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s-th power of Fibonacci number of the form $2^a + 3^b + 5^c$

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Abstract: In this paper, we solve the Diophantine equation $F_n^s = 2^a + 3^b + 5^c$, where a, b, c and s are positive integers with $1 \leq \max\{a, b\} \leq c$.
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1 Introduction

Let $(F_n)_{m\geq 0}$ be the Fibonacci sequence given by the relation $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$, $F_1 = 1$ for all $n \geq 2$. It has many amazing combinatorial identities (see [7]). Put $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then the well-known Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1}$$

holds for $n \geq 0$.

The problem of finding the different types of numbers among the terms of a linear recurrence has a long history. One of the popular results by Bageaud, Mignotte and Siksek [2] is that the integers $0,\ 1,\ 8,\ 144$ among the Fibonacci numbers and the integers $1,\ 4$ among the Lucas numbers (an associated sequence of Fibonacci) can be written in the form y^t where $t > 1$. Szalay and Luca [4] showed that there are only finitely many quadruples (n, a, b, p) such that $F_n = p^a \pm p^b + 1$ where p is a prime number. Marques and Togbé [5] determined the Fibonacci numbers and the Lucas numbers of the form $2^a + 3^b + 5^c$ under $1 \leq \max\{a, b\} \leq c$. Bertők, Hajdu, Pink and Rábani [1] removed this condition. Namely, they gave full solutions of the equation

$$U_n = 2^a + 3^b + 5^c,$$

where U_n is the n-th Fibonacci, Lucas, Pell or Pell–Lucas number. We refer to the paper of Shorey and Stewart [9] for pure powers in recurrence sequences and some related Diophantine equations.

In this work, we generalize the problem of Marques and Togbé. We solve the Diophantine equation

$$F_n^s = 2^a + 3^b + 5^c \tag{2}$$

for $s \geq 1$ integer and $1 \leq \max\{a, b\} \leq c$. Our result is following,

Theorem 1.1. *The solution of the equation (2) is* $(n, s, a, b, c) = (3, 5, 2, 1, 2)$.

2 Auxiliary results

Before going further, we present several lemmas. The following lemma was given by Matveev [6].

Lemma 2.1. *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \gamma_2, \ldots \gamma_t$ be positive real numbers of $\mathbb{K}$, and $b_1, b_2, \ldots b_t$ be rational integers. Put*

$$B \geq \max\{|b_1|, |b_2|, \ldots, |b_t|\},$$

and

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1.$$

Let $A_1, \ldots, A_t$ be real numbers such that

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \qquad i = 1, \ldots, t.$$

Then, assuming that $\Lambda \neq 0$, we have

$$\begin{aligned} |\Lambda| \;>\; & \exp\left(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2\right. \\ & \left. \times (1 + \log D)(1 + \log B) A_1 \ldots A_t\right) \end{aligned} \tag{3}$$

As usual, in the above lemma, the logarithmic height of the algebraic number η is defined as

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\left(\max\left\{\left|\eta^{(i)}, 1\right|\right\}\right)\right)$$

with d being the degree of η over $\mathbb{Q}$ and $\left(\eta^{(i)}\right)_{1\leq i\leq d}$ being the conjugates of η over $\mathbb{Q}$.

Application of the Matveev theorem gives the large upper bound. In order to reduce this bound, we use the following lemma.

Lemma 2.2. *Suppose that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of the irrational number γ such that $q > 6M$ and $\epsilon = \parallel \mu q \parallel - M \parallel \gamma q \parallel$, where μ is a real number and $\parallel \cdot \parallel$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < AB^{-m}$$

in positive integers m and n with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m < M.$$

The following lemma is in the paper [3] (the case $k = 2$).

Lemma 2.3. *For every positive integer $n \geq 2$, we have*

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1},$$

where α is the dominant root of the characteristic equation $x^2 - x - 1 = 0$.

Lemma 2.4. *There is no solution of the equation*

$$2^s = 2^a + 3^b + 5^c \tag{4}$$

for $1 \leq \max\{a, b\} \leq c,\ c \geq 6$ and s being positive integers.

