Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 25, 2019, No. 4, 8–15
DOI: 10.7546/nntdm.2019.25.4.8-15

A note on balanced numbers

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Received: 22 August 2019 **Accepted:** 26 October 2019

Abstract: A new proof of solvability of equations

$$\frac{\sigma(n)}{d(n)} = \frac{n}{2}$$

and

$$\frac{\sigma_k(n)}{d(n)} = \frac{n^k}{2},$$

for $k > 1$ are given. Connections with related problems and inequalities are pointed out, too.
Keywords: Arithmetic function, Balanced number, Inequality, Sum of divisors of a number.
2010 Mathematics Subject Classification: 11A25, 26D15.

1 Introduction

In 1963, M. V. Subbarao [9] considered the solvability of equation

$$\frac{\sigma(n)}{d(n)} = \frac{n}{2}, \tag{1}$$

where $n \geq 1$ is an integer, while $\sigma(n)$ and $d(n)$ denote, respectively, the sum and the number of the divisors of n. A number n satisfying equation (1) has been called a *balanced number.* It is shown that $n = 6$ is the single balanced number.

Around 2007, the two authors discovered independently (see [3] and [10]) new proofs of this result.

Let $\sigma_k(n)$ denote the sum of the k-th powers of the divisors of n ($k \geq 1$, integer). Let us call a number n *k-balanced number* if it satisfies the equality

$$\frac{\sigma_k(n)}{d(n)} = \frac{n^k}{2}. \tag{2}$$

In [3], it was shown that for $k > 1$ there are no k-balanced numbers. The proofs, given in [3], are based on the inequality

$$\frac{\sigma_k(n)}{d(n)} \leq \frac{n^k}{2}, \tag{3}$$

for any $k, n \geq 1$ and $\omega(n) \geq 2$, where $\omega(n)$ denotes the number of the distinct prime divisors of n. This inequality was published in 1990 in [1], and proved by the use of the Jensen–Hadamard (or Hadamard) integral inequalities of real analysis. Though not explicitly stated, in [3] the following results were pointed out.

Theorem 1. *The inequality* (3) *holds true in case* $k = 1$ *for any* $n > 1$ *if and only if* $n \neq 4$ *and* n *is a prime number. There is equality only for* $n = 6$.

Theorem 2. *The inequality* (3) *holds true for any* $n > 1$ *if and only if* n *is a prime number.*

The aim of the present paper is to offer new proofs of these two theorems. Certain new inequalities, as well as connections with related problems will be considered, too.

2 Main results

Theorem 3. *For any integers* $n > 1$ *and* $k \geq 1$, *one has*

$$\sigma_k(n) \leq n^k \left(1 - \frac{1}{2^{k-1}}\right) + \frac{n^k.d(n)}{2^k} + 1. \tag{4}$$

When $n \geq 3$ *is an odd number, one has*

$$\sigma_k(n) \leq n^k \left(1 - \frac{1}{3^k}\right) + \frac{n^k.d(n)}{3^k} + 1. \tag{5}$$

There is an equality in (4) only for prime n, or $n = 4$; and there is an equality in (5) only for n prime, or $n = 9$.

By letting $k = 1$, we get the following theorem.

Theorem 4. *For any integer* $n > 1$ *one has*

$$\sigma(n) \leq \frac{n.d(n)}{2} + 1, \tag{6}$$

with equality only for $n =$ *prime, or* $n = 4$. *If* $n \geq 3$ *is odd, then*

$$\sigma(n) \leq \frac{n.(d(n) + 1)}{3} + 1, \tag{7}$$

with equality only for n *prime, or* $n = 9$.

When $k = 1$, (6) is a simple corollary of (4), and, respectively, (7) is a simple corollary of (5). Thus, we shall prove Theorem 3.

