+4}{k^2} \end{pmatrix}\vec{j} + \begin{pmatrix} -\frac{2}{k^2} & \frac{k+4}{k^2} \\ \frac{k+4}{k^3} & -\frac{4(k+2)}{k^3} \end{pmatrix}\vec{k}
\end{aligned}$$

$$= \begin{pmatrix} P_{k,1} + P_{k,0}\vec{i} + P_{k,-1}\vec{j} + P_{k,-2}\vec{k} & k\left(P_{k,0} + P_{k,-1}\vec{i} + P_{k,-2}\vec{j} + P_{k,-3}\vec{k}\right) \\ P_{k,0} + P_{k,-1}\vec{i} + P_{k,-2}\vec{j} + P_{k,-3}\vec{k} & k\left(P_{k,-1} + P_{k,-2}\vec{i} + P_{k,-3}\vec{j} + P_{k,-4}\vec{k}\right) \end{pmatrix}.$$

Thus, we can determine the following formula

$$QV_{k,n} = \begin{pmatrix} QP_{k,\mathrm{n}+1} & k \cdot QP_{k,\mathrm{n}} \\ QP_{k,\mathrm{n}} & k \cdot QP_{k,\mathrm{n}-1} \end{pmatrix} = \begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} QP_{k,1} & k \cdot QP_{k,0} \\ QP_{k,0} & k \cdot QP_{k,-1} \end{pmatrix}$$

or

$$QV_{k,n} = \begin{pmatrix} P_{k,\mathrm{n}+1} & k \cdot P_{k,\mathrm{n}} \\ P_{k,\mathrm{n}} & k \cdot P_{k,\mathrm{n}-1} \end{pmatrix} \begin{pmatrix} QP_{k,1} & k \cdot QP_{k,0} \\ QP_{k,0} & k \cdot QP_{k,-1} \end{pmatrix},$$

for every integer $n \geq 0$. Here, we identify the same generating matrix

$$\begin{pmatrix} 2 & k \\ 1 & 0 \end{pmatrix}^n.$$

Similarly, we will develop the study of the matrix determined by the Fibonacci generalized matrix sequence, from representations of quaternions described by

$$QS_{k,n} = \begin{pmatrix} QR_{k,\mathrm{n}+1} & k \cdot QR_{k,\mathrm{n}} \\ QR_{k,\mathrm{n}} & k \cdot QR_{k,\mathrm{n}-1} \end{pmatrix},$$

where $QR_{k,n} = R_{k,n} + R_{k,n-1}\vec{i} + R_{k,n-1}\vec{j} + R_{k,n-1}\vec{k}$ are the generalized Fibonacci quaternion of n-th order. Finally, from Theorem 11, we can write that

$$QS_{k,n} = \begin{pmatrix} QR_{k,\mathrm{n}+1} & k \cdot QR_{k,\mathrm{n}} \\ QR_{k,\mathrm{n}} & k \cdot QR_{k,\mathrm{n}-1} \end{pmatrix} = V_{k,1}{}^n \left[S_{k,0} + S_{k,-1}\vec{i} + S_{k,-2}\vec{j} + S_{k,-3}\vec{k} \right],$$

for every integer $n \geq 0.$ In the following steps, we will try to determine the behavior of the matrices $QV_{k,-n}$ and $QS_{k,-n}$.

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