Proof. By (4) together with the facts that $2 < \sqrt{5}$ and $3 < 5^{0.7}$, we get

$$\left|1 - 2^s 5^{-c}\right| < \frac{2}{(1.6)^c}. \tag{5}$$

We take $\alpha_1 := 2,\ \alpha_2 := 5,\ b_1 := s,\ b_2 := c$. For this choice $D = 1,\ t = 2,\ B = s$ and $A_1 = 0.7 > \log 2$, $A_2 = 1.61 > \log 5$. The Lemma 2.1 yields that

$$\exp\left(C \cdot (1 + \log s)\right) < \left|1 - 2^s 5^{-c}\right| < \frac{2}{(1.6)^c}, \tag{6}$$

where $C := 1.4 \cdot 30^5 \cdot 2^{4.5} \cdot 0.7 \cdot 1.61$. Since $2^s < 5^{c+1}$, we have that $0.4s < c+1$. So, the inequality

$$s < 2.5 \cdot 10^{11}$$

is obtained. Let $z := |s \log 2 - c \log 5|$. Note that (5) can be written as

$$|1 - e^z| < \frac{3.2}{(1.2)^s},$$

since $0.4s < c + 1$ holds. Since $1 < 2^s 5^{-c}$, then $z > 0$ holds. We obtain that

$$0 < |s \log 2 - c \log 5| < |1 - e^z| < \frac{3.2}{(1.2)^s}.$$

Dividing both sides by $\log 5$ yields that

$$\left| s\frac{\log 2}{\log 5} - c \right| < \frac{2}{(1.2)^s}.$$

Let $\gamma := \frac{\log 2}{\log 5}$ and $[a_0, a_1, a_2, \ldots] = [0, 2, 3, 9, 2, \ldots]$ be the continued fraction of γ, and let p_k/q_k be its k-th convergent. *Mathematica* reveals that

$$q_{23} < 2.5 \cdot 10^{11} < q_{24}.$$

$a_M := \max\{a_i; i = 0, \ldots, 24\} = a_{23} = 42$. By the properties of continued fractions, we obtain

$$\frac{1}{(a_M + 2)\, s} < \left| s\frac{\log 2}{\log 5} - c \right| < \frac{2}{(1.2)^s}$$

which yields that $s \leq 45$ as $a_M = 42$. Since $5^c < 2^s$, then we deduce that $c \leq 19$. A quick inspection using *Mathematica* reveals that there is no solution of the equation (4) with $1 \leq \max\{a, b\} \leq c$ and $6 \leq c \leq 19$. □

3 Proof of Theorem 1.1

Firstly, assume that $1 \leq c \leq 5$. Then the solution of the equation (2) is given in Theorem 1.1. From now on, suppose that $c \geq 6$. Lemma 2.3 gives that $\alpha^{s(n-2)} < F_n^s < 3 \cdot 5^c < 5^{1.1c}$. So, we have the fact $s < c$. Since $a, b, c \geq 1$, then $n \geq 3$ holds. If $n = 3$, then we can rewrite formula (2) as

$$2^s = 2^a + 3^b + 5^c.$$

which is investigated in Lemma 2.4. If $n = 4$ and $n = 5$ hold, then we arrive at a contradiction since the left-hand side of the equation $F_n^s = 2^a + 3^b + 5^c$ is odd, while right-hand side is even. Therefore, we suppose that $n \geq 6$.

Using formula (1), we rewrite the equation (2) as

$$F_n^s - 5^c = 2^a + 3^b. \tag{7}$$

Since $\max\{a, b\} \geq 1$, then the right-hand side of above equation is positive. Dividing both sides of the equation (7) by 5^c, we obtain

$$\left| F_n^s 5^{-c} - 1 \right| < \frac{2}{5^{0.3c}} \tag{8}$$

where we use $2 < 3 < 5^{0.7}$.

In the application of theorem of Matveev, we take $\alpha_1 = F_n,\ \alpha_2 = 5,\ b_1 = s,\ b_2 = c$. We also take

$$\Lambda := F_n^s 5^{-c} - 1.$$

Since we assume $n \geq 6$, then it is obvious that $\Lambda \neq 0$. We can take the degree $D = 1$. Then $A_1 = \log F_n$ and $A_2 = 1.61 > \log 5$ follow. As $s < c$, then we get $B = c$ together with $t = 2$.

After applying the inequality (3) to get lower bound for the form Λ, then we have

$$e^{-C_{2,1}(1+\log c)\times 1.61\times \log F_n} < \frac{2}{5^{0.3c}}, \tag{9}$$

where $C_{2,1} = 1.4 \times 30^5 \times 2^{4.5}$. Hence, we obtain that

$$\frac{c}{\log c} < 2.5 \times 10^9 (n-1), \tag{10}$$

where we used the fact $1 + \log c < 2\log c$. It is easy to prove that $\frac{x}{\log x} < A$ yields $x < 2A\log A$. After rewriting the formula (9), we obtain

$$c < 7.3 \times 10^{10} (n-1)\log(n-1) \tag{11}$$

by the inequality $21.64 + \log(n-1) < 14.6 \times \log(n-1)$.