Proof of Theorem 3. First, we remark that when n is prime, then $d(n) = 2$ and since

$$\sigma_k(n) = n^k + 1,$$

so there is equality (4), as well as (5). Let us suppose now that $n > 1$ is composite. If s is a divisor of n, then $n = s.k$, where $k \geq 1$. Now, let us suppose that $s \neq 1$ and $s \neq n$. Then, it is clear that $k \neq 1$, so $k \geq 2$. This gives that $d(n) > 2$ and

$$s = \frac{n}{k} \leq \frac{n}{2}.$$

This implies at once

$$\sigma_k(n) = 1 + n^k + \sum_{1<r<n, r|n} r^k \leq 1 + n^k + \left(\frac{n}{2}\right)^k .(d(n) - 2)$$

$$= 1 + n^k + \left(\frac{n}{2}\right)^k .d(n) - 2\left(\frac{n}{2}\right)^k = n^k\left(1 - \frac{1}{2^{k-1}}\right) + \frac{n^k.d(n)}{2^k} + 1.$$

This gives the inequality (4). Now, for the case of equality, besides the primes, remark that all divisors of n, distinct from 1 and n should be equal to $\frac{n}{2}$. Thus, n should be even, i.e., $n = 2^k.N$, where N is an odd number. But then, if $N > 1$, by $\frac{n}{2} = 2^{k-1}N$, at least one divisor of N would be distinct from $\frac{n}{2}$. Thus $N = 1$.

If $k = 1$, then $n = 2$. If $k \geq 3$, $2 \neq \frac{n}{2}$ would be another divisor of $\frac{n}{2}$. Thus, $n = 2^2 = 4$.

The proof of (5) is similar, by remarking that when $n \geq 3$ is odd, then if $s \neq 1, s \neq n$ is a divisor of n, then $n = s.k$, so $k \geq 3$, and this implies

$$s = \frac{n}{k} \leq \frac{n}{3}.$$

One gets

$$\sigma_k(n) = 1 + n^k + \sum_{1<r<n, r|n} r^k \leq 1 + n^k + \left(\frac{n}{3}\right)^k .(d(n) - 2)$$

and (5) follows. The case of equality follows at once by similarity. □

Theorem 5. *For any $k > 1$, one has*

$$\sigma_k(n) < n^k\zeta(k), \tag{8}$$

where $\zeta(k)$ is the value at $s = k$ of the Riemann zeta-function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Proof. The result is well-known, see, e.g., [5]; however, here we offer a simple proof of (8).

$$\sigma_k(n) = \sum_{1<r<n, r|n} r^k = \sum_{1<r<n, r|n} \left(\frac{n}{r}\right)^k = n^k \sum_{1<r<n, r|n} \frac{1}{r^k}$$

$$\leq n^k \sum_{1<r\leq n} \frac{1}{r^k} < n^k \sum_{r=1}^{\infty} \frac{1}{r^k} = n^k\zeta(k).$$ □

Proof of Theorem 1. First, let $n \geq 3$ be odd, but not a prime number, i.e., $n \geq 9$. Then, by using inequality (7), it will be sufficient to prove that

$$\frac{n}{3}.(d(n)+1)+1 \leq \frac{n.d(n)}{2}. \tag{9}$$

It is seen immediately that (9) can be written as

$$2n+6 < n.d(n). \tag{10}$$

Really, in (10) n is not prime, i.e., $d(n) \geq 3$, from where $n.d(n) \geq 3n > 2n+6$. Now, let $n \geq 2$ be even. So, we write $n = 2^k N$, where $N \geq 1$ is an odd number. If $N = 1$, then $n = 2^k$ and the inequality

$$\sigma(n) < \frac{n.d(n)}{2}$$

becomes $2^{k+1} - 1 < 2^{k-1}(k+1)$ or $1 + 2^{k-1}(k+1) > 4.2^{k-1}$. This is true for $k \geq 3$; for $k = 1$, $n = 2$; and for $k = 2, \sigma(4) = 7 < 8 = 4.2^{2-1}$.

Thus, we may suppose $N \geq 3$. Then, by inequality (7) we get

$$\sigma(n) = (2^{k+1}-1)\sigma(N) \leq (2^{k+1}-1)\left(\frac{N.d(N)}{3} + \frac{N}{3} + 1\right).$$

On the other hand,

$$\frac{n.d(n)}{2} = 2^{k-1}N(k+1)d(N).$$

Now, we see that

$$2^{k+1}(N.d(N) + N + 3) < 2^{k+1}(N.d(N) + k.N.d(N))$$

or

$$N.d(N)(3k-1) > 4(N+3).$$

Now, let $k \geq 2$. As $N \geq 3$ and $d(N) \geq 2$, the above inequality, written as

$$2N(3k-1) > 4(N+3),$$

or $N(k-1) \geq 2$, follows. Let $k = 1$, i.e., $n = 2N$. Now, we have to prove that

$$\sigma(n) = 3\sigma(N) \leq \frac{nd(n)}{2} = \frac{2N.2d(N)}{2} = 2N.d(N).$$

As by (8), one has

$$\sigma(N) \leq \frac{N.d(N)}{3} + \frac{N}{3} + 1$$

for $N \geq 3$. So, we have to show that

$$N.d(N) + N + 3 \leq 2N.d(N)$$

or $N + 3 \leq N.d(N)$.

As $d(N) \geq 2$, this is true by $2N \geq N+3$, i.e., $N \geq 3$. There is an equality when $N = 3$. But then $n = 2N = 6$ and Theorem 1 is proved. □

First proof of Theorem 2. By applying relation (3), we get

$$\frac{\sigma_k(n)}{d(n)} \le \frac{n^k}{2^k} + \frac{1 + n^k\left(1 - \frac{1}{2^{k-1}}\right)}{d(n)}.$$

As $d(n) \ge 3$, it is sufficient to verify that

$$\frac{n^k}{2^k} + \frac{1 + n^k\left(1 - \frac{1}{2^{k-1}}\right)}{3} < \frac{n^k}{2} \tag{11}$$

for $k > 1$. This can be written also as

$$\frac{1}{2^k} + \frac{\frac{1}{n^k} + 1 - \frac{1}{2^{k-1}}}{3} < \frac{1}{2}.$$

Now, n cannot be 2 or 3. So, $n \ge 4$ and hence $\frac{1}{n^k} \le \frac{1}{4^k}$, and we have to verify that

$$\frac{1}{2^k} + \frac{1}{3.4^k} + \frac{1}{3} - \frac{1}{3.2^{k-1}} < \frac{1}{2}$$

or

$$\frac{1}{2^k} + \frac{1}{3.4^k} - \frac{1}{3.2^{k-1}} < \frac{1}{6}$$

or

$$\frac{1}{3.2^k} + \frac{1}{3.4^k} < \frac{1}{6}$$

or

$$\frac{1}{2^k} + \frac{1}{4^k} < \frac{1}{2}$$

that is obviously true because $k \ge 2$. □

Second proof of Theorem 2. Remark that for $k \ge 2$ one has $\zeta(k) \le \zeta(2) = \frac{\pi^2}{6}$, so by Theorem 5 we can write $\sigma_k(n) < n^k.\frac{\pi^2}{6}$. Thus (4) is true if

$$d(n) \ge 2.\frac{\pi^2}{6} > 2\zeta(k).$$

As $\frac{\pi^2}{6} = 3.28\ldots$, this is true if $d(n) \ge 4$.

Thus, we have to verify inequality (4) only when $d(n) = 3$. Now, this is possible only if $n = p^2$ (p is prime), in which case (4) becomes $\frac{1 + p^k + p^{2k}}{3} < \frac{p^{2k}}{2}$ or $(p^k - 1)^2 > 3$.

As $p \ge 2, k \ge 2$, this is trivially true. □

3 Related results

1. Inequality (7) refines the famous Landford inequality (see, e.g., [2]) for any $n \ge 2$.

$$\frac{\sigma(n)}{d(n)} \le \frac{n+1}{2}, \tag{12}$$

i.e., one has

$$\sigma(n) \le \frac{nd(n)}{2} + 1 \le \left(\frac{n+1}{2}\right) d(n). \tag{13}$$

Indeed, the second inequality of (13) becomes $d(n) \ge 2$. There is an equality only when n is prime.

2. Relations (7) and (8) can be extended as follows

Theorem 6. *Let $n > 1$ and denote by $p(n)$ the least prime factor of n. Then one has*

$$\sigma(n) \le \frac{n(d(n) + p(n) - 2)}{p(n)} + 1. \tag{14}$$

Proof. This is similar to the proof of Theorem 3, by remarking that for $r \ne 1$ and $r \ne n$, in $n = rk$ one has $k \ge p(n)$, so $r = \frac{n}{k} \le \frac{n}{p(n)}$. Therefore,

$$\sigma(n) \le n + 1 + \frac{n}{p(n)}.(d(n) - 2),$$

and (14) follows. □

Remark 1. Clearly, by this method, an extension of (14) for σ_k can be formulated:

Theorem 7. *For any $n, k \ge 1$,*

$$\sigma_k(n) \le n^k + 1 + \left(\frac{n}{p(n)}\right)^k.(d(n) - 2). \tag{15}$$

Remark 2. In fact, (14) is a sharpening of (7), as

$$\frac{d(n) + p(n) - 2}{p(n)} \le \frac{d(n)}{2}. \tag{16}$$

Indeed, (16) may be written equivalently as

$$(d(n) - 2)(p(n) - 2) \ge 0,$$

which is true, as $d(n) \ge 2$ and $p(n) \ge 2$.