Assume that $n \in [6, 233]$. Label $z := s\log F_n - c\log 5$. Hence, by the equation (8)

$$0 < z < e^z - 1 < \frac{2}{5^{0.3c}} \tag{12}$$

follows. Dividing both sides by $\log 5$, we obtain

$$0 < s\gamma - c < 1.25 \times 5^{-0.3c}$$

where $\gamma := \frac{\log F_n}{\log 5}$. Let $[a_0, a_1, a_2, \ldots]$ be the continued fraction of γ, and let p_k/q_k be its k-th convergent. We have

$$s < c < 9.23 \times 10^{13}$$

by the inequality (11). A quick inspection using *Mathematica* reveals that $q_{40} > M$. Moreover, $a_M := \max\{a_i, i = 0, 1, \ldots, 40\} = 3996$. From the properties of continued fractions, we get that

$$|s\gamma - c| > \frac{1}{(a_M + 2)\, s}. \tag{13}$$

Comparing the estimates (12) and (13) we get

$$\frac{1}{3998s} < 1.25 \times 5^{-0.3c} \Rightarrow 5^{0.3s} < 5^{0.3c} < 4997.5s,$$

which yields that $s \leq 23$. Hence, $c \leq 1581$ follows. In order to decrease the upper bound for c, we use that $\nu_5\left(F_n^s - 2^a - 3^b\right) = c$. Thus, *Mathematica* returns $\nu_5\left(F_n^s - 2^a - 3^b\right) = c \leq 12$ for $1 \leq s \leq 23,\ 6 \leq n \leq 233$ and $c > \max\{a, b\} \geq 1$. Therefore, $c \leq 12$ gives that $n \leq 44$. Then the solutions of the equation (2) are given Theorem 1.1.

From now on, assume that $n > 233$. In order to find the upper bound for c, we use the key argument in the paper [8]. Let $x := \frac{s}{\alpha^{2n}}$. From the above inequality (11), it follows that

$$x < \frac{7.3 \times 10^{10} (n-1)\log(n-1)}{\alpha^{2n}} < \frac{2}{\alpha^n},$$

where it holds for $n > 233$. We now write

$$F_n^s = \frac{\alpha^{ns}}{5^{\frac{s}{2}}}\left(1 - \frac{(-1)^n}{\alpha^{2n}}\right)^s. \tag{14}$$

In the paper of Luca and Oyono [8], it was proven that

$$\left|\left(1 - \frac{(-1)^n}{\alpha^{2n}}\right)^s - 1\right| < \frac{2}{\alpha^n}. \tag{15}$$

Let $\Lambda_2 := 5^{c+\frac{s}{2}}\alpha^{-ns} - 1$. From the formulas (2) and (14) together with the inequality (15), we have

$$|\Lambda_2| < \frac{2}{\alpha^n} + \frac{\left(2^a + 3^b\right)5^{\frac{s}{2}}}{\alpha^{ns}}. \tag{16}$$

For the inequality (16) the facts that $2^a + 3^b < 2 \times 5^{0.7c}$ and $n > 233$ yield that

$$|\Lambda_2| < 0.8.$$

The last inequality gives that $\frac{5^{\frac{s}{2}}}{\alpha^{ns}} < \frac{2}{5^c}$. The inequality (16) yields

$$|\Lambda_2| < \frac{2}{\alpha^c} + \frac{2}{\alpha^n} = \frac{4}{\alpha^l},$$

where $l = \min\{n, c\}$. We use again the theorem of Matveev. We take $k = 2$, $\alpha_1 := \alpha$, $\alpha_2 := 5$, $b_1 := ns$, $b_2 := c + \frac{s}{2}$. As in the previous application of Matveev's result, we can take $D := 2$, $A_1 := 0.5$, $A_2 := 1.61$. Note that $\alpha^c < 5^c < \alpha^{s(n-1)}$ gives $c + \frac{s}{2} < c + s < ns$. So, we take $B := ns$. We thus get that

$$\exp\left(-C_{2,2}\left(1 + \log ns\right) \times 0.5 \times 1.61\right) < \frac{4}{\alpha^l},$$

where $C_{2,2} = 1.4 \times 30^5 \times 2^{4.5} \times 4\left(1 + \log 2\right)$. This leads to