3. One can obtain lower bounds also for $\sigma(n)$ and $\sigma_k(n)$ by remarking that for any divisor $r \ne 1$ and $r \ne n$ of n one has $r \ge 2$. Indeed, if $n = rk$, where $k \ne 1, k \ne n$ also, then clearly $r \ge 2, k \ge 2$. Thus,

$$\sigma_k(n) = n^k + 1 + \sum_{1<r<n, r|n} r^k \ge n^k + 1 + 2^k(d(n) - 2).$$

Theorem 8. *For any $n > 1$,*

$$\sigma_k(n) \ge 2^k d(n) + n^k - 2^{k+1} + 1, \tag{17}$$

with equality only if n is prime.

For $k = 1$ we get the following theorem.

Theorem 9. *For any* $n > 1$,

$$\sigma(n) \geq 2d(n) + n - 3, \tag{18}$$

with equality only if n *is prime.*

Remark 3. If $n \geq 3$ is odd, then we get

$$\sigma(n) \geq 3d(n) + n - 5. \tag{19}$$

The proof is similar, by remarking that any divisor $r \neq 1$, $r \neq n$ if $n \geq 3$.

Remark 4. Inequality (18) can be written also as

$$\frac{\sigma(n)}{d(n)} \geq 2 + \frac{n-3}{d(n)}.$$

Now, the inequality $2+ \frac{n-3}{d(n)} > \sqrt{n}$ can be written as $d(n) < \frac{n-3}{\sqrt{n}-2}$. The inequality $\frac{n-3}{d(n)} > \sqrt{n}$ can be written as $2\sqrt{n} > 3$, which is valid for $n \geq 3$. Therefore, if

$$d(n) < \sqrt{n} \tag{20}$$

holds true, one has

$$\frac{\sigma(n)}{d(n)} \geq 2 + \frac{n-3}{d(n)} > \sqrt{n}. \tag{21}$$

In [4], it is proved that (20) is true for any $n \geq 1262$. Thus, (21) holds for such value of n. Clearly, the second inequality of (21) holds true for any prime n. The inequality

$$\frac{\sigma(n)}{d(n)} > \sqrt{n} \tag{22}$$

holds true for any $n > 1$ (see [2]), and (21) gives an improvement of this relation.

4 Final remarks

1. In 2009, in [6], the first author proved the inequality ($n > 1$):

$$\sigma(n) \geq n + 1 + \sqrt{n}(d(n) - 2), \tag{23}$$

with equality only if $n = p$ or $n = p^2$ for p being prime.

This is stronger than (18), as the inequality

$$n + 1 + \sqrt{n}(d(n) - 2) \geq 2d(n) + n - 3$$

is equivalent to $d(n) \geq 2$, which is valid for any $n \geq 2$.

From (23), we get

$$\frac{\sigma(n)}{d(n)} \geq \sqrt{n} + \frac{(\sqrt{n}-1)^2}{d(n)} > \sqrt{n} \tag{24}$$

for $n \geq 1$, which is another refinement of (22).

2. In paper [7], it has been shown that

$$d(n) < 4\sqrt[3]{n} \tag{25}$$

for any $n > 1$, which improves inequality (20) if $4\sqrt[3]{n} < \sqrt{n}$, i.e., if $n > 4^6 = 4096$. We also note that (25) improves the classical Sierpinski's inequality (see, [2]) $d(n) < 2\sqrt{n}$, if $4\sqrt[3]{n} < \sqrt{n}$, i.e., $n > 2^6 = 64$.

3. In 2014, in paper [8], by using other methods, the first author proved the following refinement of Theorem 2: *For $k > 1$ and $n > 1$ not prime, one has*

$$\frac{\sigma_k(n)}{n^k} < \frac{2n}{n+\varphi(n)} < \frac{d(n)}{2} \tag{26}$$

where $\varphi(n)$ is the Euler's totient function.

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