$$l < \frac{C_{2,2}\left(\log ns\right) \times 1.61}{\log \alpha}.$$

If $l = n$, then the last inequalities

$$\begin{aligned} n &< \frac{C_{2,2}\left(\log ns\right) \times 1.61}{\log \alpha} \\ &< \frac{C_{2,2}\left(\log n\left(7.3 \times 10^{10}\left(n-1\right)\log\left(n-1\right)\right)\right) \times 1.61}{\log \alpha} \\ &< \frac{C_{2,2}\left(\log\left(7 \times 10^{10} n^3\right)\right) \times 1.61}{\log \alpha} \end{aligned}$$

give that $n < 1.92 \times 10^{12}$. By the inequality (11), we get

$$c < 7.3 \times 10^{10}\left(n-1\right)\log\left(n-1\right) < 4 \times 10^{24}.$$

If $l = c$, then we have that

$$c < \frac{C_{2,2}\left(\log ns\right) \times 1.61}{\log \alpha} < \frac{C_{2,2}\left(\log 6.8c\right) \times 1.61}{\log \alpha}$$

yields $c < 6 \times 10^{11}$, where we used the fact

$$\alpha^{ms} < 5^{1.2c}\alpha^{2s} < \alpha^{4.8c}\alpha^{2s} = \alpha^{6.8c}.$$

At any rate, we get

$$c < 4 \times 10^{24}.$$

Next we take $\Gamma := (\frac{s}{2} + c)\log 5 - ns\log\alpha$. Observe that $\Lambda_2 = e^{\Gamma} - 1$. Since $|\Lambda_2| < 0.8$, then we have $\left|e^{\Gamma} - 1\right| < 0.8$, which yields that $e^{|\Gamma|} < 2$. Hence,

$$|\Gamma| \leq e^{|\Gamma|}\left|e^{\Gamma} - 1\right| < 2\,|\Lambda_2| < \frac{2}{\alpha^c} + \frac{2}{\alpha^n}.$$

This leads to

$$\begin{aligned}\left|\frac{\log\alpha}{\log 5} - \frac{c + \frac{s}{2}}{ns}\right| = \left|\frac{\log\alpha}{\log 5} - \frac{2c + s}{2ns}\right| &< \frac{1}{ns\log 5}\left(\frac{2}{\alpha^c} + \frac{2}{\alpha^n}\right) \\ &< \frac{1}{374s}\left(\frac{2}{\alpha^c} + \frac{2}{\alpha^n}\right) \qquad (17)\end{aligned}$$

since $n > 233$. Assume that $c \geq 30$. In this case, note that $\alpha^n > 160c^2$ (as $n < ns < 6.8c$) and $\alpha^c > 160c^2$. Hence, we get that by the inequality (17) by the fact $\alpha^{ns} < \alpha^{6.8c}$,

$$\begin{aligned}\left|\frac{\log\alpha}{\log 5} - \frac{2c + s}{2ns}\right| &< \frac{1}{14960sc^2} < \frac{1}{14960c^2} \qquad (18) \\ &< \frac{6.8^2}{14960\,(ns)^2} < \frac{1}{80\,(2ns)^2}.\end{aligned}$$

By a criterion of Legendre, the rational number $\frac{2c+s}{2ns}$ converts to $\gamma := \frac{\log\alpha}{\log 5}$.

Let $[a_0, a_1, a_2, \ldots] = [0, 3, 2, 1, \ldots]$ be the continued fraction of γ, and p_k/q_k be its k-th convergent. Assume that $\frac{2c+s}{2ns} = \frac{p_t}{q_t}$ for some t. We have $q_{49} > 4 \times 10^{24}$. Thus, $t \in \{0, 1, \ldots, 49\}$. Furthermore, $a_k \leq 59$, for $k = 0, 1, \ldots, 49$. From the well-known properties of continued fractions, we get that

$$\left|\gamma - \frac{2c + s}{2ns}\right| = \left|\gamma - \frac{p_t}{q_t}\right| > \frac{1}{(a_t + 2)\,q_t^2} > \frac{1}{(a_t + 2)\,(2ns)^2} > \frac{1}{4 \times 61 \times 6.8^2 \times c^2}. \qquad (19)$$

After combining the inequalities (18) and (19), then

$$\frac{1}{4 \times 61 \times 6.8^2 \times c^2} < \frac{1}{14960c^2s}$$

gives $s < 1$. But, this is not possible.

Therefore, c is at most 29. By the inequality $ns < 6.8c$, we obtain that $ns \leq 197$, which is false as $n > 233$.

Hence, the proof theorem is completed.

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